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Graded morphisms of $G$-modules

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1. Introduction.

During the 1987 meeting in honor of J. K. Koszul, Steve Halperin explained to us the following conjecture (motivated by the study of the spectral sequence associated to a homogeneous space).

1.1. Conjecture. — If \( f_1, f_2, \ldots, f_n \) is a regular sequence in the polynomial ring \( \mathbb{C}[x_1, x_2, \ldots, x_n] \), the connected component of the automorphism group of the (finite dimensional) algebra \( \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) is solvable.

In this paper we prove a weak form of this (Corollary 4.3) which implies the conjecture at least when the \( f_i \)'s are homogeneous (Remark 4.4).

2. Preliminaries.

Our base field is \( \mathbb{C} \), the field of complex numbers, or any other algebraically closed field of characteristic zero.

2.1. Definition. — A morphism \( \varphi : V \to W \) between finite dimensional vector spaces \( V \) and \( W \) is called graded if there is a basis of \( W \) such that the components of \( \varphi \) are all homogeneous polynomials.

Let us denote by \( \mathcal{O}(V), \mathcal{O}(W) \) the ring of regular functions on \( V \) and \( W \). These \( \mathbb{C} \)-algebras are naturally graded by degree: \( \mathcal{O}(V) = \bigoplus \mathcal{O}(V)_i \). A subspace \( S \subset \mathcal{O}(V) \) is called graded if \( S = \bigoplus_i S \cap \mathcal{O}(V)_i \).

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If $\varphi : V \to W$ is a morphism and $\varphi^* : \mathcal{O}(W) \to \mathcal{O}(V)$ the corresponding comorphism we have the following equivalence:

$$\varphi \text{ is graded } \iff \varphi^*(W^*) \text{ is a graded subspace of } \mathcal{O}(V).$$

2.2. Lemma. — For any graded morphism $\varphi : V \to W$ there is a unique decomposition $W = \bigoplus W_v$ and homogeneous morphisms $\varphi_v : V \to W_v$ of degree $v$ such that

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \ldots) : V \to W_0 \oplus W_1 \oplus W_2 \oplus \cdots.$$

(This is clear from the definitions.)

2.3. Remark. — Let $G$ be an algebraic group. Assume that $V$ and $W$ are $G$-modules and that $\varphi : V \to W$ is graded and $G$-equivariant. Then in the notations of lemma 2.2 all $W_v$ are submodules and all components $\varphi_v$ are $G$-equivariant.

2.4. Remark. — If $\varphi : V \to W$ is graded and dominant with $\varphi^{-1}(0) = \{0\}$, then $\varphi$ is a finite surjective morphism. In fact given a finitely generated graded algebra $A = \bigoplus A_i$ with $A_0 = \mathbb{C}$ and a graded subspace $S \subset A$ such that the radical $\text{rad}(S)$ of the ideal generated by $S$ is the homogeneous maximal ideal $\bigoplus A_i$ of $A$, then $A$ is a finitely generated module over the subalgebra $\mathbb{C}[S]$ generated by $S$ (see [1, II.4.3 Satz 8]).

3. The Main Theorem.

3.1. Theorem. — Let $G$ be a connected reductive algebraic group and let $V, W$ be two $G$-modules. Assume that $V$ and $W$ do not contain 1-dimensional submodules. Then any graded $G$-equivariant dominant morphism with finite fibres is a linear isomorphism.

We first prove this for $G = \text{SL}_2$ and then reduce to this situation.

For any $C^*$-module $V$ we have the weight decomposition

$$V = \bigoplus_j V_j, \quad V_j := \{v \in V \mid t(v) = t^j \cdot v\}.$$

We say that $V$ has only positive weights if $V = \bigoplus_{j > 0} V_j$. 
3.2. Lemma. — Let $V$, $W$ be two $C^*$-modules with only positive weights, and let $\varphi : V \to W$ be a $C^*$-equivariant graded morphism with finite fibres. For all $k \geq 0$ we have

$$\varphi^{-1}\left( \bigoplus_{j \leq k} W_j \right) \subseteq \bigoplus_{j \leq k} V_j,$$

and the inclusion is strict for at least one $k$ in case $\varphi$ is not linear.

