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# ON THE DISCREPANCY OF SEQUENCES ASSOCIATED WITH THE SUM-OF-DIGITS FUNCTION

by N. KOPECEK, G. LARCHER, R.F. TICHY and G. TURNWALD

## 1. Introduction.

In a series of papers J. Coquet *et al.* investigated the distribution modulo 1 of sequences  $(x \cdot s_\alpha(n))_{n=0}^\infty$  where  $x$  is an irrational number and  $s_\alpha(n)$  denotes the sum of digits in the  $\alpha$ -adic expansion of  $n$  (cf. [1], [2], [3], [5])<sup>(\*)</sup>. We will give a quantitative refinement and a generalization to the multi-dimensional case.

Let  $(y_n)_{n=0}^\infty$  be a sequence of elements of  $\mathbf{R}^d$  ( $d \geq 1$ ). Then the discrepancy mod 1 of  $(y_n)$  is defined by

$$D_N(y_n) = \sup_I \left| \frac{A(I, N, y_n)}{N} - \text{vol}(I) \right|, \quad (1.1)$$

where the supremum is extended over all  $d$ -dimensional subintervals of  $[0, 1]^d$  of the form  $I = \{(t_1, \dots, t_d) : a_j \leq t_j < b_j \text{ for } 1 \leq j \leq d\}$ ,

$\text{vol}(I)$  means the volume  $\prod_{j=1}^d (b_j - a_j)$  of  $I$ , and  $A(I, N, y_n)$

denotes the number of indices  $n$  ( $0 \leq n < N$ ) such that the fractional part of the  $j$ -th component of  $y_n$  belongs to the interval  $[a_j, b_j[$  for  $j = 1, \dots, d$ . The sequence  $(y_n)$  is uniformly distributed mod 1 if and only if

$$\lim_{N \rightarrow \infty} D_N(y_n) = 0;$$

cf. the monographs [4] and [7].

(\*) These investigations were initiated by M. Mendès-France [*J. Analyse Math.*, 20 (1967), 1-56].

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Let  $\alpha$  be an irrational number with continued fraction expansion  $[a_0; a_1, a_2, \dots]$ . Let  $q_0 = 1, q_1 = a_1$ , and

$$q_{k+2} = a_{k+2} q_{k+1} + q_k \quad (k \geq 0).$$

We define the  $\alpha$ -adic expansion of a positive integer by

$$n = \sum_{k=0}^{L(n)} \epsilon_k(n) q_k \quad (\epsilon_{L(n)}(n) \neq 0), \quad (1.2)$$

where the digits  $\epsilon_k(n)$  satisfy the following conditions:

$$(i) \quad 0 \leq \epsilon_0(n) < a_1,$$

$$(ii) \quad 0 \leq \epsilon_k(n) \leq a_{k+1} \quad (k \geq 1),$$

and

$$(iii) \quad \epsilon_k(n) = a_{k+1} \text{ implies } \epsilon_{k-1}(n) = 0.$$

In the following we consider the sequence  $y_n = x s_\alpha(n)$  for a fixed vector  $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ , where

$$s_\alpha(n) = \sum_{k=0}^{L(n)} \epsilon_k(n).$$

By [2], the one-dimensional sequence  $(x s_\alpha(n))$  is uniformly distributed mod 1 if  $x$  is an irrational number. In order to obtain estimates for the discrepancy  $D_N(y_n)$ , we need information concerning the diophantine approximation properties of  $x = (x_1, \dots, x_d)$ . Let  $\psi: [0, \infty) \rightarrow [0, \infty)$  be a continuous strictly increasing function with  $\psi(0) = 0$  and  $\psi(t) \geq t$ . We say that  $x$  is of approximation type  $< \psi$  if there exists a positive constant  $c = c(x, \psi)$  such that

$$\|h \cdot x\| \geq \frac{c}{\psi(r(h))} \quad (1.3)$$

for all lattice points  $h = (h_1, \dots, h_d) \in \mathbf{Z}^d, h \neq (0, \dots, 0)$ ;  $\|t\|$  denotes the distance from the real number  $t$  to the nearest integer

and  $r(h) = \prod_{j=1}^d \max(|h_j|, 1)$ . We will prove the following results:

**THEOREM 1.** — *Let  $x = (x_1, \dots, x_d)$  be of approximation type  $< \psi$  and  $\alpha = [a_0; a_1; a_2, \dots]$  an irrational number. Then for every  $\epsilon > 0$  there exists a constant  $c = c(x, \psi, \epsilon, \alpha)$  such that*

$$D_N(x s_\alpha(n)) \leq \frac{c}{(\psi^*(L(N)^{1/2-\epsilon}))^{1/d}}$$

for all integers  $N \geq a_1$ . ( $\psi^*$  denotes the inverse function of  $\psi$ .)

