Isao Nakai

Topological stability theorem for composite mappings


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TOPOLOGICAL STABILITY THEOREM
FOR COMPOSITE MAPPINGS

by Isao NAKAI

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0.1. The main theorems and some other known results.

Let \( G = (V, L, \Lambda) \) be an oriented graph where \( V \) is the set of vertices, \( L \) the set of edges and \( \Lambda = (\alpha, \beta) : L \rightarrow V \times V \) is the orientation. Let \( M = (M_v)_{v \in V} \) be a collection of \( C^\infty \) manifolds. A diagram of smooth (proper) mappings on \( (G, M) \) is a family \( f = (f_\ell)_{\ell \in L} \) of (proper) mappings \( f_\ell : M_{\alpha(\ell)} \rightarrow M_{\beta(\ell)} \). We denote the set of those diagrams by

\[
C^\infty(G, M) = \prod_{\ell \in L} C^\infty(M_{\alpha(\ell)}, M_{\beta(\ell)}), \quad C^\infty_p(G, M) = \prod_{\ell \in L} C^\infty_p(M_{\alpha(\ell)}, M_{\beta(\ell)}).
\]

Two diagrams \( f, g \in C^\infty(G, M) \) are \( C^r \) equivalent (topologically equivalent, if \( r = 0 \)) if there are \( C^r \) diffeomorphisms \( \phi_v \) of \( M_v \) such that \( \phi_{\beta(\ell)} \circ f_\ell = g_\ell \circ \phi_{\alpha(\ell)} \) for \( \ell \in L \). The \( C^r \) equivalence class of \( f \) is denoted \( \Theta^r(F) \) and \( f \) is \( C^r \) stable if \( \Theta^r(F) \) is a neighbourhood of \( f \) in the Whitney topology.

Our first question is:

Is \( C^r \) stability a generic property?

It is easy to see that the answer to this question depends deeply on the combinatorial type of the underlying graph \( G \) and manifolds \( M_v, v \in V \). For example if \( G \) is of the types either \( \bigcirc \) (cycle) or \( \nabla \) (divergent) it is known that topological stability does not hold in general by the study of discrete dynamics and Web geometry [Ca, Du 2-3]. We will touch on these counter examples later in this section and also in Appendix 2.

The graphs which we study in this paper are the (finite) convergent graphs:

\[
\xrightarrow{\begin{array}{c}
\vdots \\
\xrightarrow{\ldots} \\
\xrightarrow{\vdots}
\end{array}}
\]

defined below. We will establish a foundation for differential calculus of convergent diagrams of smooth mappings for such graphs.
The relation $\alpha(\ell) < \beta(\ell)$, $\ell \in L$ generates the partial order $<$ of vertices for an oriented tree. A finite oriented tree $G$ is *convergent* if there is only one maximal vertex $v_0$: the root (sink) of $G$. If $G$ is convergent, then for each vertex $v \neq v_0$, there is a unique edge $\ell_v$ with source $\alpha(\ell_v) = v$. We define $\beta(v)$ by $\beta(\ell_v)$ for $v \neq v_0$. The *height* of a vertex $v \in V$ is defined inductively $h(v_0) = 0$; otherwise $h(v) = 1 + h(\beta(v))$. Each vertex $v$ of $G$ defines a *branch* $G_v$, which is the subgraph consisting of all vertices $v' \leq v$ and edges $\ell$ with $\beta(\ell) \leq v$.

In this paper we call also a union of trees a *tree*.

Our goal in this paper is to prove

**Theorem.** — Let $G = (V, L, \Lambda)$ be a finite convergent tree and let $M = (M_v)_{v \in V}$ be a collection of smooth manifolds and $P = (\dim M_v)_{v \in V}$. If $P$ satisfies the condition $G$ defined in Section 2.1, then topologically stable mappings are dense in $C^\infty_p(G, M)$ with the Whitney topology.

