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STRUCTURE OF A LEAF OF SOME CODIMENSION ONE
RIEMANNIAN FOLIATION

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1. Introduction.

Let $M$ be a smooth, connected, open manifold of dimension $n$ and let $\mathcal{F}$ be a smooth codimension-one complete Riemannian (that is $(M, \mathcal{F})$ admits a bundle like metric $g$ in the sense of [6]) foliation of $M$. Let $E \subset TM$ be the tangent bundle of $\mathcal{F}$ and let $D \subset TM$ be the distribution orthogonal to $E$ i.e. $D = E^\perp$ and $TM = E \oplus D$. Let all leaves of $\mathcal{F}$ be open, orientable manifolds and let $M$ be also orientable. Then there exists a normal field of unit vectors $n(x)$ and all leaves of $\mathcal{F}$ have trivial holonomy ([6] cor. 4 p. 130). For a vector $v \in T_x M$ and for a real number $e$ let $g(x, v, e)$ denote the geodesic arc issuing from $x$ whose length is $|e|$ and whose initial vector is $v$ or $-v$ according as $c > 0$ or $< 0$. By $(x, v, c)$ we will denote its terminal point. Let $\mathcal{F}$ be a totally geodesic foliation. Now, since $D$ is integrable, every leaf of $\mathcal{F}$ meets every leaf of the horizontal foliation $\mathcal{H}$ determined by $D$ ([3], lemme (1.9) p. 230). Let $\mathcal{L}(x)$ and $\mathcal{H}(x)$ be the leaves through $x \in M$ of $\mathcal{F}$ and $\mathcal{H}$ respectively. Let $I(x)$ denote the set $\mathcal{L}(x) \cap \mathcal{H}(x)$.

**DEFINITION 1.** — Let $x_0 \in \mathcal{L}(x)$ and let $N(x_0)$ denote the set of all positive numbers $s$ such that at least one of two points

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(x, ±n(x), s) belongs to \( L(x) \). If \( N(x_0) \) is non-empty we denote the greatest lower bound of \( N(x_0) \) by \( \rho(x_0) \). If \( N(x_0) \) is empty we put \( \rho(x_0) = \infty \). So \( 0 \leq \rho(x_0) \leq \infty \).

**Definition 2.** — If \( I(x) - x_0 \) is non-empty then the greatest lower bound of \( d_L(x_0, x) \) for \( x \in I(x_0) - x_0 \) is called the range of \( x_0 \) and is denoted by \( e_L(x_0) \). Here \( d_L(x_0, x) \) denotes the length of a minimizing geodesic joining \( x_0 \) to \( x \) in the \( L \)-submanifold.

If \( 0 < \rho(x) < \infty \) then lemma (4.3) of [4] asserts that at least one of two points \( (x_0, \pm n(x), \rho) \) belongs to \( L(x_0) \). Also for each \( x \in L(x_0) \), \( \rho(x) = \rho(x_0) \) (lemma (3.2) of [4]). Hence we denote \( \rho(x_0) \) by \( \rho(L(x_0)) \) and call it the distance of \( L \). As a matter of fact for any leaves \( L, L_1 \) of \( F \), \( \rho(L) = \rho(L_1) \) ([4] p. 136). Although \( e_L(x) \) has no such property we can show the following:

**Proposition 1.** — Let \( e_L(x_0) \) be a finite non-equal to zero number. Then

a) there exists an element \( x \in I(x_0) \) such that \( d_L(x_0, x) = e_L(x_0) \)

b) for every \( x \in I(x_0) \), \( e_L(x) = e_L(x_0) \) i.e. the ranges of \( \mathcal{H} \)-equivalent points of \( L \) are the same.

**Proposition 2.** — Let \( L \) be a map \( f : L \to L \) given by \( f(x) = (x, n(x), m\rho) \). If for some \( m \in \mathbb{Z}^+ \) and for some \( x_0 \in L \), \( d_L(x_0, f(x_0)) = e_L(x_0) \) then for every \( x \in L \) we have \( d_L(x, f(x)) = e_L(x) \).