Proof. — By lemma 2.2 and remark 2.3 we have $\varphi = \sum_{v \geq 1} \varphi_v$ where $\varphi_v : V \to W_v$ is homogeneous of degree $v$ and $C^*$-equivariant. Let

$$v = \sum_{j=1}^{k} v_j \in \bigoplus_{j > 0} V_j = V$$

with $v_k \neq 0$. Then

$$\lim_{\lambda \to 0} \lambda^{k} \cdot t_\lambda^{-1}(v) = v_k.$$ (Here $t_\lambda$ denotes the action of $C^*$. ) Since $\varphi_v$ is homogeneous of degree $v$ and $C^*$-equivariant we obtain

$$(1) \quad \lim_{\lambda \to 0} \lambda^{v_k} \cdot t_\lambda^{-1}(\varphi_v(v)) = \varphi_v(v_k).$$

This implies that $\varphi_v(v) \in \bigoplus_{j \leq v_k} W_j$ for all $v$, proving the first claim.

If $\varphi$ is not linear, i.e. $\varphi \neq \varphi_1$, then there is a $v > 1$, an index $k$ and an element $v \in V_k$ such that $\varphi_v(v) \neq 0$. But $\varphi_v(v) \in W_{v_k}$ by (1) and so $v \notin \varphi^{-1}\left( \bigoplus_{j \leq k} W_j \right)$. $\square$

3.3. Corollary. — Under the assumptions of lemma 3.2 suppose that $\varphi$ is surjective. Put $\lambda_j := \dim V_j$ and $\mu_j := \dim W_j$. Then for all $k \geq 1$ we have

$$(2) \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k.$$ If $\varphi$ is not linear the inequality is strict for at least one $k$.

(This is clear.)

3.4. Proposition. — Let $V$, $W$ be two $\text{SL}_2$-modules containing no fixed lines. Let $\varphi : V \to W$ be a graded $\text{SL}_2$-equivariant morphism, which is dominant and has finite fibres. Then $\varphi$ is a linear isomorphism.
**Proof.** - Consider the maximal unipotent subgroup

\[ U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \text{SL}_2 \]

and the maximal torus

\[ T := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^* \right\} \cong \mathbb{C}^*. \]

By assumption \( \varphi \) is finite and surjective (Remark 2.4), and \( \varphi^{-1}(W^U) = V^U \). Hence the induced morphism

\[ \varphi|_{V^U} : V^U \rightarrow W^U \]

is graded, \( T \)-equivariant, finite and surjective too. Furthermore all weights \( \lambda_j \) of \( V^U \) and \( \mu_j \) of \( W^U \) are positive. It follows from (2) that

\[ \lambda_k + \lambda_{k+1} + \cdots \leq \mu_k + \mu_{k+1} + \cdots \]

for all \( k \), because \( \sum \lambda_j = \dim V^U = \dim W^U = \sum \mu_j \). From this we get

\[ \dim V = 2\lambda_1 + 3\lambda_2 + \cdots + (n+1)\lambda_n \]

\[ \leq 2\mu_1 + 3\mu_2 + \cdots + (n+1)\mu_n = \dim W \]

for all \( n \) which are big enough. (Remember that an irreducible \( \text{SL}_2 \)-module of highest weight \( j \) is of dimension \( j + 1 \)). If \( \varphi \) is not linear this inequality is strict (Corollary 3.3), contradicting the fact that \( \varphi \) is finite and surjective. \( \square \)

### 3.5. Proof of the Theorem.

- Assume that \( \varphi : V \rightarrow W \) is not linear, i.e. there is a \( \nu_0 > 1 \) such that the component \( \varphi|_{\nu_0} : V \rightarrow W|_{\nu_0} \) is non-zero. Then there is a homomorphism \( \text{SL}_2 \rightarrow G \) and a non-trivial irreducible \( \text{SL}_2 \)-submodule \( M \subset V \) such that \( \varphi|_M \neq 0 \). (In fact the intersection of the fixed point sets \( V^{(\text{SL}_2)} \) for all homomorphisms \( \iota : \text{SL}_2 \rightarrow G \) is zero.) Now consider the \( G \)-stable decompositions \( V = V^{\text{SL}_2} \oplus V' \) and \( W = W^{\text{SL}_2} \oplus W' \) and the following morphism:

\[ \varphi' : V' \hookrightarrow V \xrightarrow{\varphi} W \xrightarrow{\text{pr}} W'. \]

Since \( V' \) and \( W' \) are sums of isotypic components the morphism \( \varphi' \) is again graded. Furthermore \( \varphi^{-1}(W^{\text{SL}_2}) = V^{\text{SL}_2} \), hence \( \varphi^{-1}(0) = V^{\text{SL}_2} \cap V' = \{0\} \). This implies that \( \varphi' : V' \rightarrow W' \) is dominant
with finite fibres and satisfies therefore the assumptions of proposition 3.4. As a consequence $\varphi'$ is linear. Since $\varphi|_{V'} : V' \to W$ is graded too we have $\varphi_v|_{V'} = 0$ for all $v > 1$. This contradicts the facts that $M \subseteq V'$ and $\varphi_v|_{M} \neq 0$ (see the construction above).

