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# STRUCTURE OF A LEAF OF SOME CODIMENSION ONE RIEMANNIAN FOLIATION

par Krystyna BUGAJSKA

## 1. Introduction.

Let $M$ be a smooth, connected, open manifold of dimension $n$ and let $\mathcal{F}$ be a smooth codimension-one complete Riemannian (that is $(M,\mathcal{F})$ admits a bundle like metric $g$ in the sense of [6]) foliation of $M$. Let $E \subset TM$ be the tangent bundle of $\mathcal{F}$ and let $\mathcal{D} \subset TM$ be the distribution orthogonal to $E$ i.e. $\mathcal{D} = E^{\perp}$ and $TM = E \oplus \mathcal{D}$. Let all leaves of $\mathcal{F}$ be open, orientable manifolds and let $M$ be also orientable. Then there exists a normal field of unit vectors $n(x)$ and all leaves of $\mathcal{F}$ have trivial holonomy ([6] cor. 4 p. 130). For a vector $v \in T_xM$ and for a real number $e$ let $g(x,v,c,)$ denote the geodesic arc issuing from $x$ whose length is $|c|$ and whose initial vector is $v$ or $-v$ according as $c > 0$ or $< 0$. By $(x,v,c)$ we will denote its terminal point. Let $\mathcal{F}$ be a totally geodesic foliation. Now, since $\mathcal{D}$ is integrable, every leaf of $\mathcal{F}$ meets every leaf of the horizontal foliation $\mathcal{H}$ determined by $\mathcal{D}$ ([3], lemme (1.9) p. 230). Let $\mathcal{L}(x)$ and $\mathcal{H}(x)$ be the leaves through $x \in M$ of $\mathcal{F}$ and $\mathcal{H}$ respectively. Let $I(x)$ denote the set $\mathcal{L}(x) \cap \mathcal{H}(x)$.

DEFINITION 1. — *Let $x_0 \in \mathcal{L}(x)$ and let $N(x_0)$ denote the set of all positive numbers $s$ such that at least one of two points*

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$(x, \pm n(x), s)$ *belongs to* $\mathcal{L}(x)$. *If* $N(x_0)$ *is non-empty we denote the greatest lower bound of* $N(x_0)$ *by* $\rho(x_0)$. *If* $N(x_0)$ *is empty we put* $\rho(x_0) = \infty$ . *So* $0 \leq \rho(x_0) \leq \infty$.

DEFINITION 2. — *If* $I(x) - x_0$ *is non-empty then the greatest lower bound of* $d_{\mathcal{L}}(x_0, x)$ *for* $x \in I(x_0) - x_0$ *is called the range of* $x_0$ *and is denoted by* $e_{\mathcal{L}}(x_0)$. *Here* $d_{\mathcal{L}}(x_0, x)$ *denotes the length of a minimazing geodesic joining* $x_0$ *to* $x$ *in the* $\mathcal{L}$*-submanifold.*

If $0 < \rho(x) < \infty$ then lemma (4.3) of [4] asserts that at least one of two points $(x_0, \pm n(x), \rho)$ belongs to $\mathcal{L}(x_0)$. Also for each $x \in \mathcal{L}(x_0)$, $\rho(x) = \rho(x_0)$ (lemma (3.2) of [4]). Hence we denote $\rho(x_0)$ by $\rho(\mathcal{L}(x_0))$ and call it the distance of $\mathcal{L}$. As a matter of fact for any leaves $\mathcal{L}$, $\mathcal{L}_1$ of $\mathcal{F}$, $\rho(\mathcal{L}) = \rho(\mathcal{L}_1)$ ([4] p. 136). Although $e_{\mathcal{L}}(x)$ has no such property we can show the following :

PROPOSITION 1. — *Let* $e_{\mathcal{L}}(x_0)$ *be a finite non-equal to zero number. Then*

a) *there exists an element* $x \in I(x_0)$ *such that* $d_{\mathcal{L}}(x_0, x) = e_{\mathcal{L}}(x_0)$

b) *for every* $x \in I(x_0)$, $e_{\mathcal{L}}(x) = e_{\mathcal{L}}(x_0)$ *i.e. the ranges of* $\mathcal{H}$*-equivalent points of* $\mathcal{L}$ *are the same.*

PROPOSITION 2. — *Let* $\mathcal{L}$ *be a map* $f : \mathcal{L} \to \mathcal{L}$ *given by* $f(x) = (x, n(x), m\rho)$. *If for some* $m \in \mathbf{Z}^+$ *and for some* $x_0 \in \mathcal{L}$, $d_{\mathcal{L}}(x_0, f(x_0)) = e_{\mathcal{L}}(x_0)$ *then for every* $x \in \mathcal{L}$ *we have* $d_{\mathcal{L}}(x, f(x)) = e_{\mathcal{L}}(x)$.