As a consequence of the above theorem and Theorem 2 in the paper [N2], we have

**Corollary.** — Let $G, M, P$ be as above.

If $p_0$ satisfies one of the following conditions, for any $v \in V$:

1. $p_v \leq p_{\beta^n(v)}$ for $0 \leq n \leq h(v)$
2. $p_{\beta(v)} \leq p_{\beta^n(v)}$ for $1 \leq n \leq h(v)$
3. $p_v, p_{\beta(v)} \geq p_{\beta^2(v)} \leq p_{\beta^n(v)}$ for $2 \leq n \leq h(v)$

and the pair $(p_v - p_{\beta^2(v)}, p_{\beta(v)} - p_{\beta^2(v)})$ is semi-nice, i.e.

$$2^2 \sigma(p_v - p_{\beta^2(v)}, p_{\beta(v)} - p_{\beta^2(v)}) \geq p_v - p_{\beta^2(v)}.$$

Then topologically stable diagrams are dense in $C^\infty_p(G, W)$. Here $2^2 \sigma(n, p)$ is the function defined by Mather [M2] (see also [W2-3]).

The main theorem above generalizes well known topological stability theorem for single mappings due to Mather [M4] and also gives a partially affirmative answer to a conjecture by Baas and Mather [B1-3, L-T]: topologically stable diagrams are dense in $C^\infty_p(G, M)$ if $G$ is a finite convergent tree.

We now recall some known results on the $C^r$ stability problem respectively for various types of diagrams.
Case 1: is an arrow $\rightarrow$. In this case our problem turns into the ordinary singularity theory of smooth mappings between two manifolds. We recall the main global results:

1. $C^\infty$ stable mappings are dense in $C^\infty_{pr}(N^n, P^p)$ if the dimension pair $(n, p)$ is nice, i.e. $\sigma(n, p) > n$ [M2].

2. $C^0$ stable mappings are dense in $C^0_{pr}(N, P)$ for any smooth manifolds $N, P$ [M4, Gi].

3. The $C^0$ stability and the $C^\infty$ stability are equivalent in $C^\infty(N, P)$ if $(n, p)$ is a nice pair and $N$ is compact (see e.g. [Da]).

4. The complement of the union of equivalence classes $C^\infty(f)$ with finite codimension in $C^\infty(N, P)$ has infinite codimension for compact $N$, if and only if $(n, p)$ is a semi-nice pair [P, W2].

5. $C^1$ stable mappings are dense in $C^\infty_{pr}(N, P)$ if and only if $(n, p)$ is a nice pair [W1].

A survey of these nice and semi-nice properties is available in the paper [W2] and the complete determination of those ranges is given by Mather [M3] and Wall [W3], respectively.

Case 2: $G$ is the composition $\rightarrow \rightarrow$. In this case $C^1$ stability does not hold generically even for some triples $(M, N, P)$ of manifolds of small dimensions. In fact du Plessis showed that

6. $C^1$ stable compositions are not dense in the space of proper composite mappings $C^\infty_{pr}(M^3 \rightarrow N^4 \rightarrow P^2)$.

We will give a proof for this in Appendix 2. On the other hand, the triples (3.4.2) satisfy the condition (3) of the above corollary. Since the pair $(1, 2)$ is nice and in particular semi-nice, $C^0$ stable mappings are dense in this space of compositions.

A technical reason for the restriction to the case of convergent diagrams is that the Malgrange-Mather division theorem does not hold for the other cases. In fact the nature of the space of diagrams $C^\infty(G, M)$ presents a remarkable difference between the convergent and the other types. Some of these aspects will be found in the following two typical non convergent cases.

Case 3: $G$ is a cycle $\bigcirc$. In this case our problem corresponds to the theory of endomorphisms of manifolds, which have been long studied by many mathematicians. It is known that $C^0$ stability is not a generic property. This phenomenon is caused by the topological
structure of orbits of endomorphisms $f : M \rightarrow N$. The structure of compositions of their singularities along orbits is the same as that of their developments $f : M \rightarrow N \rightarrow P$ (covering $f$), for which it seems that the argument in this paper remains effective. So the topological structure of endomorphisms may be described by a certain combination of singularities and the orbit structure of $f$.

Case 4: $G$ is the divergent graph $\xi$. In this case our problem is related to envelope theory in the papers [A, Ca, Du 2-3, Th]. The recent results by Carneiro [Ca] and Dufour [Du 2-3] present a new aspect of the topological classification problem for diagrams of this type by using a topological method in web geometry. Namely, Dufour proved that

(7) In the divergent mapping space $C^\infty(M^1 \rightarrow N^2 \rightarrow P^2)$, $C^0$ stable diagrams are not dense [Du2-3].