4. Some Consequences.

We add some corollaries of the theorem. Let $G$ be a connected reductive group. For every $G$-module $V$ we have the canonical $G$-stable decomposition $V = V^o \oplus V'$ where $V^o$ is the sum of all 1-dimensional representations (i.e. $V^o = V^{(G,G)}$) and $V'$ the sum of all others. The proof of the theorem above easily generalizes to obtain the following result:

4.1. Theorem. – Let $\varphi : V \to W$ be a graded $G$-equivariant dominant morphism with finite fibres. Then $\varphi$ induces a linear isomorphism

$$\varphi|_{V'} : V' \cong W'.$$

4.2. Corollary. – Let $\mathcal{O}(V)$ be the ring of regular functions on a $G$-module $V$, and let $f_1, \ldots, f_n$ be a regular sequence of homogenous elements of $\mathcal{O}(V)$ such that the linear span $\langle f_1, \ldots, f_n \rangle$ is $G$-stable. Then $\langle f_1, \ldots, f_n \rangle$ contains all non-trivial representations of $(G,G)$ in $\mathcal{O}(V)_1$, the linear part of $\mathcal{O}(V)$.

Proof. – The regular sequence $f_1, \ldots, f_n$ defines a $G$-equivariant finite morphism $\varphi : V \to W$, $W := \langle f_1, \ldots, f_n \rangle^*$. By the theorem above the restriction $\varphi'|_{V'} : V' \to W'$ is a linear isomorphism which means that every non-trivial $(G,G)$-submodule of $\langle f_1, \ldots, f_n \rangle$ is contained in the linear part $\mathcal{O}(V_1)$ of $\mathcal{O}(V)$.

4.3. Recall that a finite dimensional $\mathbb{C}$-algebra is called a complete intersection if it is of the form $\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ with a regular sequence $f_1, \ldots, f_n$.

Corollary. – Let $A$ be a finite dimensional local $\mathbb{C}$-algebra with maximal ideal $m$ and let $\text{gr}_mA$ be the associated graded algebra (with respect to the $m$-adic filtration). If $\text{gr}_mA$ is a complete intersection then the connected component of the automorphism group of $A$ is solvable.
Proof. — Let $G$ and $\mathcal{G}$ be the connected components of the automorphism groups of $A$ and of $\text{gr}_mA$ respectively. Since the $m$-adic filtration of $A$ is $G$-stable we have a canonical homomorphism $\rho : G \to \mathcal{G}$. It is easy to see that $\ker \rho$ is unipotent, so it remains to show that $\mathcal{G}$ is solvable.

Assume that $\mathcal{G}$ is not solvable. Then $\mathcal{G}$ contains a (non-trivial) semisimple subgroup $H$. By assumption we have an isomorphism

$$\text{gr}_mA \cong \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

with a regular sequence $f_1, \ldots, f_n$ where all $f_i$ are homogeneous of degree $\geq 2$. Clearly the action of $\mathcal{G}$ on $\text{gr}_mA$ is induced from a (faithful) linear representation on $\mathbb{C}[x_1, \ldots, x_n] \subset \mathbb{C}[x_1, \ldots, x_n]$. Hence it follows from corollary 4.2 that $\langle f_1, \ldots, f_n \rangle$ contains all non-trivial $H$-submodules of $\mathbb{C}[x_1, \ldots, x_n]$, contradicting the fact that all $f_i$ have degree $\geq 2$. 

4.4. Remark. — The corollary above implies that conjecture 1.1 is true in case all $f_i$ are homogeneous, i.e. if the algebra

$$A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

is finite dimensional and graded with all $x_i$ of degree 1.

4.5. Remark. — Another formulation of our result is the following: Let $V$ be a representation of a connected algebraic group $G$ and $Z \subset V$ a $G$-stable graded subscheme, which is a complete intersection supported in $\{0\}$. Then $(G, G)$ acts trivially on $Z$.

BIBLIOGRAPHY


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