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ON FUNCTIONS WITH BOUNDED REMAINDER

by P. HELLEKALEK & G. LARCHER

0. Introduction.

Let $\lambda$ denote normalized Haar measure on the one-dimensional torus $\mathbb{R}/\mathbb{Z}$. The following two classes of $\lambda$-preserving measurable transformations on $\mathbb{R}/\mathbb{Z}$ are important in ergodic theory as well as in the theory of uniform distribution modulo one.

Let $\alpha$ be an irrational number and $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $Tx := \{x + \alpha\}$, $\{\cdot\}$ the fractional part. $T$ is called an "irrational rotation" on $\mathbb{R}/\mathbb{Z}$.

Let $q \geq 2$ be an integer and $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $Tx := x - (1 - q^{-k}) + q^{-(k+1)}$, whenever $x \in [1 - q^{-k}, 1 - q^{-(k+1)}[$, $k = 0, 1, \ldots$. $T$ is called a "$q$-adic von Neumann-Kakutani adding machine transformation" on $\mathbb{R}/\mathbb{Z}$. In the following, $T$ will be called a "$q$-adic transformation".

Let $\varphi : [0, 1] \to \mathbb{R}$ be a Riemann-integrable function with $\int_0^1 \varphi(t) \, dt = 0$ and let $T$ be either an irrational rotation or a $q$-adic transformation on $\mathbb{R}/\mathbb{Z}$. Define

$$\varphi_n(x) := \sum_{k=0}^{n-1} \varphi(T^k x),$$

where $x \in \mathbb{R}/\mathbb{Z}$ and $n \in \mathbb{N}$ (we shall always identify $\mathbb{R}/\mathbb{Z}$ with $[0, 1]$).

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The following two questions are of importance in ergodic theory – for the study of skew products – as well as for the study of irregularities in the distribution of sequences in \( \mathbb{R}/\mathbb{Z} \):

1. Under which conditions (on \( \varphi \) and \( x \)) one has \( \sup \limits_{n} |\varphi_n(x)| < +\infty \)?

2. What can be said about limit points of \( (\varphi_n(x))_{n \geq 1} \)?

The classical example. — Let \( \varphi(x) = 1_{[0,\beta]}(x) - \beta \), \( 0 < \beta < 1 \). In this now “classical” example, the first question leads to the study of irregularities in the distribution of the sequence \((T^n x)_{k \geq 0}\), \( \varphi_n(x) \) being the so-called discrepancy function. For \( x = 0 \) one gets well-known sequences: in the first case \( \{k\alpha\}_{k \geq 0} \), in the second case the Van-der-Corput-sequence to the base \( q \).

For this example, the first question has been solved completely by elementary and by ergodic methods (for the first type of \( T \) see Kesten [8] and Petersen [11], for the second type Faure [2] and Hellekalek [4]). The numbers \( \beta \) with \( \sup \limits_{n} |\varphi_n(0)| < +\infty \), respectively \( \sup \limits_{n} |\varphi_n(x)| < +\infty \), are all known.

The second question is closely related to ergodicity of the skew product (cylinder flow) \( T_\varphi : T_\varphi(x,y) = (Tx,y + \varphi(x)) \) on the cylinder \( \mathbb{R}/\mathbb{Z} \times \mathbb{R} \) (see Oren [10] and Hellekalek [5]). In exactly this context Oren has solved the problem.

In this paper we shall be interested in question 1,2 and ergodicity of the cylinder flow \( T_\varphi \) on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R} \) in the case of a \( q \)-adic transformation \( T \) and \( \varphi \in C^1([0,1]) \).

1. Results.

Throughout this paper we shall assume \( q \geq 2 \) to be an integer and \( T \) to be a \( q \)-adic transformation on \( \mathbb{R}/\mathbb{Z} \).

**Theorem 1.** — Let \( \varphi \in C^1([0,1]) \), let \( \int_0^1 \varphi(t) \, dt = 0 \) and \( \varphi(1) \neq \varphi(0) \). Then every number \( c \) such that \( |c| \leq |\varphi(1) - \varphi(0)|/2 \) is a limit point of the sequence \((\varphi^k(x))_{k \geq 0}\) for almost all \( x \in \mathbb{R}/\mathbb{Z} \), in particular for any \( x \) normal to base \( q \).