Let  $\eta \geq 1$  be a real number; then we say that  $x = (x_1, \dots, x_d)$  is of finite approximation type  $\eta$  if (1.3) holds with  $\psi(t) = t^{\eta+\delta}$  for every  $\delta > 0$ . Obviously,  $1, x_1, \dots, x_d$  must be linearly independent over the rationals; conversely, by a famous theorem of W.M. Schmidt [8], under this assumption  $x = (x_1, \dots, x_d)$  is of finite approximation type  $\eta = 1$ , if  $x_1, \dots, x_d$  are algebraic numbers. Hence we obtain

**COROLLARY.** — Let  $x = (x_1, \dots, x_d)$  be of finite approximation type  $\eta$ . Then we have (in the notation of the theorem)

$$D_N(x s_\alpha(n)) \leq c'(x, \eta, \epsilon, \alpha) L(N)^{-\frac{1}{2d\eta} + \epsilon} \quad \text{for every } \epsilon > 0.$$

If  $1, x_1, \dots, x_d$  are algebraic and linearly independent over the rationals then

$$D_N(x \cdot s_\alpha(n)) \leq c''(x, \epsilon, \alpha) L(N)^{-\frac{1}{2d} + \epsilon} \quad \text{for every } \epsilon > 0.$$

At last we consider more exactly the case  $d = 1$  and we show that the result of the theorem is, apart from the constant best possible, if we assume that  $\alpha$  has bounded continued fraction coefficients.

*Remark.* — In [9] the authors have established a corresponding result (for dimension  $d = 1$ ) for the sequence  $(x \cdot s(q; n))$ , where  $s(q; n)$  denotes the sum of digits of  $n$  in the usual  $q$ -adic expansion ( $q \geq 2$  integral).

**THEOREM 2.** — Let  $x \in \mathbf{R}$ ,  $c$  and  $\psi$  be such that

$$\|h \cdot x\| \leq \frac{c}{\psi(h)}$$

for infinitely many  $h \in \mathbf{N}$ , and  $\alpha = [a_0; a_1, a_2, \dots]$  an irrational number with  $a_i \leq K$  for all  $i$ , then there is a constant  $c_1 = c_1(x, \psi, c, \alpha)$  such that

$$D_N(x \cdot s_\alpha(n)) \geq \frac{c_1}{(\psi^*(L(N)^{1/2}))}$$

for infinitely many  $N$ .

## 2. Auxiliary results.

Our main tool for estimating the discrepancy of a sequence is the inequality of Erdős-Turan-Koksma ([6], cf. [7]):

LEMMA 1. — Let  $(y_n)_{n=0}^\infty$  denote a sequence of elements of  $\mathbf{R}^d$ . Then for an arbitrary integer  $H \geq 1$  we have

$$D_N(y_n) \leq C_d \cdot \left( \frac{1}{H} + \sum_{\substack{\mathbf{h}=(h_1, \dots, h_d) \in \mathbf{Z}^d \\ 0 < \max(|h_1|, \dots, |h_d|) < H}} r(\mathbf{h})^{-1} \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i(\mathbf{h} \cdot y_n)) \right| \right),$$

for some constant  $C_d$  only depending on  $d$  ( $\exp t := e^t$ ).

A useful instrument in the proof of our Theorem 1 is the following elementary inequality:

LEMMA 2. — For non-integral  $t$  and integral  $n \geq 2$  we have

$$\left| \frac{1 - \exp(2\pi i n t)}{1 - \exp(2\pi i t)} \right| \leq \frac{n}{1 + \pi \|t\|^2}.$$

*Proof.* — The left-hand side is equal to  $\left| \frac{\sin n\pi t}{\sin \pi t} \right|$ . Since

$$\left| \frac{\sin 2\pi t}{\sin \pi t} \right| = 2 |\cos \pi t| \leq \frac{2}{2 - |\cos \pi t|},$$

the inequality

$$\left| \frac{\sin n\pi t}{\sin \pi t} \right| \leq \frac{n}{2 - |\cos \pi t|} \tag{2.1}$$

holds for  $n = 2$ . Suppose that (2.1) holds for some  $n \geq 2$ . Then

$$\left| \frac{\sin (n+1) \pi t}{\sin \pi t} \right| = \left| \cos n \pi t + \frac{\sin n \pi t}{\sin \pi t} \cos \pi t \right|$$

$$\leq 1 + \frac{n}{2 - |\cos \pi t|} |\cos \pi t| \leq \frac{n+1}{2 - |\cos \pi t|};$$

thus, by induction, (2.1) holds for all  $n \geq 2$ . Next we observe that  $|\cos \pi t| = \cos \pi \|t\|$ . Hence the assertion of Lemma 2 follows from (2.1) and the inequality  $\cos \pi \|t\| \leq 1 - \pi \|t\|^2$  (which is valid since

$$\cos \pi \|t\| = 1 - \int_0^{\pi \|t\|} \sin u \, du \leq 1 - \int_0^{\pi \|t\|} \frac{2}{\pi} u \, du = 1 - \pi \|t\|^2).$$

In order to apply Lemma 1 we have to derive estimates for the exponential sums

$$\sum_{n=0}^{N-1} \exp (2 \pi i \mathbf{h} \cdot \mathbf{x} s_{\alpha}(n)).$$

LEMMA 3. — Put  $\vartheta_0 = \frac{1}{1 + \pi \|\mathbf{h} \cdot \mathbf{x}\|^2}$ ,  $\vartheta = \frac{\vartheta_0 + 4}{5}$ , and

$$S_k = \sum_{0 \leq n < q_k} \exp (2 \pi i \mathbf{h} \cdot \mathbf{x} s_{\alpha}(n)).$$

If  $\mathbf{h} \cdot \mathbf{x}$  is non-integral, then  $|S_k| \leq \vartheta^{k-1} q_k$  for  $k \geq 0$ .