**Corollary 1.** — There exists a vector field \( v \) on \( L \) such that \( f(x) = \exp_x e_L(x)v(x) \). So, to any point \( x \in L \) we can relate a piece of the geodesic \( g(x, v(x), e_L(x)) \).

Since the elements of a holonomy along a horizontal curve are local isometries of the induced Riemannian metrics of the leaves of \( F \) ([1] p. 383) the map \( f \) determines the partition of \( L \) onto mutually isometric subspaces.
COROLLARY 2. — \( \mathcal{L} \) is of fibred type over a complete Riemannian manifold \( N \) with boundary. A fiber contains a countable number of elements and projection is a local isometry. If \( C_x \) is a maximal, open subset of \( \mathcal{L} \) containing \( x \) and such that \( C_x \cap f(C_x) = \emptyset \) then \( N \cong C_x \cup (\tilde{C}_x \cap f(\tilde{C}_x)) \).

Let us assume that the vector field \( v \) which determined by \( f \) is a parallel one. Then we have

COROLLARY 3. — Leaf \( \mathcal{L} \) is diffeomorphic to \( \mathcal{L}' \times \mathbb{R} \) and has non-positive curvature.

I would like to thank the referee for indicating me my error.

2. Proofs.

It is easy to see that for each \( x' \in \mathcal{H}(x_0) \cap \mathcal{L}(x_0), \ d_{\mathcal{H}}(x_0, x') = m\rho \) for some \( m \in \mathbb{Z} \). Now let us suppose that a point \( x \in I(x_0) \) such that \( e_{\mathcal{L}}(x_0) = d_{\mathcal{L}}(x_0, x) \) does not exist. However we can find a sequence of points \( \{y_\lambda; \lambda = 1, 2, \ldots\} \) belonging to \( I(x_0) \) such that \( \lim_{\lambda \to \infty} d_{\mathcal{L}}(x_0, y_\lambda) = e_{\mathcal{L}}(x_0) \). Since \( \mathcal{L} \) is a complete Riemannian manifold, an accumulation point \( y \) of \( \{y_\lambda\} \) belongs to \( \mathcal{L} \). Let \( [y_\lambda, y] \) denote the geodesic arc in \( \mathcal{L} \). Let us displace parallelly \( g(y_\lambda, n(y_\lambda), s_{\lambda, \lambda+1}) \) along \( [y_\lambda, y] \). Here \( s_{\lambda, \lambda+1} \) denotes a parameter on the \( \mathcal{H}(x_0) \) geodesic such that \( (y_\lambda, n(y_\lambda), s_{\lambda, \lambda+1}) = y_{\lambda+1} \); \( s_{\lambda, \lambda+1} = m(\lambda)\rho \). We obtain the geodesic arcs \( g(y, n(y), m(\lambda)\rho) \) with \( y_\lambda' \) as their terminal points. So we see that \( y \) is an accumulation point of \( y_\lambda' \in I(y) \) relative to \( \mathcal{L} \). However if \( e_{\mathcal{L}}(x_0) > 0 \) then \( e_{\mathcal{L}}(x) > 0 \) for each \( x \in \mathcal{L} \) ([4], lemma (4.1)). So we come to a contradiction which proves (a) of proposition 1.

For (b) let \( y_0 \in I(x_0) \) have the property that \( d_{\mathcal{L}}(x_0, y_0) = e_{\mathcal{L}}(x_0) \). Let \( y_0 = (x_0, n(x_0), m\rho) \). Since \( \mathcal{L} \) is complete there exists a minimal \( \mathcal{L} \)-geodesic \( g(x_0, n_0, e_{\mathcal{L}}(x_0)) \) which joins \( x_0 \) and \( y_0 \). Let us express \( \mathcal{H}(x_0) \) by \( z(s), -\infty < s < \infty \), where \( z(0) = x_0 \) and \( s \) denotes the arclength. Let us displace \( U_0 \) parallelly along the curve \( z(x) \). Then corresponding to each \( s \) we get a vector \( n(s) \) at \( z(s) \).
tangent to the leaf \( \mathcal{L}(z(s)) \) with \( g(z(s), n(s), e_\mathcal{L}(x_0)) \subseteq \mathcal{L}(z(s)) \). Let \( y_0 = z(s_0) \). Taking a finite system of coordinate neighborhoods of \( z(s) \) for \( 0 \leq s \leq s_0 \), we see that the point \( (z(s_0), n(s_0), e_\mathcal{L}(x_0)) \in \mathcal{L} \) also belongs to \( \mathcal{H}(x_0) \). Let us denote this point by \( y_1 \). We have \( d_\mathcal{L}(x_0, y_0) = d_\mathcal{L}(y_0, y_1) \). Let us suppose that \( d_\mathcal{L}(y_0, y_1) \neq e_\mathcal{L}(y_0) \).