**Theorem 2.** — Let \( \varphi \in C^1([0,1]) \), let \( \int_0^1 \varphi(t) \, dt = 0 \) and let \( \varphi' \) be Lipschitz continuous on \([0,1]\). Then
(1) $\varphi(0) = \varphi(1) \Rightarrow \sup_{n} |\varphi_n(x)| < \infty$ for all $x \in \mathbb{R}/\mathbb{Z}$;

(2) $\sup_{n} |\varphi_n(x)| < \infty$ for some $x \in \mathbb{R}/\mathbb{Z} \Rightarrow \varphi(0) = \varphi(1)$;

(3) $\varphi(1) < \varphi(0) \Rightarrow -\infty < \liminf_{n \to \infty} \varphi_n(0)$ and $\limsup_{n \to \infty} \varphi_n(0) = +\infty$;

(4) $\varphi(1) > \varphi(0) \Rightarrow -\infty = \liminf_{n \to \infty} \varphi_n(0)$ and $\limsup_{n \to \infty} \varphi_n(0) < +\infty$;

(if $\omega(\delta) := \sup\{||\varphi'(x) - \varphi'(y)|| : |x-y| < \delta$, $0 \leq x, y \leq 1\}$, $\delta > 0$, denotes the modulus of continuity of $\varphi'$, then $\varphi'$ called Lipschitz-continuous if $\omega(\delta) \leq L \cdot \delta$, $\forall \delta > 0$, $L$ a positive constant).

The reader might want to compare theorem 2 (1) with theorem 7.8 in [7], and theorem 2 (3) and (4) with results on the one-sided boundedness of the discrepancy function (see [1]).

**THEOREM 3.** — Let $\varphi \in C^1([0,1])$ and let $\int_0^1 \varphi(t) \, dt = 0$. Then $\varphi(1) \neq \varphi(0) \Rightarrow \forall \varepsilon \in \mathbb{R}/\mathbb{Z}$ normal to base $q : (\varphi_n(x))_{n \geq 1}$ is dense in $\mathbb{R}$.

In particular, if $\varphi(1) \neq \varphi(0)$ and if $x$ is normal to base $q$, then
\[\liminf_{n \to \infty} \varphi_n(x) = -\infty \text{ and } \limsup_{n \to \infty} \varphi_n(x) = +\infty.\]

The reader might want to compare theorem 3 with corollary C in [10].

**THEOREM 4.** — Let $\varphi$ be as in theorem 3 and let $T_\varphi : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, $T_\varphi(x, y) = (T x, y + \varphi(x))$. Then

(1) $\varphi(1) \neq \varphi(0) \Rightarrow T_\varphi$ ergodic;

(2) let $\varphi'$ be Lipschitz-continuous on $[0,1]$. Then $T_\varphi$ is ergodic if and only if $\varphi(1) \neq \varphi(0)$.

**2. The proofs.**

Let $A(q) = \left\{ \sum_{i=0}^{\infty} z_i q^i : z_i \in \{0,1,\ldots,q-1\} \right\}$ denote the compact Abelian group of $q$-adic integers with the metric
\[\rho(z, z') := q^{-\min\{i : z_i \neq z'_i\}}\]
for $z = \sum_{i=0}^{\infty} z_i q^i \neq z' = \sum_{i=0}^{\infty} z'_i q^i$ and $\rho(z, z) := 0$. 

The homeomorphism $S : A(q) \to A(q)$, $Sz = z + 1$ $(z \in A(q))$, $1 := 1 \cdot q^0 + 0 \cdot q^1 + 0 \cdot q^2 + \cdots$ has a unique invariant Borel probability measure on $A(q)$: the normalized Haar measure. The dynamical system $(A(q), S)$ is minimal (see [4]).

The map $\Phi : A(q) \to \mathbb{R}/\mathbb{Z}$, $\Phi\left(\sum_{i=0}^{\infty} z_i q^i\right) := \sum_{i=0}^{\infty} z_i q^{-(i+1)} \mod 1$, is measure preserving, continuous and surjective.