*Proof.* — The inequality holds for  $k = 0$  since  $0 < \vartheta \leq 1$ , and is trivial for  $k = 1$ . For  $k \geq 2$  we split up the range of summation  $0 \leq n < q_k = a_k q_{k-1} + q_{k-2}$  into the intervals

$$0 \leq n < q_{k-1}, q_{k-1} \leq n < 2q_{k-1}, \dots, (a_k - 1) q_{k-1} \leq n < a_k q_{k-1},$$

and  $a_k q_{k-1} \leq n < a_k q_{k-1} + q_{k-2}$ . Since

$$s_{\alpha}(m q_{k-1} + r) = m + s_{\alpha}(r)$$

for  $m < a_k$  and  $r < q_{k-1}$ , and  $s_{\alpha}(a_k q_k + r) = a_k + s_{\alpha}(r)$  for  $r < q_{k-2}$ , this yields

$$\begin{aligned}
S_k &= (1 + \exp(2\pi ih \cdot x) + \dots + (\exp(2\pi ih \cdot x))^{a_k-1}) S_{k-1} \\
&\quad + (\exp(2\pi ih \cdot x a_k)) S_{k-2} \\
&= \frac{1 - \exp(2\pi ih \cdot x a_k)}{1 - \exp(2\pi ih \cdot x)} S_{k-1} + \exp(2\pi ih \cdot x a_k) S_{k-2}.
\end{aligned} \tag{2.2}$$

Hence, by Lemma 2, we obtain ( $k \geq 2$ )

$$|S_k| \leq \vartheta_0 a_k |S_{k-1}| + |S_{k-2}| \quad \text{for } a_k \neq 1, \tag{2.3}$$

$$|S_k| \leq |S_{k-1}| + |S_{k-2}| \quad \text{for } a_k = 1.$$

If  $k = 2$ , we have

$$|S_2| \leq \vartheta_0 a_2 q_1 + 1 \leq \frac{1 + \vartheta_0}{2} (a_2 q_1 + 1) = \frac{1 + \vartheta_0}{2} q_2 \leq \vartheta q_2$$

for  $a_2 \neq 1$  or  $a_1 \neq 1$ ;

$$|S_2| = |1 + \exp(2\pi ih \cdot x)| \leq 2 \vartheta_0 \leq 2 \vartheta = \vartheta q_2$$

(by (2.2) and Lemma 2) for  $a_1 = a_2 = 1$ . For  $k \geq 3$  the assertion of Lemma 3 will be proved by induction. Assume that

$$|S_m| \leq \vartheta^{m-1} q_m \tag{2.4}$$

for  $0 \leq m < k$ .

*Case (i):*  $a_k \neq 1$ . Applying (2.3) we have

$$|S_k| \leq \vartheta_0 a_k |S_{k-1}| + |S_{k-2}|.$$

Hence by (2.4)

$$\begin{aligned}
|S_k| &\leq \vartheta_0 a_k \vartheta^{k-2} q_{k-1} + \vartheta^{k-3} q_{k-2} \\
&= \vartheta^{k-1} (a_k q_{k-1} + q_{k-2}) \\
&\quad - \vartheta^{k-3} ((\vartheta^2 - \vartheta_0 \vartheta) a_k q_{k-1} + (\vartheta^2 - 1) q_{k-2}) \\
&\leq \vartheta^{k-1} q_k - \vartheta^{k-3} (\vartheta^2 - \vartheta_0 \vartheta) a_k \\
&\quad + (\vartheta^2 - 1) q_{k-1} \leq \vartheta^{k-1} q_k;
\end{aligned}$$

the least inequality holds since

$$\begin{aligned}
& (\vartheta^2 - \vartheta_0 \vartheta) a_k + (\vartheta^2 - 1) \geq (\vartheta^2 - \vartheta_0 \vartheta) + (\vartheta^2 - 1) \\
& = 2\vartheta^2 - (5\vartheta - 4)\vartheta - 1 = (1 - \vartheta)(3\vartheta - 1) \geq 0
\end{aligned}$$

(note that  $1 \geq \vartheta \geq \vartheta_0 \geq \frac{1}{2}$ ).