By definition \( e_\mathcal{L}(y_0) < d_\mathcal{L}(y_0, y_1) \). By (a) there exists \( y_2 \in I(x_0) \) such that \( d_\mathcal{L}(y_0, y_2) = e_\mathcal{L}(y_0) \). Let us displace parallelly a minimal geodesic \([y_0, y_2]\) along \( z(s) \). For \( z(0) = x_0 \) we obtain some point \( x \in I(x_0) \) which satisfies \( d_\mathcal{L}(x_0, x) < d_\mathcal{L}(y_0, y_1) = e_\mathcal{L}(x_0) \). So we come to a contradiction, hence \( e_\mathcal{L}(x_0) = e_\mathcal{L}(y_0) \). However this implies that \( e_\mathcal{L}(x) = e_\mathcal{L}(x_0) \) for each \( x \in I(x_0) \) and completes the proof of (b).

For the horizontal curve \( z(s) \) there exists a family of diffeomorphisms \( \phi_s : U_0 \to U_s ; s \in (-\infty, \infty) \), such that

1. \( U_s \) is a neighborhood of \( z(s) \) in the leaf \( \mathcal{L}(z(s)) \) for all \( s \in (-\infty, \infty) \)
2. \( \phi_s(z(0)) = z(s) \) for all \( s \in (-\infty, \infty) \)
3. for \( x \in U_0 \), the curve \( s \to \phi_s(x) \) is horizontal
4. \( \phi_0 \) is the identity map of \( U_0 \), i.e. \( z(s) \) uniquely determines germs of local diffeomorphisms from one leaf to another. According to [5] we call this family of diffeomorphisms an element of holonomy along \( z(s) \). However in our case of totally geodesic foliation \( \mathcal{F} \) these local diffeomorphisms are local isometries. Moreover we can extend them to \( a \)-neighborhoods \( U_\mathcal{F}(z(s), a) \), where \( a < \frac{1}{2} e_\mathcal{L}(y) \) for all \( y \in U_\mathcal{F}(z(s), a) ; s \in (-\infty, \infty) \).

Let us consider a map \( d : U_\mathcal{F}(x_0, a) \to R \) given by \( d(x) = d_\mathcal{F}(x, f(x)) \). Since \( d \) is continuous we have \( \forall \varepsilon > 0, \exists \delta \text{s.t.} \left| (d(x) - d(y)) \right| < \varepsilon \) if \( d_\mathcal{F}(x, y) < \delta \); \( x, y \in U_\mathcal{F}(x_0, a) \). Let \( \delta < \frac{1}{2} a \) i.e. the ball \( U_\mathcal{F}(x_0, 2\delta) \subseteq U_\mathcal{F}(x_0, a) \). Let \( d(x_0) = e_\mathcal{F}(x_0) \). Suppose that for some \( x \in U_\mathcal{F}(x_0, \delta) \), \( d(x) \neq e_\mathcal{F}(x) \). Then we have \( d(x) = e_\mathcal{F}(x) + b \) with \( b > 0 \). By (a) of proposition 1 there exists \( x' \in I(x) \) such that \( d_\mathcal{F}(x, x') = e_\mathcal{F}(x) \), \( x' = (x, n(x), m' p) \) with \( m' \neq m \). Let \( f' : \mathcal{L} \to \mathcal{L} \) be given as \( f'(x) = (x, n(x), m' p) \) and let \( d' \) be analogous to \( d \) map with \( f' \) instead of \( f \). We have \( d'(x_0) = d(x_0) + \tau, \tau > 0 \). (If \( \tau = 0 \),
the property \( U_L(x_0, 2\delta) \subset U_L(x_0, \delta) \) allows us to interchange the role of the maps \( f \) and \( f' \) as well as \( x_0 \) and \( x \). For this it is enough to consider the case with \( \tau > 0 \). Now, for each \( x \in U_L(x_0, \delta) \) we have \( d(x_0) = d(x) \pm \mathcal{H} ; \) \( d'(x_0) = d'(x) \pm \mathcal{H}' \) with \( \mathcal{H}, \mathcal{H}' < \varepsilon \). So \( d'(x) = (d(x_0) + \tau) \mp \mathcal{H}' \). For \( \varepsilon < \frac{1}{2} \tau \) we come to a contradiction since \( d'(x) \neq d(x) \). Hence for all \( x \in U_L(x_0, \delta) \), \( d(x) = e_L(x) \).