The $q$-adic representation of an element $x$ of $\mathbb{R}/\mathbb{Z}$, $x = \sum_{i=0}^{\infty} x_i q^{-(i+1)}$ with digits $x_i \in \{0, 1, \ldots, q-1\}$, is unique under the condition $x_i \neq q-1$ for infinitely many $i$. From now on we shall assume this uniqueness condition to hold for all $x$. Numbers $x$ with $x_i \neq 0$ for infinitely many $i$ will be called non-$q$-adic. In the following $z = z(x)$ will denote the element

$$z = z(x) := \sum_{i=0}^{\infty} x_i q^i$$

of $A(q)$ associated with $x$. One has

$$Tx = \Phi(z + 1)$$

and it is elementary to see:

- $T \circ \Phi(z) = \Phi \circ S(z)$, $\forall z \in A(q)$
- $x \in [aq^{-k}, (a+1)q^{-k}]$, $0 \leq a < q^k$, $k = 1, 2, \ldots$ $\Rightarrow$ $T^k x \in [aq^{-k}, (a+1)q^{-k}]$ and therefore $|T^k x - x| < q^{-k}$.
- $T$ permutes the open elementary $q$-adic intervals $]aq^{-k}, (a+1)q^{-k}[$, $0 \leq a < q^k$, of length $q^{-k}$, $k = 1, 2, \ldots$.

**Proposition 1.** Let $\varphi$ be continuously differentiable on the closed interval $[0,1]$ and let $\int_0^1 \varphi(t) \, dt = 0$. If $\omega$ denotes the modulus of continuity of $\varphi'$, then for all $k \in \mathbb{N}$ and for all $x \in \mathbb{R}/\mathbb{Z}$

$$\varphi_{q^k}(x) = (\varphi(1) - \varphi(0))(\rho_k + \sigma_k - 1/2) + O(\omega(q^{-k}))$$

$$(1) + O(\rho_k \cdot \omega(c(q) \cdot (q^k - z(k)))^{-1} \log(q^k - z(k))))$$

$${} + O(\sigma_k \cdot \omega(c(q) \cdot z(k))^{-1} \log z(k)))$$,

where

$$x = \sum_{i=0}^{\infty} x_i q^{-(i+1)}$$

$$z = z(x) := \sum_{i=0}^{\infty} x_i q^i$$
z(k) := \sum_{i=0}^{k-1} x_i q^i \quad k = 1, 2, \ldots
\rho_k := (q^k - z(k)) \cdot \Phi(z - z(k))
\sigma_k := z(k) \cdot \Phi(z - z(k) + q^k)

and c(q) is a constant that depends only on q. The \(O\)-constants that appear in identity (1) are all bounded from above by a constant that depends only on q and \(\varphi\).

Proof. — It is easy to prove

\[ \varphi_k(x) = \sum_{i=0}^{q-1} \varphi(a_i q^{-k}) + \sum_{i=0}^{q-1} \varphi'(a_i q^{-k})(T^i x - a_i q^{-k}) + O(\omega(q^{-k})) , \]

where \(a_i\) is the uniquely determined integer with \(0 \leq a_i < q^k\) and \(T^i x \in [a_i q^{-k}, (a_i + 1)q^{-k}]\). From proposition 1 in [6] it follows that

\[ \sum_{i=0}^{q^k-1} \varphi(a_i q^{-k}) = -(\varphi(1) - \varphi(0))/2 + O(\omega(q^{-k})) . \]

Further

\[ T^i x - a_i q^{-k} = \begin{cases} 
\Phi(z - z(k)) & 0 \leq i < q^k - z(k) \\
\Phi(z - z(k) + q^k) & q^k - z(k) \leq i < q^k .
\end{cases} \]

By theorem 5.4, chapter 2 of [9]

\[ (q^k - z(k))^{-1} \sum_{i=0}^{q^k-z(k)-1} \varphi'(a_i q^{-k}) = \varphi(1) - \varphi(0) + O(\omega(D_{q^k-z(k)})) , \]

where \(D_{q^k-z(k)}\) denotes the discrepancy of \((a_i q^{-k})_{i=0}^{q^k-z(k)-1}\). As \(a_i q^{-k} = \Phi(z(k) + i)\), this is a string in the Van-der-Corput-sequence to base q, and therefore the following discrepancy estimate holds (see [9] chapter 2, theorem 3.5 for the idea of the proof):

\[ D_{q^k-z(k)} \leq c(q)(q^k - z(k))^{-1} \log(q^k - z(k)) , \quad k = 1, 2, \ldots , \]

\(c(q)\) a constant that depends only on q.