*Case (ii):*  $a_k = 1$  and  $a_{k-1} \neq 1$ . By a double application of (2.3) we have

$$|S_k| \leq (1 + \vartheta_0 a_{k-1}) |S_{k-2}| + |S_{k-3}|.$$

Hence by (2.4)

$$\begin{aligned}
|S_k| & \leq (1 + \vartheta_0 a_{k-1}) \vartheta^{k-3} q_{k-2} + \vartheta^{k-4} q_{k-3} \\
& = \vartheta^{k-1} ((1 + a_{k-1}) q_{k-2} + q_{k-3}) - \vartheta^{k-4} ((\vartheta^3 (1 + a_{k-1}) \\
& \quad - \vartheta (1 + \vartheta_0 a_{k-1})) \cdot q_{k-2} + (\vartheta^3 - 1) q_{k-3}) \\
& \leq \vartheta^{k-1} q_k - \vartheta^{k-4} (\vartheta^3 (1 + a_{k-1}) - \vartheta (1 + \vartheta_0 a_{k-1}) \\
& \quad + \vartheta^3 - 1) q_{k-2} \leq \vartheta^{k-1} q_k;
\end{aligned}$$

the last inequality holds since

$$\begin{aligned}
& \vartheta^3 (1 + a_{k-1}) - \vartheta (1 + (5\vartheta - 4) a_{k-1}) + \vartheta^3 - 1 \\
& = (\vartheta^3 - 5\vartheta^2 + 4\vartheta) a_{k-1} + (2\vartheta^3 - \vartheta - 1) \\
& \geq \vartheta (1 - \vartheta) (4 - \vartheta) 2 + (2\vartheta^3 - \vartheta - 1) \\
& = 4(1 - \vartheta) \left( \vartheta - \frac{3 - \sqrt{5}}{4} \right) \left( \frac{3 + \sqrt{5}}{4} - \vartheta \right) \geq 0.
\end{aligned}$$

*Case (iii):*  $a_k = a_{k-1} = 1$ . By a double application of (2.2) we have

$$\begin{aligned}
|S_k| & = |(1 + \exp(2\pi i h \cdot x)) S_{k-2} + \exp(2\pi i h \cdot x) S_{k-3}| \\
& \leq 2\vartheta_0 |S_{k-2}| + |S_{k-3}|
\end{aligned}$$

(applying Lemma 2 for  $n = 2$ ).

Hence by (2.4)



$$\begin{aligned}
|S_k| &\leq 2\vartheta_0 \vartheta^{k-3} q_{k-2} + \vartheta^{k-4} q_{k-3} \\
&\leq \vartheta^{k-1} (2q_{k-2} + q_{k-3}) - \vartheta^{k-4} ((2\vartheta^3 - 2\vartheta_0 \vartheta) q_{k-2} \\
&\quad + (\vartheta^3 - 1) q_{k-3}) \\
&\leq \vartheta^{k-1} q_k - \vartheta^{k-4} (3\vartheta^3 - 2\vartheta_0 \vartheta - 1) q_{k-2} \leq \vartheta^{k-1} q_k;
\end{aligned}$$

the last inequality holds since

$$3\vartheta^3 - 2\vartheta(5\vartheta - 4) - 1$$

$$= 3(1 - \vartheta) \left( \vartheta - \frac{7 - \sqrt{37}}{6} \right) \left( \frac{7 + \sqrt{37}}{6} - \vartheta \right) \geq 0.$$

Thus, by induction, (2.4) holds for all  $m \geq 0$  and the proof of Lemma 3 is completed.

LEMMA 4. — *Let  $\vartheta$  be defined as in Lemma 3. If  $\mathbf{h} \cdot \mathbf{x}$  is non-integral then we have*

$$\left| \sum_{0 \leq n < N} \exp(2\pi i \mathbf{h} \cdot \mathbf{x} s_\alpha(n)) \right| \leq \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^{L(N)} N.$$

*Proof.* — Put  $u_n = \exp(2\pi i \mathbf{h} \cdot \mathbf{x} s_\alpha(n))$  and  $N = \sum_{k=0}^L \epsilon_k q_k$

(compare (1.2)). Splitting up the range of summation  $0 \leq n < N$  into the intervals

$$0 \leq n < \epsilon_L q_L, \epsilon_L q_L \leq n < \epsilon_L q_L$$

$$+ \epsilon_{L-1} q_{L-1}, \dots, \epsilon_L q_L + \dots + \epsilon_1 q_1 \leq n < \epsilon_L q_L + \dots + \epsilon_0 q_0$$

we obtain

$$\begin{aligned}
\left| \sum_{0 \leq n < N} u_n \right| &= \left| \sum_{0 \leq n < \epsilon_L q_L} u_n \right. \\
&\quad + \exp(2\pi i \mathbf{h} \cdot \mathbf{x} \epsilon_L) \sum_{0 \leq n < \epsilon_{L-1} q_{L-1}} u_n + \dots \\
&\quad \left. + \exp(2\pi i \mathbf{h} \cdot \mathbf{x} (\epsilon_L + \dots + \epsilon_1)) \sum_{0 \leq n < \epsilon_0 q_0} u_n \right| \\
&\leq \sum_{k=0}^L \left| \sum_{0 \leq n < \epsilon_k q_k} u_n \right|
\end{aligned}$$

(cf. the first lines of the proof of Lemma 3).