Now, let \( y \) be an element of \( L \) and \( [x_0, y] \) a minimal geodesic joining \( x_0 \) and \( y \). We can take a finite sequence of points \( y_i, i = 0, 1 \ldots N \) on \( [x_0, y] ; y_0 = x_0, y_N = y \) and \( U_L(y_i, \delta_i) \cap [x_0, y] \cap U_L(y_{i+1}, \delta_{i+1}) \neq \emptyset \) for all \( i \in (0 \ldots N) \). We repeat the above consideration for each \( U_L(u_i, \delta_i) \). This completes the proof of proposition 2.

Let \( \tilde{C}_x = \tilde{C}_x - C_x \). Then any element \( x' \in C_x \) cannot be \( \mathcal{H} \)-equivalent to any element \( y \in \tilde{C}_x \). For this let \( z_i \in C_x \) be a sequence of elements such that \( \lim_{\mathcal{L}} z_i = y \). Let us suppose that \( y' \in C_x \) is \( \mathcal{H} \)-equivalent to \( y \). Then there exists a sequence of elements \( z_i' \notin e_x \), \( \mathcal{H} \)-equivalent to \( z_i \), for each \( i \), with \( \lim_{\mathcal{L}} z_i' = y' \). This is a contradiction since \( C_x \) is open in \( L \). Similarly we can see that for each \( y \in \tilde{C}_x \) there exists an \( \mathcal{H} \)-equivalent point \( y' \in \tilde{C}_x \). By proposition 2 we can define \( W_x = f(\tilde{C}_x) \cap \tilde{C}_x \) which is the border of \( N \).

We can define the action of \( Z \) on \( L \) by isometries : \( m(x) = f^m(x), m \in Z \). This action is free and properly discontinuous. It implies that the quotient space \( \frac{L}{Z} \) has a structure of differentiable manifold and the projection \( L \rightarrow \frac{L}{Z} \) is differentiable. When \( L \) is simply connected then the isometry group of \( \frac{L}{Z} \) is isomorphic to \( \frac{N(Z)}{Z} \) ![5] where \( N(Z) \) is the normaliser of \( Z \) in the group of isometries of \( L \).

If we assume that the vector field \( v \) is a parallel one then it has to be a complete Killing vector field. Welsh [7] has proven that if a Riemannian manifold admits a complete parallel vector field then either \( L \) is diffeomorphic to the product of an Euclidean space with some other manifold \( L' \) or else there is a circle action on \( L \) whose orbits are not real homologous to zero. In our case the one-
parameter subgroup of isometries generated by $v$ cannot induce an $S^1$ action (in this case its orbits are closed geodesics) so the latter possibility is excluded. (It is in agreement to Yau result [8] that the identity component of the isometry group of an open Riemannian manifold $X$ is compact if $X$ is not diffeomorphic to the product of an Euclidean space with some other manifold.) On the other hand we have Gromoll and Meyer result [2] that the isometry group of a complete open manifold with positive curvature is compact and that a Killing vector field cannot have non-closed geodesic orbits. In this way the corollary 3 is proven.

BIBLIOGRAPHIE