In the final section we will show that if $P^2$ is orientable and $N^2$ is not then all topological equivalence classes have infinite codimension. Surprisingly it was proved quite recently by Dufour [Du 4] that even in the space of pairs of functions $C^\infty(M^1 \rightarrow N^2 \rightarrow P^1)$, $C^0$ stability does not hold in general.

0.2. Sketch of the proof of the theorem.

First we begin by recalling the idea due to Thom and Mather for topological study of singularities of mappings, known as the theory of canonical stratification.

A stratification of a smooth mapping $f : N \rightarrow P$ is a pair $(\mathcal{S}_N, \mathcal{S}_P)$ of stratifications of manifolds $N, P$ such that $f$ restricts on each stratum $X \in \mathcal{S}_N$ to a submersion $f : X \rightarrow Y$ to some stratum $Y \in \mathcal{S}_P$. Thom's second isotopy lemma (Theorem 3.2.4) says that if a family of proper mappings $(f_t \times \text{id}, \text{Pr}) : N \times \mathbb{R} \rightarrow P \times \mathbb{R} \rightarrow \mathbb{R}$ is simultaneously stratified by a triple $(\mathcal{S}_N, \mathcal{S}_P, \mathcal{R})$ of stratifications of $N \times \mathbb{R}$, $P \times \mathbb{R}$ and $\mathbb{R}$, and $\mathcal{S}_N$ satisfies Thom's condition $A_{f_0}$ then the family $f_t$ is locally topologically trivial. This suggests that topological stability of mappings may be deduced from a certain stability of their $A_f$ regular stratifications under small perturbations. A canonical stratification was explicitly constructed by Mather [M4] by using his highly systemized method in papers in a series, where the finite determinacy theorem and the unfolding theory played a crucial role.
Using the same basic idea as above, a fundamental part of the proof of the topological stability theorem for convergent diagrams will be a construction of their stratifications in a canonical way. For the simplest case of two-compositions \((f, g) : M \to N \to P\), this may be done by refining a canonical stratification \(\mathcal{S}_N(f)\) of \(N\) for \(f\) to a stratification \(\mathcal{S}_N'(f)\) such that for some stratification \(\mathcal{S}_P\) of \(P\) the pair \((\mathcal{S}_N'(f), \mathcal{S}_P)\) is \(A_g\) regular. Thus this problem is called the problem of the second stratification by Thom. In the following we will explain how the second stratifications of convergent diagrams are constructed in a canonical way.

Let \(f_{v'} : M_{v'} \to M_v\) denote the composition of \(f\) along the oriented path from \(v'\) to \(v\). The main technical problem in this paper is to give an intrinsic notion for the singularities of convergent diagrams \(f\) involving these compositions.

Given a diagram \(f \in C^\infty(G, M)\) and another convergent graph \(\Gamma\), a diagram of \(f\) of type \(\Gamma\) consists of

i) a morphism \(i : (V_{\ell}, L_{\ell}) \to (V_G, L_G)\) with \(\alpha_G \circ i = i \circ \alpha_{\ell}\), \(\beta_G \circ i = i \circ \beta_{\ell}\),

ii) points \(x_{\ell} \in M_{i(\ell)}\) for \(\ell \in V_{\ell}\) such that \(f_{i(\ell)}(x_{\ell}) = x_{\ell}\) for \(\ell \in L_{\ell}\).

We shall seek to understand the singularities of \(f\) in terms of its multi germs along such diagrams.

A diagram of \(f\) is determined by the set \(X = \{x_{\ell} | \ell \in L\}\), so we denote it simply by \(f_X\). The Collection of multi germs \(f_{X_{\ell}}\), \(\ell \in L_G\), where \(X = \bigcup_{\ell \in L} X_{\ell} \subset \bigcup_{\ell \in L} M_{\ell}\). Note that \(V_{\ell} \cong \bigcup_{\ell \in L} (X_{\ell} \cup f_{\ell}(X_{\ell}))\).

We first explain the role of trees for the case of a single mapping. Here a diagram \(f_X, X \subset N \) of \(f : N \to P\) is an oriented graph of height 1 consisting of \(\# f(X)\) disjoint trees: forest. The germs of canonical stratifications \(\mathcal{S}_N, \mathcal{S}_P\) of \(f\) at \(f^{-1}(y)\), \(y\) are characterized by the multi germ \(f_{X_y}\) at \(X_y = \Sigma(f) \cap f^{-1}(y)\) [M4].