Similarly we derive

$$\begin{aligned} \left| \sum_{0 \leq n < \epsilon_k q_k} u_n \right| &= \left| \sum_{0 \leq n < q_k} u_n + \dots + \sum_{(\epsilon_k - 1)q_k \leq n < \epsilon_k q_k} u_n \right| \\ &= \left| (1 + e^{2\pi i h \cdot x} + \dots + e^{2\pi i h \cdot x (\epsilon_k - 1)}) \sum_{0 \leq n < q_k} u_n \right| \leq \epsilon_k |S_k|. \end{aligned}$$

Applying Lemma 3 thus yields

$$\left| \sum_{0 \leq n < N} u_n \right| \leq \sum_{k=0}^L \epsilon_k \vartheta^{k-1} q_k.$$

In order to complete the proof of Lemma 4 it remains to show

$$\sum_{k=0}^l \epsilon_k \vartheta^{k-1} q_k \leq \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^l \sum_{k=0}^l \epsilon_k q_k \quad (2.5)$$

for  $l = L$ . For  $l = 0$  (2.5) holds trivially; inductively we assume that (2.5) holds for  $l < L$ . Then

$$\begin{aligned} \sum_{k=0}^L \epsilon_k \vartheta^{k-1} q_k &= \sum_{k=0}^{L-1} \epsilon_k \vartheta^{k-1} q_k + \epsilon_L q_L \vartheta^{L-1} \\ &\leq \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^{L-1} \sum_{k=0}^{L-1} \epsilon_k q_k + \epsilon_L q_L \vartheta^{L-1} \\ &= \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^L \sum_{k=0}^L \epsilon_k q_k \\ &\quad - \frac{1}{\vartheta} \left( \left( \frac{1 + \vartheta}{2} \right)^L - \left( \frac{1 + \vartheta}{2} \right)^{L-1} \right) \sum_{k=0}^{L-1} \epsilon_k q_k \\ &\quad + \epsilon_L q_L \left( \vartheta^{L-1} - \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^L \right) \leq \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^L \sum_{k=0}^L \epsilon_k q_k \\ &\quad - \frac{1}{\vartheta} \left( \left( \frac{1 + \vartheta}{2} \right)^L - \left( \frac{1 + \vartheta}{2} \right)^{L-1} \right) q_L \\ &\quad + \left( \vartheta^{L-1} - \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^L \right) q_L \leq \frac{1}{\vartheta} \left( \frac{1 + \vartheta}{2} \right)^L \sum_{k=0}^L \epsilon_k q_k; \end{aligned}$$

the last inequality follows from

$$\begin{aligned} \vartheta^L - \left(\frac{1+\vartheta}{2}\right)^L &\leq \vartheta \left(\frac{1+\vartheta}{2}\right)^{L-1} - \left(\frac{1+\vartheta}{2}\right)^L \\ &= \left(\frac{1+\vartheta}{2}\right)^L - \left(\frac{1+\vartheta}{2}\right)^{L-1}. \end{aligned}$$

Thus the proof of Lemma 4 is completed.

### 3. Proof of Theorem 1.

From Lemma 4 and (1.3) we obtain (with  $L = L(N)$ )

$$\begin{aligned} \left| \frac{1}{N} \sum_{0 \leq n < N} \exp(2\pi i \mathbf{h} \cdot \mathbf{x} s_\alpha(n)) \right| &\leq 2 \left( \frac{\vartheta_0 + 9}{10} \right)^L \\ &\leq 2 \cdot \left( \frac{9c_1 + 10\psi(r(\mathbf{h}))^2}{10c_1 + 10\psi(r(\mathbf{h}))^2} \right)^L \\ &= 2 \left( 1 - \frac{c_1 L / 10c_1 + 10\psi(r(\mathbf{h}))^2}{L} \right)^L \\ &\leq 2 \exp \left( - \frac{c_1 L}{10c_1 + 10\psi(r(\mathbf{h}))^2} \right) \end{aligned} \tag{3.1}$$

for some constant  $c_1 = c_1(\mathbf{x}, \psi) > 0$ ; the last inequality holds since

$\left(1 - \frac{1}{u}\right)^u \leq \frac{1}{e}$  for  $u \geq 1$ . Hence Lemma 1 yields (for some  $c_2 = c_2(d)$ )

$$D_N(\mathbf{x} s_\alpha(n)) \leq c_2 \left( \frac{1}{H} + (\log(H+1))^d \exp \left( - \frac{c_1 L}{10c_1 + 10\psi(H^d)^2} \right) \right) \tag{3.2}$$

where we have used  $r(\mathbf{h}) \leq H^d$  and

$$\begin{aligned} \sum_{\substack{\mathbf{h} = (h_1, \dots, h_d) \in \mathbf{Z}^d \\ 0 < \max(|h_1|, \dots, |h_d|) \leq H}} r(\mathbf{h})^{-1} &= \left( 1 + 2 \sum_{h=1}^H \frac{1}{h} \right)^d - 1 \\ &\leq 6^d (\log(H+1))^d. \end{aligned}$$