COROLLARY 1. — *There exists a vector field* $v$ *on* $\mathcal{L}$ *such that* $f(x) = \exp_x e_{\mathcal{L}}(x) v(x)$. *So, to any point* $x \in \mathcal{L}$ *we can relate a piece of the geodesic* $g(x, v(x), e_{\mathcal{L}}(x))$.

Since the elements of a holonomy along a horizontal curve are local isometries of the induced Riemannian metrics of the leaves of $\mathcal{F}$ ([1] p. 383) the map $f$ determines the partition of $\mathcal{L}$ onto mutually isometric subspaces.

COROLLARY 2. — *$\mathcal{L}$ is of fibred type over a complete Riemannian manifold $N$ with boundary. A fiber contains a countable number of elements and projection is a local isometry. If $\mathcal{C}_x$ is a maximal, open subset of $\mathcal{L}$ containing $x$ and such that $\mathcal{C}_x \cap f(\mathcal{C}_x) = \emptyset$ then $N \cong \mathcal{C}_x \cup (\bar{\mathcal{C}}_x \cap f(\bar{\mathcal{C}}_x))$.*

Let us assume that the vector field $v$ which determined by $f$ is a parallel one. Then we have

COROLLARY 3. — *Leaf $\mathcal{L}$ is diffeomorphic to $\mathcal{L}' \times \mathbf{R}$ and has non-positive curvature.*

I would like to thank the referee for indicating me my error.

## 2. Proofs.

It is easy to see that for each $x' \in \mathcal{H}(x_0) \cap \mathcal{L}(x_0)$, $d_{\mathcal{H}}(x_0, x') = m\rho$ for some $m \in \mathbf{Z}$. Now let us suppose that a point $x \in I(x_0)$ such that $e_{\mathcal{L}}(x_0) = d_{\mathcal{L}}(x_0, x)$ does not exist. However we can find a sequence of points $\{y_\lambda; \lambda = 1, 2, \ldots\}$ belonging to $I(x_0)$ such that $\lim_{\lambda \to \infty} d_{\mathcal{L}}(x_0, y_\lambda) = e_{\mathcal{L}}(x_0)$. Since $\mathcal{L}$ is a complete Riemannian manifold, an accumulation point $y$ of $\{y_\lambda\}$ belongs to $\mathcal{L}$. Let $[y_\lambda, y]$ denote the geodesic arc in $\mathcal{L}$. Let us displace parallelly $g(y_\lambda, n(y_\lambda), s_{\lambda,\lambda+1}$ along $[y_\lambda, y]$ . Here $s_{\lambda,\lambda+1}$ denotes a parameter on the $\mathcal{H}(x_0)$ geodesic such that $(y_\lambda, n(y_\lambda), s_{\lambda,\lambda+1}) = y_{\lambda+1}$ ; $s_{\lambda,\lambda+1} = m(\lambda)\rho$. We obtain the geodesic arcs $g(y, n(y), m(\lambda)\rho)$ with $y'_\lambda$ as their terminal points. So we see that $y$ is an accumulation point of $y'_\lambda \in I(y)$ relative to $\mathcal{L}$. However if $e_{\mathcal{L}}(x_0) > 0$ then $e_{\mathcal{L}}(x) > 0$ for each $x \in \mathcal{L}$ ([4], lemma (4.1)). So we come to a contradiction which proves (a) of proposition 1.