For a general convergent tree \(G\), our first problem is to describe the singularity type of convergent diagrams of smooth mapping, in other words to seek the smallest subset \(X \subset \bigcup_{\ell \in L} M_{\ell}\) with \(y \in f(X)\) for \(\ell \in L\) which the germ \(f_X\) characterizes the property of the germ of \(f\) along the fibres \(f_{v'}(y)\), \(v' < v\) on \(y \in M_v\).

In Section 0.3, we define the critical point sets \(C_{\alpha_{\ell}}(f) \subset M_{\alpha_{\ell}}\), and the critical values sets \(D_{\beta}(f) = \bigcup_{\beta_{\ell} = \beta} f_{\ell}(C_{\alpha_{\ell}}(f))\) for convergent diagrams.
$f \in C^\infty(G,M)$, using the notion of trees. The restriction $f_z = (f_\ell : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f))$ is considered as a skeleton of $f$. In fact $f_z$ contains complete information about the singularities of $f$. The author would suggest these sets as good candidates for the notion of singularity for diagrams, in proving the $C^\infty$ stability theorem (Thm. 2.3.1) and in constructing a canonical stratification for diagrams in a certain class $f \in T_\infty \subset A_\infty$ (Theorems 2.1.2, 3.1.2).

The fundamental question for these critical sets is: whether the restrictions $f_\ell : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f), \ell \in L$ are proper and finite-to-one. We say a convergent diagram $f \in C^\infty(G,M)$ is a good representative of a tree $f_x$ if the restrictions $f : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f)$ are all proper and finite-to-one and also satisfy a certain additional condition on maximal trees (see Section 0.3 for the definition).

In the paper [N2] we proved that a (finite) tree $f_x$ of a convergent diagram admits a good representative if the $I_0$ codimension of $f_x$ is finite (Proposition 1.4.1 [N2]). Under the conditions $f \in A_\infty \cap C^\infty_{pr}(G,M)$ that $f_\ell$ are proper and all trees have finite $I_0$ codimension, it is proved that the critical sets $C_{\alpha(\ell)}(f), D_{\beta(\ell)}(f)$ are closed and the restrictions $f_\ell : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f)$ are all proper and finite-to-one (Theorem 2.2.1).

Now we are ready to explain the construction of stratification of diagrams $f \in C^\infty_{pr}(G,M)$. By Theorem 0.3.2, a tree $f_x$ with finite $I_0$ codimension admits an (infinitesimally) stable unfolding $F_x = (f_{x,x} : M_{\alpha(\ell)} \times \mathbb{R}', x \times 0 \to M_{\beta(\ell)} \times \mathbb{R}', f(x) \times 0)_{x \in X}$ of the form $F_{x,x}(x,u) = (f_{x,x}(x), u), f_{x,x} = f_x$. By the finite determinacy of stable diagrams (Theorem 0.3.1), we may suppose $F_x$ is a diagram of polynomial mappings. Then by a standard technique in the theory of semi-algebraic sets, we can construct a critical value stratification (CVS) $\mathcal{C}(F_x) = (\mathcal{C}_{x}(F_x))_{x \in F(x)}$, which yields immediately a Thom $A_{F_x}$ regular stratification $\mathcal{S}(F_x) = (\mathcal{S}_{x}(F_x))_{x \in X \cup f(X)}$ of $F_x$. A tree $f_x$ is topologically transversal if all inclusions $i_x : M_{\alpha x} \hookrightarrow M_{\alpha x} \times \mathbb{R}'$ are transversal to $\mathcal{S}(F_x)$. Then the pullbacks $i^*_{x} \mathcal{S}(F_x)$ give the stratification of $f_x$ denoted by $\mathcal{S}(f_x)$. A diagram $f \in A_\infty$ is topologically transversal if so are all trees in it (it is sufficient to consider topological transversality of maximal trees). The set of those diagrams is denoted by $T_\infty$. By the naturality of $\mathcal{C}(f_x)$, $\mathcal{S}(f_x)$ with respect to coordinate transformations (Proposition 1.2.1) and the coherence of maximal trees and branches (Theorem 2.2.1), the CVS $\mathcal{C}_{x}(f_x), x \in X \cup f(X)$ glue up to give a stratification of $M_\circ$ denoted $\mathcal{C}(f) = (\mathcal{C}_{o}(f))$, from which we obtain immediately a canonical stratification $\mathcal{S}(f)$ of $f$. 