With the same arguments one proves

\[ z(k)^{-1} \sum_{i=q^k-z(k)}^{q^k-1} \varphi'(a_i q^{-k}) = \varphi(1) - \varphi(0) + O(\omega(c(q)z(k)^{-1} \log z(k))) . \]
COROLLARY 1. — Let \( n \in \mathbb{N} \), \( n = \sum_{i=0}^{s} n_i q^i \), with \( n_i \in \{0,1,\ldots, q-1\} \), \( 0 \leq i \leq s \), \( n_s \neq 0 \), and let \( n(k) := \sum_{i=0}^{k-1} n_i q^i \) if \( k = 1, \ldots, s+1 \), \( n(0) := 0 \).

If \( \sum_{k=0}^{s} \) denotes \( \sum_{k \in k \neq 0} \) then

\[
\varphi_n(x) = \sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} q^{k-1} \sum_{j=0}^{\ell} \varphi(T^{n(k)+\ell q^k+j} x).
\]

Let

\[
T^{n(k)+\ell q^k} x := x_{k,\ell} = \sum_{i=0}^{\infty} x_i^{k,\ell} q^{-(i+1)}
\]

\[
z^{k,\ell} := \sum_{i=0}^{\infty} x_i^{k,\ell} q^i
\]

\[
z^{k,\ell}(m) := \sum_{i=0}^{m-1} x_i^{k,\ell} q^i \quad (m = 1, 2, \ldots)
\]

\[
\rho_{k,\ell} := (q^k - z^{k,\ell}(k)) \cdot \Phi(z^{k,\ell} - z^{k,\ell}(k))
\]

\[
\sigma_{k,\ell} := z^{k,\ell}(k) \cdot \Phi(z^{k,\ell} - z^{k,\ell}(k) + q^k).
\]

Then proposition 1 implies:

\[
\varphi_n(x) = (\varphi(1) - \varphi(0)) \sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} (\rho_{k,\ell} + \sigma_{k,\ell} - 1/2) + \mathcal{O}\left(\sum_{k=0}^{s} n_k \omega(q^{-k})\right)
\]

\[
+ \mathcal{O}\left(\sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} (\rho_{k,\ell} \omega(c(q)(q^k - z^{k,\ell}(k))^{-1} \log(q^k - z^{k,\ell}(k)))
\]

\[
+ \sigma_{k,\ell} \omega(c(q)z^{k,\ell}(k)^{-1} \log z^{k,\ell}(k))\right).
\]

The \( \mathcal{O} \)-constants in identity (2) are bounded from above by a constant that depends only on \( q \) and \( \varphi \).

Proof of theorem 1. — Let \( x \) be normal to base \( q \) and let \( d = 0, d_0 d_1 d_2 \cdots \) be an arbitrary number in \([0,1[\). For any index \( k \) such that
\( x_k < q - 1 \) we have
\[
\rho_k + \sigma_k = (q^k - z(k)) \sum_{i \geq k} x_i q^{-i-1} + z(k) \left( \sum_{i \geq k} x_i q^{-i-1} + q^{-k-1} \right)
\]
\[
= \sum_{i \geq 0} x_i q^{-|i-k|-1} .
\]

Let \( \varepsilon > 0 \) be arbitrary. Choose \( m \) such that \( q^{-m} < \varepsilon \). As \( x \) is normal there are infinitely many \( k \) such that \( x_k < q - 1 \)
\[
|\rho_k + \sigma_k - d| = |0, x_k x_{k+1} x_{k+2} \cdots + 0, 0x_{k-1} x_{k-2} \cdots x_0 - d| < q^{-m}
\]
(this imposes a condition on the digits \( x_k, x_{k\pm 1}, \ldots, x_{k\pm m} \))
\[
\quad x_{k-m} = q - 1 , \quad x_{k-m-1} = 0 .
\]
Then
\[
z(k) \geq q^{k-m} , \quad q^k - z(k) \geq q^{k-m-1}
\]
and, if we choose \( k \) sufficiently large,
\[
\omega(q^{-k}) < \varepsilon \quad \text{and} \quad \omega(c(q)q^{-k+m+1} \log q^k) < \varepsilon .
\]
If we put \( c := (\varphi(1) - \varphi(0))(d - 1/2) \), then it follows directly that
\[
|\varphi_{q^k}(x) - c| = O(\varepsilon) .
\]