For (b) let $y_0 \in I(x_0)$ have the property that $d_{\mathcal{L}}(x_0, y_0) = e_{\mathcal{L}}(x_0)$. Let $y_0 = (x_0, n(x_0), m\rho)$. Since $\mathcal{L}$ is complete there exists a minimal $\mathcal{L}$-geodesic $g(x_0, n_0, e_{\mathcal{L}}(x_0))$ which joins $x_0$ and $y_0$. Let us express $\mathcal{H}(x_0)$ by $z(s)$, $-\infty < s < \infty$, where $z(0) = x_0$ and $s$ denotes the arclength. Let us displace $U_0$ parallelly along the curve $z(x)$. Then corresponding to each $s$ we get a vector $n(s)$ at $z(s)$

tangent to the leaf $\mathcal{L}(z(s))$ with $g(z(s), n(s), e_{\mathcal{L}}(x_0)) \subset \mathcal{L}(z(s))$. Let $y_0 = z(s_0)$. Taking a finite system of coordinate neighborhoods of $z(s)$ for $0 \leq s \leq s_0$, we see that the point $(z(s_0), n(s_0), e_{\mathcal{L}}(x_0)) \in \mathcal{L}$ also belongs to $\mathcal{H}(x_0)$. Let us denote this point by $y_1$. We have $d_{\mathcal{L}}(x_0, y_0) = d_{\mathcal{L}}(y_0, y_1)$. Let us suppose that $d_{\mathcal{L}}(y_0, y_1) \neq e_{\mathcal{L}}(y_0)$. By definition $e_{\mathcal{L}}(y_0) < d_{\mathcal{L}}(y_0, y_1)$. By (a) there exists $y_2 \in I(x_0)$ such that $d_{\mathcal{L}}(y_0, y_2) = e_{\mathcal{L}}(y_0)$. Let us displace parallelly a minimal geodesic $[y_0, y_2]$ along $z(s)$. For $z(0) = x_0$ we obtain some point $x \in I(x_0)$ which satisfies $d_{\mathcal{L}}(x_0, x) < d_{\mathcal{L}}(y_0, y_1) = e_{\mathcal{L}}(x_0)$. So we come to a contradiction, hence $e_{\mathcal{L}}(x_0) = e_{\mathcal{L}}(y_0)$. However this implies that $e_{\mathcal{L}}(x) = e_{\mathcal{L}}(x_0)$ for each $x \in I(x_0)$ and completes the proof of (b).

For the horizontal curve $z(s)$ there exists a family of diffeomorphisms $\phi_s : U_0 \to U_s$ ; $s \in (-\infty, \infty)$, such that

1 – $U_s$ is a neighborhood of $z(s)$ in the leaf $\mathcal{L}(z(s))$ for all $s \in (-\infty, \infty)$

2 – $\phi_s(z(0)) = z(s)$ for all $s \in (-\infty, \infty)$

3 – for $x \in U_0$, the curve $s \to \phi_s(x)$ is horizontal

4 – $\phi_0$ is the identity map of $U_0$,

i.e. $z(s)$ uniquely determines germs of local diffeomorphisms from one leaf to another. According to [5] we call this family of diffeomorphisms an element of holonomy along $z(s)$. However in our case of totally geodesic foliation $\mathcal{F}$ these local diffeomorphisms are local isometries. Moreover we can extend them to $a$-neighborhoods $U_{\mathcal{L}}(z(s), a)$, where $a < \frac{1}{2} e_{\mathcal{L}}(y)$ for all $y \in U_{\mathcal{L}}(z(s), a)$ ; $s \in (-\infty, \infty)$.