We put  $H = [(\psi^*(L^{1/2-\epsilon}))^{\frac{1}{d}}]$  for some fixed  $\epsilon$  with  $0 < \epsilon < \frac{1}{2}$  ( $[t]$  denotes the greatest integer  $\leq t$ ). Let  $N$  be sufficiently large so that we can assume  $\psi^*(L^{1/2-\epsilon}) \geq 2^d$ ; hence by (3.2)

$$D_N(x s_\alpha(n)) \leq c_2 \left( 2(\psi^*(L^{1/2-\epsilon}))^{-\frac{1}{d}} + (2 \log((\psi^*(L^{1/2-\epsilon}))^{1/d}))^d \times \exp\left(-\frac{c_1 L}{10c_1 + 10(L^{1/2-\epsilon})^2}\right) \right).$$

Since, for sufficiently large  $N \geq N_0 = N_0(x, \psi, \epsilon)$

$$\begin{aligned} \psi^*(L^{1/2-\epsilon})^{-1/d} (\log \psi^*(L^{1/2-\epsilon}))^{-d} \exp\left(\frac{c_1 L}{10c_1 + 10L^{1-2\epsilon}}\right) \\ \geq \psi^*(L^{1/2-\epsilon})^{-1/d-\epsilon} \exp(L^\epsilon) \\ \geq (L^{1/2-\epsilon})^{-1/d-\epsilon} \exp(L^\epsilon) \geq 1, \end{aligned}$$

we have

$$D_N(x s_\alpha(n)) \leq c_3 (\psi^*(L^{1/2-\epsilon}))^{-1/d} \quad \text{for } N \geq N_0.$$

If  $N \geq a_1$  then  $\psi^*(L^{1/2-\epsilon}) \neq 0$  (since  $L = L(N) > 0$ ). Hence choosing  $c \geq c_3$  such that

$$D_N(x s_\alpha(n)) \leq c (\psi^*(L^{1/2-\epsilon}))^{-1/d} \quad (c = c(x, \psi, \epsilon, \alpha)) \quad (3.3)$$

holds for the finitely many  $N$  with  $a_1 \leq N < N_0$ , (3.3) is valid for all  $N \geq a_1$ . Thus the proof of the theorem is complete.

#### 4. Proof of Theorem 2.

In the following, we need three further Lemmas:

LEMMA 5. — For a sequence  $(y_n)_{n=0}^\infty$  in  $\mathbf{R}$ , we have for every  $h \in \mathbf{N}$ :

$$D_N(y_n) \geq \frac{1}{2\pi \cdot h \cdot N} \cdot \left| \sum_{n=0}^{N-1} \exp(2\pi i h \cdot y_n) \right|.$$

*Proof.* — This is a special case of the inequality of Koksma ([7], page 142).

LEMMA 6. – For  $t \in \mathbf{R}$  and all integers  $n \geq 1$  with  $0 < n \cdot |t| < \frac{1}{4}$  we have

$$\left| \frac{1 - \exp(2\pi int)}{1 - \exp(2\pi it)} \right| \geq n \cdot (1 - (n\pi t)^2) \geq 1 - (n\pi t)^2.$$

*Proof.* – The assertion is clearly true for  $n = 1$ . By using the inequality

$$\cos \pi x = 1 - \int_0^{\pi x} \sin u \, du \geq 1 - \int_0^{\pi x} u \, du = 1 - \frac{\pi^2 x^2}{2}$$

and because  $0 < |t| \leq (n-1) \cdot |t| < n \cdot |t| < \frac{1}{4}$  we get for  $n \geq 2$  by induction:

$$\begin{aligned} \left| \frac{\sin \pi n t}{\sin \pi t} \right| &= \left| \cos(n-1) \cdot \pi t + \frac{\sin(n-1) \pi t}{\sin \pi t} \cdot \cos \pi t \right| \\ &= \cos(n-1) \pi t + \frac{\sin(n-1) \cdot \pi t}{\sin \pi t} \cdot \cos \pi t \\ &\geq 1 - \frac{((n-1) \cdot \pi t)^2}{2} + (n-1) \cdot (1 - ((n-1) \cdot \pi t)^2) \\ &\quad \cdot \left( 1 - \frac{(\pi t)^2}{2} \right) \geq n \cdot \left( 1 - (\pi t)^2 \cdot \left( \frac{n}{2} + (n-1)^2 + \frac{1}{2} \right) \right) \\ &\geq n \cdot (1 - (n\pi t)^2). \end{aligned}$$

LEMMA 7. – Let  $z_k = v_k \cdot e^{2\pi i t_k}$ ,  $k = 1, 2$  be two complex numbers not equal to zero with  $|t_1 - t_2| < \frac{1}{4}$  and  $z_1 + z_2 = v \cdot e^{2\pi i t}$ ; then

a) If we choose  $t$  such, that  $-\frac{1}{2} \leq t_1 - t < \frac{1}{2}$ , then:

$$|t_1 - t| \leq \frac{1}{1 + \frac{2v_1}{\pi v_2}} \cdot |t_1 - t_2|$$

b)  $v \geq (1 - (2\pi \cdot |t_1 - t_2|)^2) \cdot (v_1 + v_2)$ .