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By the argument outlined above, the genericity of topological stability is deduced from the openness and density of the sets $A_\infty$ and $T_\infty$. These properties are proved in Theorem 3.2.2 and Theorem 2.1.2 under Condition $G$ defined below by arguments using transversality of jet sections. We explain these briefly.

A multi jet $z \in \prod_{\ell \in \mathcal{L}} \mathcal{J}^{\infty}(M_{\alpha(\ell)}, M_{\beta(\ell)})$ is a collection of jets $Jf_{\ell}(x_{\ell}) \in J(M_{\alpha(\ell)}, M_{\beta(\ell)})$, $\ell \in \mathcal{L}$, $i = 1, \ldots, m$. We regard again $z$ as a combinatorial tree $\Gamma_z$ (possibly a union of many connected components) consisting of the vertices $x_{\ell_i}$, $f_{\ell_i}(x_{\ell_i})$ and edges $Jf_{\ell_i}(x_{\ell_i}) : x_{\ell_i} \to f_{\ell_i}(x_{\ell_i})$.

We say the dimension function $P = (\dim M_\alpha)$ satisfies Condition $G$ if, for any $m$, $z \in \prod_{\ell \in \mathcal{L}} \mathcal{J}^{\infty}(M_{\alpha(\ell)}, M_{\beta(\ell)})$ off a subset of infinite codimension with any combinatorial type, $\Gamma_z$ is finitely $I_0$ determined. Some range of such dimensions $P$ is presented in the paper [N2].

The canonical stratification $S$ of $\prod_{\ell \in \mathcal{L}} \mathcal{J}^{\infty}(G, M)$ is roughly the partition by topological types of the stratification of stable unfoldings of those trees $\Gamma_z$.

Proposition 1.3.3 and the argument in Section 3.2 say that topological transversality of $f \in A_\infty$ is equivalent to the transversality of the multi jet section $\prod_{\ell \in \mathcal{L}} Jf_{\ell}$ to the $S$ for sufficiently large $m$. Therefore $T_\infty$ is a countable intersection of open dense subsets by the transversality theorem (Theorem 0.3.5), hence it is a dense subset by the Baire property of $C^{\infty}(G, M)$. The openness of $T_\infty$ is shown in Theorem 3.2.2 in the same way as that of $A_\infty$ in Theorem 2.1.2.

0.3. Terminology and Preliminaries.

For a tuple of positive integers $P = (p_v)$, let

$$\mathcal{E}(G, P) = \bigoplus_{\ell \in \mathcal{L}} m(p_{\alpha(\ell)}, p_{\beta(\ell)})$$

 denote the set of diagrams of map germs $f_{\ell} : \mathbb{R}^{p_{\alpha(\ell)}}, 0 \to \mathbb{R}^{p_{\beta(\ell)}}, 0$. Here $\mathcal{E}(n)$ is the local ring of smooth function germs on $\mathbb{R}^n$ at 0 with maximal ideal $m(n)$ and $\mathcal{E}(n, p) = \bigoplus_{v \in \mathcal{V}} \mathcal{E}(n)$. Let $\theta(P) = \bigoplus_{v \in \mathcal{V}} \theta(p_v)$, $\theta(f) = \bigoplus_{\ell \in \mathcal{L}} \theta(f_{\ell})$ and define the morphism $T(f) : \theta(P) \to \theta(f)$ by

$$T(f) \left( \bigoplus_{v \in \mathcal{V}} \chi_v \right) = \bigoplus_{\ell \in \mathcal{L}} \omega f_{\ell}(\chi_{p_{\alpha(\ell)}}) - tf_{\ell}(\chi_{p_{\beta(\ell)}}).$$
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(For these notions, see [M1-4]). We say \( f \) is infinitesimally stable or simply stable if \( T(f) \) is surjective, and \( f \) is trivial if

\[
T(f) \left( \bigoplus_{v \neq v_0} \Theta(p_v) \right) = \Theta(f).
\]

The \( I_0 \) codimension \( C_{I_0}(f) \) of \( f \) is defined by

\[
C_{I_0}(f) = \dim_0 \Theta(f)/\text{Im } T(f) + \bigoplus_{\ell \in L} f^*_{\pi(\ell)w_0} m(p_{v_0})\Theta(f_\ell),
\]

where \( f_{\pi(\ell)w_0} \) denotes the composition of \( f_\ell \) along the oriented path from \( v \) to the root \( v_0 \).