Let us consider a map $d : U_{\mathcal{L}}(x_0, a) \to R$ given by $d(x) = d_{\mathcal{L}}(x, f(x))$. Since $d$ is continuous we have $\forall \varepsilon > 0$, $\exists \delta$ s.t. $|(d(x) - d(y)| < \varepsilon$ if $d_{\mathcal{L}}(x, y) < \delta$ ; $x, y \in U_{\mathcal{L}}(x_0, a)$. Let $\delta < \frac{1}{2} a$ i.e. the ball $U_{\mathcal{L}}(x_0, 2\delta) \subset U_{\mathcal{L}}(x_0, a)$. Let $d(x_0) = e_{\mathcal{L}}(x_0)$. Suppose that for some $x \in U_{\mathcal{L}}(x_0, \delta)$, $d(x) \neq e_{\mathcal{L}}(x)$. Then we have $d(x) = e_{\mathcal{L}}(x) + b$ with $b > 0$. By (a) of proposition 1 there exists $x' \in I(x)$ such that $d_{\mathcal{L}}(x, x') = e_{\mathcal{L}}(x)$, $x' = (x, n(x), m'\rho)$ with $m' \neq m$. Let $f' : \mathcal{L} \to \mathcal{L}$ be given as $f'(x) = (x, n(x), m'\rho)$ and let $d'$ be analogous to $d$ map with $f'$ instead of $f$. We have $d'(x_0) = d(x_0) + \tau, \tau > 0$. (If $\tau = 0$, the property $U_{\mathcal{L}}(x_0, 2\delta) \subset U_{\mathcal{L}}(x_0, a)$ allows us to interchange the role of the maps $f$ and $f'$ as well as $x_0$ and $x$. For this it is enough to consider the case with $\tau > 0$). Now, for each $x \in U_{\mathcal{L}}(x_0, \delta)$ we have $d(x_0) = d(x) \pm \mathcal{H}$; $d'(x_0) = d'(x) \pm \mathcal{H}'$ with $\mathcal{H}$, $\mathcal{H}' < \varepsilon$. So $d'(x) = d(x_0) + \tau \mp \mathcal{H}'$. For $\varepsilon < \frac{1}{2}\tau$ we come to a contradiction since $d'(x) \stackrel{df}{=} e_{\mathcal{L}}(x) > d(x)$. Hence for all $x \in U_{\mathcal{L}}(x_0, \delta)$, $d(x) = e_{\mathcal{L}}(x)$. Now, let $y$ be an element of $\mathcal{L}$ and $[x_0, y]$ a minimal geodesic joining $x_0$ and $y$. We can take a finite sequence of points $y_i$, $i = 0, 1 \ldots N$ on $[x_0, y]$ ; $y_0 = x_0$ , $y_N = y$ and $U_{\mathcal{L}}(y_i, \delta_i) \cap [x_0, y] \cap U_{\mathcal{L}}(y_{i+1}, \delta_{i+1}) \neq \emptyset$ for all $i \in (O \ldots N)$. We repeat the above consideration for each $U_{\mathcal{L}}(u_i, \delta_i)$. This completes the proof of proposition 2.

Let $\tilde{C}_x = \bar{C}_x - C_x$. Then any element $x' \in C_x$ cannot be $\mathcal{H}$-equivalent to any element $y \in \tilde{C}_x$. For this let $z_i \in C_x$ be a sequence of elements such that $\lim_{\mathcal{L}} z_i = y$. Let us suppose that $y' \in C_x$ is $\mathcal{H}$-equivalent to $y$. Then there exists a sequence of elements $z_i' \notin e_x$, $\mathcal{H}$-equivalent to $z_i$, for each $i$, with $\lim_{\mathcal{L}} z_i' = y'$. This is a contradiction since $C_x$ is open in $\mathcal{L}$. Similarly we can see that for each $y \in \tilde{C}_x$ there exists an $\mathcal{H}$-equivalent point $y' \in \tilde{C}_x$. By proposition 2 we can define $W_x = f(\tilde{C}_x) \cap \tilde{C}_x$ which is the border of $N$.

We can define the action of $\mathbf{Z}$ on $\mathcal{L}$ by isometries : $m(x) = f^m(x)$, $m \in \mathbf{Z}$. This action is free and properly discontinuous. It implies that the quotient space $\frac{\mathcal{L}}{\mathbf{Z}}$ has a structure of differentiable manifold and the projection $\mathcal{L} \to \frac{\mathcal{L}}{\mathbf{Z}}$ is differentiable. When $\mathcal{L}$ is simply connected then the isometry group of $\frac{\mathcal{L}}{\mathbf{Z}}$ is isomorphic to $\frac{N(\mathbf{Z})}{\mathbf{Z}}$ [5] where $N(\mathbf{Z})$ is the normaliser of $Z$ in the group of isometries of $\mathcal{L}$.

If we assume that the vector field $v$ is a parallel one then it has to be a complete Killing vector field. Welsh [7] has proven that if a Riemannian manifold admits a complete parallel vector field then either $\mathcal{L}$ is diffeomorphic to the product of an Euclidean space with some other manifold $\mathcal{L}'$ or else there is a circle action on $\mathcal{L}$ whose orbits are not real homologous to zero. In our case the one-parameter subgroup of isometries generated by $v$ cannot induce an $S^1$ action (in this case its orbits are closed geodesics) so the latter possibility is excluded. (It is in agreement to Yau result [8] that the identity component of the isometry group of an open Riemannian manifold $X$ is compact if $X$ is not diffeomorphic to the product of an Euclidean space with some other manifold.) On the other hand we have Gromoll and Meyer result [2] that the isometry group of a complete open manifold with positive curvature is compact and that a Killing vector field cannot have non-closed geodesic orbits. In this way the corollary 3 is proven.

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