*Proof.* — a) We have  $\operatorname{sgn}(t_1 - t) = -\operatorname{sgn}(t_2 - t)$ , so  $|t_1 - t_2| = |t_1 - t| + |t - t_2|$  and  $|t_i - t| < \frac{1}{4}$ .

Since  $v_1 \cdot \sin(2\pi|t - t_1|) = v_2 \cdot \sin(2\pi|t - t_2|)$ , we have  $v_1 \cdot \frac{2}{\pi}(2\pi|t - t_1|) \leq v_2 \cdot 2\pi|t - t_2|$  and the assertion a) follows.

b) We have  $v = v_1 \cdot \cos(2\pi(t_1 - t)) + v_2 \cdot \cos(2\pi(t_2 - t))$ ,  $|t_i - t| \leq |t_1 - t_2|$  and therefore

$$\cos(2\pi(t_i - t)) \geq 1 - (2\pi|t_1 - t_2|)^2$$

and the assertion b) follows.

To complete the proof of Theorem 2 we proceed as follows. For a complex  $z = v \cdot e^{2\pi i u}$  we define  $\arg z := u$ , then we take  $t > 0$  so small that  $K \cdot t \leq \frac{1}{\sqrt{648} \cdot \pi}$  and then we first show by induction that for the exponential sums

$$S_n = \sum_{0 \leq n < a_n} \exp(2\pi i t \cdot s_\alpha(n))$$

we have

$$\|\arg(S_{n+1}) - \arg(S_n)\| \leq \frac{15}{2} \cdot K \cdot t \quad \text{for } n \geq 0.$$

We have  $S_0 = 1$ ,  $S_1 = \frac{1 - \exp(2\pi i t a_1)}{1 - \exp(2\pi i t)}$ , so  $\arg(S_1) = \frac{t \cdot (a_1 - 1)}{2}$

and  $\|\arg(S_1) - \arg(S_0)\| < \frac{15}{2} \cdot K \cdot t$ .

Now by formula (2.2):

$$S_{k+1} = \frac{1 - \exp(2\pi i t a_{k+1})}{1 - \exp(2\pi i t)} \cdot S_k + \exp(2\pi i t a_{k+1}) \cdot S_{k-1}.$$

If we assume that our assertion is true for  $k < n$  then for  $k < n$ :

$$\begin{aligned} \|\arg\left(\frac{1 - \exp(2\pi i t a_{k+2})}{1 - \exp(2\pi i t)} \cdot S_{k+1}\right) - \arg(\exp(2\pi i t a_{k+2}) \cdot S_k)\| \\ \leq \frac{3t}{2} a_{k+2} + \frac{15}{2} K \cdot t < 9t \cdot K < \frac{1}{4} \quad (4.1) \end{aligned}$$

and therefore especially because of  $|z_1 + z_2| \geq \max(|z_1|, |z_2|)$  if  $|\arg(z_1) - \arg(z_2)| < \frac{1}{4}$ , and because of (2.2) and Lemma 6 we have :

$$|S_n| \geq \left| \frac{1 - \exp(2\pi i t a_n)}{1 - \exp(2\pi i t)} \right| \cdot |S_{n-1}| \geq (1 - (K\pi t)^2) \cdot |S_{n-1}|,$$

and further

$$\begin{aligned} \left| \frac{1 - \exp(2\pi i t a_{n+1})}{1 - \exp(2\pi i t)} \right| \cdot |S_n| &\geq (1 - (K\pi t)^2)^2 \cdot |S_{n-1}| \\ &> (1 - 2(K\pi t)^2) \cdot |(\exp(2\pi i t a_{n+1})) \cdot S_{n-1}|, \end{aligned}$$

and so because of (4.1) and Lemma 7a) :

$$\begin{aligned} \|\arg(S_{n+1}) - \arg(S_n)\| &\leq \frac{3t}{2} \cdot K \\ &+ \|\arg(S_{n+1}) - \arg\left(\left(\frac{1 - \exp(2\pi i t a_{n+1})}{1 - \exp(2\pi i t)}\right) \cdot S_n\right)\| \\ &\leq \frac{3t}{2} \cdot K + \frac{1}{1 + \frac{2}{\pi} \cdot (1 - 2(K\pi t)^2)} \cdot 9 \cdot K \cdot t. \end{aligned}$$