Proof of theorem 2. — (1) : Let \( \varphi(1) = \varphi(0) \). It is \( \Phi(z - z(k)) < q^{-k} \)
and \( \Phi(z - z(k) + q^k) < q^{-k} , k = 1, 2, \ldots \). Hence for the third term in
identity (2) we get the estimate
\[
\left( 3 \sum_{k=0}^{s} \sum_{t=0}^{n_k-1} (\rho_k, t \cdots + \cdots \log z^k, t(k)) \right) \leq 2q Lc(q) \sum_{k=0}^{\infty} q^{-k} \log q^k < +\infty .
\]
Thus the first part of the theorem is proved.

(2) : Let \( \sup |\varphi_n(x)| < +\infty \) for some \( x \in \mathbb{R}/\mathbb{Z} \) and let \( z := z(x) \). The
map \( \varphi \circ \Phi : \mathbb{A}(q)^n \rightarrow \mathbb{R} \) is continuous and \( (\mathbb{A}(q), S) \) is a minimal (topological)
dynamical system. We have
\[
\sup_n |\varphi_n(x)| = \sup_n \left| \sum_{k=0}^{n-1} \varphi \circ \Phi(S^k z) \right| < +\infty .
\]
By theorem 14.11 of [3] there is a continuous function \( g : \mathbb{A}(q) \rightarrow \mathbb{R} \) such that
\( \varphi \circ \Phi(z) = g(z) - g(Sz) , \forall z \in \mathbb{A}(q) \). Hence
\[
-(\varphi(1) - \varphi(0))/2 = \lim_{k \to \infty} \varphi_{q^k}(0) = \lim_{k \to \infty} \sum_{i=0}^{q^k-1} \varphi \circ \Phi(S^i 0)
\]
\[
= \lim_{k \to \infty} (g(0) - g(q^k)) = 0 ;
\]
(here we use proposition 1 in [6] to prove the first equality).

(3) : We shall prove \(-\infty < \liminf_{n \to \infty} \varphi_n(0)\), then part (2) will imply the remaining statement. Because of identity (2) and inequality (3) it is enough to show, for \(x = 0\),

\[ \Sigma_n := \sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} (\rho_k,\ell + \sigma_k,\ell - 1/2) \leq K , \forall n \in \mathbb{N} \]

with some constant \(K\). If \(x = 0\) then \(z^k,\ell = n(k) + \ell q^k\) and \(z^k,\ell(k) = n(k)\). Hence \(\rho_k,\ell = (q^k - n(k))\ell q^{-(k+1)}\) and \(\sigma_k,\ell = n(k)(\ell + 1)q^{-(k+1)}\). Thus

\[ \Sigma_n = \sum_{k=0}^{s} n_k((n_k - 1)/(2q) + n(k)q^{-(k+1)} - 1/2) . \]

The statement then follows because \((n_k - 1)/(2q) + n(k)q^{-(k+1)} - 1/2 < 0\).

(4): The idea of the proof is the same as in (3).

Remark. — In theorem 2 (1), (3) and (4) one can weaken the condition on the modulus of continuity of \(\varphi'\) to \(\omega(\delta) = O((\log \delta)^{-1-\varepsilon})\) with some \(\varepsilon > 0\).