Hence, because  $t$  is so small that  $\frac{2}{\pi} \cdot (1 - 2(K\pi t)^2) > \frac{1}{2}$ , this is

less than  $\frac{15K}{2} \cdot t$ . By Lemma 7b), by (4.1) and by Lemma 6 we have :

$$\begin{aligned} |S_{n+1}| &\geq (1 - (18\pi K t)^2) \cdot (a_{n+1} \cdot (1 - (\pi K t)^2) \cdot |S_n| + |S_{n-1}|) \\ &\geq (1 - 648 \cdot \pi^2 K^2 \cdot t^2) \cdot (a_{n+1} \cdot |S_n| + |S_{n-1}|). \end{aligned}$$

We take  $\gamma := 648 \pi^2 \cdot K^2$  and because  $t \leq \frac{1}{\sqrt{\gamma}}$  by induction now it is easy to show that

$$\left| \frac{S_n}{q_n} \right| \geq (1 - \gamma \cdot t^2)^n \quad \text{for all } n.$$

This is true for  $n = 0$  and  $n = 1$  and so :

$$\begin{aligned} \left| \frac{S_{n+1}}{q_{n+1}} \right| &\geq (1 - \gamma \cdot t^2) \cdot \left| \frac{a_{n+1} \cdot S_n + S_{n-1}}{a_{n+1} \cdot q_n + q_{n-1}} \right| \\ &\geq (1 - \gamma t^2) \cdot \left| \frac{a_{n+1} \cdot q_n \cdot (1 - \gamma \cdot t^2)^n + q_{n-1} \cdot (1 - \gamma \cdot t^2)^{n-1}}{a_{n+1} \cdot q_n + q_{n-1}} \right| \\ &\geq (1 - \gamma \cdot t^2)^{n+1}. \end{aligned}$$

It we take now  $h$  such that

$$\|hx\| \leq \frac{c}{\psi(h)} \quad \text{and} \quad \frac{c}{\psi(h)} < \frac{1}{\sqrt{648 \cdot \pi \cdot K}},$$

then by Lemma 5 we have :

$$\begin{aligned} D_{q_n}(x \cdot s_\alpha(n)) &\geq \frac{1}{2\pi \cdot h \cdot q_n} \cdot \left| \sum_{k=0}^{q_n-1} \exp(2\pi i h \cdot x \cdot s_\alpha(k)) \right| \\ &\geq \frac{1}{2\pi \cdot h \cdot q_n} (1 - \gamma \cdot \|hx\|^2)^n \cdot q_n \geq \frac{1}{2\pi \cdot h} \left(1 - \frac{\gamma \cdot c^2}{\psi^2(h)}\right)^n. \end{aligned}$$

If we take  $N = q_n$  and  $n = L(N)$  such that  $n - 1 \leq \psi^2(h) < n$ , then  $h \leq \psi^*(n^{1/2}) = \psi^*(L(N)^{1/2})$  and

$$D_N \geq \frac{1}{2\pi \cdot \psi^*(L(N)^{1/2})} \cdot \left(1 - \frac{\gamma \cdot c^2}{\psi^2(h)}\right)^{\psi^2(h)+1} \geq \frac{c_1(x, \psi, c, \alpha)}{\psi^*(L(N)^{1/2})}.$$

Since we can do this for infinitely many  $h$ , the proof is finished.

*Remark.* – Formula (2.2) yields

$$|S_n| \leq \left(\frac{2}{\|hx\|} + 1\right)^n,$$

and so  $|S_n| \leq \left(\frac{2h}{c} + 1\right)^n$ , if  $\|hx\| \geq \frac{c}{h}$  for all  $h = 1, 2, \dots$ ,

and a  $c > 0$ . From the proof of Lemma 4 we have

$$\left| \sum_{n=0}^{N-1} e^{2\pi i n x s_\alpha(n)} \right| \leq \sum_{k=0}^{L(N)} \epsilon_k (2h + 1)^k.$$



If we choose now  $\alpha = [0; 1, 2, 3, 4, \dots]$ , then for every  $N$  sufficiently large, and with absolute constants  $c_i$  by Lemma 1 and by taking  $H = \frac{c}{4} N^{1/(L+1)}$  we get:

$$\begin{aligned} D_N(x \cdot s_\alpha(n)) &\leq c_0 \cdot \left( \frac{1}{H} + \sum_{\substack{h=-H \\ h \neq 0}}^H \frac{1}{|h|} \cdot \frac{1}{N} \cdot \sum_{k=0}^L (k+1) \left( \frac{2|h|}{c} + 1 \right)^k \right) \\ &\leq c_1 \cdot \left( \frac{1}{H} + \frac{1}{N} \cdot \sum_{k=0}^L (k+1) \frac{1}{(k+1)} \left( \frac{4}{c} \cdot H \right)^k \right) \\ &\leq c_2 \cdot \left( \frac{1}{H} + \frac{1}{N} \cdot \left( \frac{4}{c} \cdot H \right)^L \right) \leq \frac{c_3}{N^{1/(L+1)}} \end{aligned}$$

and because of  $N \geq L!$  this is less than

$$\frac{c_3}{(L!)^{1/(L+1)}} \leq \frac{c_4}{L(N)},$$

and therefore it can be seen that the lower bound of Theorem 2 does not hold for every  $\alpha$ .

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