Proof of theorem 3. — The idea of the proof is as follows. Let \((k_m)_{m \geq 1}\) be a strictly increasing sequence of positive integers. If \(n = q^{k_1} + \cdots + q^{k_s}\) then

\[ \varphi_n(x) = (\varphi(1) - \varphi(0)) \sum_{m=1}^{s} (\rho_m + \sigma_m - 1/2) + \mathcal{O}\left(\sum_{m=1}^{s} \omega(q^{-k_m})\right) \]

\[ + \mathcal{O}\left(\sum_{m=1}^{s} \rho_m \omega(c(q)(q^{k_m} - z^{k_m}(k_m))^{-1} \log(q^{k_m} - z^{k_m}(k_m)))\right) \]

\[ + \sigma_m \omega(c(q)(z^{k_m}(k_m))^{-1} \log z^{k_m}(k_m))) \]

with \(x = 0, x_0 x_1 x_2 \cdots, z = z(x) = \sum_{i=0}^{\infty} x_i q^i, z^{k_m} = z + q^{k_1} + \cdots + q^{k_{m-1}}\) and, if \(x_m \leq q - 2\),

\[ \rho_m + \sigma_m = 0 , \ x_k x_{k+1} \cdots + 0 , \ 0 x_{k-1} x_{k-2} \cdots x_0 . \]

Now, let \(d \in \mathbb{R}, \varepsilon > 0\) and \(x \in [0,1]\) normal to base \(q\) be given. We shall prove that there is a positive integer \(m_0\) and a strictly increasing sequence \((k_m)_{m \geq m_0}\) such that

\[ |\varphi_n(x) - d| < \varepsilon \text{ for all } n = q^{k_{m_0}} + \cdots + q^{k_s} \text{ sufficiently large.} \]
Let $m_0$ be such that $\sum_{m \geq m_0} q^{-m} < \varepsilon$. Let $(a_m)_{m \geq m_0}$ be a sequence in $[0,1[$ such that
\[
d = (\varphi(1) - \varphi(0)) \sum_{m \geq m_0} (a_m - 1/2).
\]
The number $x$ is normal to base $q$. Hence there are infinitely many $k = k(m)$ such that
1. $x_k \leq q - 2$
2. $x_{k-2m} = 1$
   
   $x_{k-2m-1} = x_{k+2m} = x_{k+2m+1} = 0$
3. $|\rho_k + \sigma_k - a_m| < q^{-m}(\varphi(1) - \varphi(0))^{-1}$, $\forall m \geq m_0$;
\[
\text{(this condition defines a string of digits } x_{k-2m+1}, \ldots, x_{k+2m-1}).
\]
Hence we may choose a strictly increasing sequence $(k_m)_{m \geq m_0}$ such that these three conditions hold for every $k_m$ and such that
4. $k_m + 2m + 1 < k_{m+1}$
5. $\sum_{m \geq m_0} \omega(q^{-k_m}) < \varepsilon$
6. $\sum_{m \geq m_0} \omega(c(q)q^{-k_m+2m+1}\log q^{k_m}) < \varepsilon$.

Then if $n = q^{k_{m_0}} + \cdots + q^{k_s}$ ($s \geq m_0$),
\[
|\varphi_n(x) - d| = O(\varepsilon),
\]
and therefore the sequence $(\varphi_n(x))_{n \geq 1}$ is dense in $\mathbb{R}$.

Remark. — Theorem 3 gives an alternative to the proof of theorem 2 (2), this time without a condition on the modulus of continuity of $\varphi'$:

If $\sup_n |\varphi_n(x)| < \infty$ for some $x \in [0,1[$, then this holds for all $x$ by the theorem of Gottschalk and Hedlund. Hence $\varphi(1) = \varphi(0)$, otherwise a contradiction to theorem 3 would arise for any $x$ normal to base $q$.

Proof of theorem 4.

(1) is proved in the very same way as the theorem of [6].

(2): Let $L_2$ stand for $L_2(\mathbb{R}/\mathbb{Z}, \lambda)$. Then $\varphi(1) = \varphi(0)$ implies $\sup_n \|\varphi_n\|_{L_2} < +\infty$. By Lemma 2.2 in [4] there exists an element $g$ of $L_2$ such that $\varphi = g - g \circ T$ (mod $\lambda$). This implies that $(x,y) \mapsto$
\((Tx, y + \varphi(x) \mod 1)\) is not ergodic on \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\) and therefore \(T_\varphi\) cannot be ergodic on \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}\) (see [5], part. I : remarks).

\[\square\]

**BIBLIOGRAPHIE**


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