A diagram \( F \in \mathcal{E}(G, P + r) \), \( P + r = (p_v + r) \) is called an \( r \) parameter unfolding of \( f \in \mathcal{E}(G, P) \) if there are smooth imbeddings \( \iota_v : \mathbb{R}^p \rightarrow \mathbb{R}^{p_v + r} \) such that \( F_\ell \circ \iota_{\pi(\ell)} = \iota_{\pi(\ell)} \circ f_\ell \) and \( \iota_{\pi(\ell)} \) is transversal to \( f_\ell \). Two unfoldings \( F, G \) of \( f \) are equivalent as unfoldings if there are germs of diffeomorphisms \( \phi_v \) of \( \mathbb{R}^{p_v} \) such that \( F_\ell \circ \phi_{\pi(\ell)} = \phi_{\pi(\ell)} \circ G_\ell \) and \( \phi_v \circ \iota_{\pi(\ell)} = \iota_{\pi(\ell)} \), where \( \iota_{v_\ell}, \iota_{vG} \) are the transversal inclusions of \( f \) to \( G, F \).

The equivalence relation \( I_0 \) introduced in the paper [N1] is defined for diagrams \( f \) with \( C_{I_0}(f) < \infty \). We say that \( f, g \) are \( I_0 \) equivalent if and only if they have unfoldings \( F, G \) which are equivalent as diagrams (see Theorem 4.2.1 [N1]). If \( F, G \) are unfoldings of the same dimension of \( f, g \) respectively then \( f \) and \( g \) are \( I_0 \) equivalent if and only if \( F \) and \( G \) are \( I_0 \) equivalent. The \( I_0 \) equivalence classes \( \mathcal{O}^{I_0}(f) \) project to locally \( C^\infty \) trivial semialgebraic manifolds in the jet space \( J^k(G, P) = \prod_{\ell \in L} J^k(p_{\pi(\ell)}, p_{\pi(\ell)}) \) denoted \( \mathcal{O}^{I_0k}(f) \) (Proposition 2.4.2-2 [N1]).

Let \( F \in \mathcal{E}(G, P + r) \) be an unfolding of \( f \) of the normal form:

\[ F_\ell(x, u) = (f_{\ell u}(x), u), \ x \in \mathbb{R}^{p_{\pi(\ell)}}, \ u \in \mathbb{R}^r. \]

Then the jet section \( J^kF: \prod \mathbb{R}^{p_{\pi(\ell)}} \times \mathbb{R}^r \rightarrow J^k(G, \mathbb{R}^p) = \prod_{\ell \in L} J^k((\mathbb{R}^{p_{\pi(\ell)}}) \times \mathbb{R}^{p_{\pi(\ell)}}) \) is defined by

\[ J^kF((x_{\pi(\ell)}, u)) = (J^k f_{\ell u}(x_{\pi(\ell)})). \]

Let \( \Delta_G \subset \prod_{\ell \in L} \mathbb{R}^{p_{\pi(\ell)}} \times \mathbb{R}^{p_{\pi(\ell)}} \) denote the diagonal set \( \{(x_\ell, y_\ell)_{\ell \in L} | x_\ell = y_\ell, \alpha(\ell) = \beta(\ell') \} \).

Theorem 0.3.1 (Theorem 3.1.1 [N1]). \( C^\infty \) stable diagrams \( f \in E(G, P) \) are finitely determined : there is a function \( e(G, P) \) such that any diagram \( g \) with the same \( e(G, P) + 1 \) jet as \( f \) is equivalent to \( f \).
Theorem 0.3.2 (Theorem 5.1.1 [N1]). - The following conditions are equivalent:

1. $F$ is infinitesimally stable,
2. $J^{e(G, P + r)}F$ is transversal to $\Delta_G \times \partial f_{\eta(G, P + r)}(f)$,
3. $(\partial f_{\ell u}/\partial u_i(u=0))_{\ell \in L}, i = 1, \ldots, r$ span $\text{coker } T(f)$,
4. $(\partial f_{\ell u}/\partial u_i(u=0))_{\ell \in L}, i = 1, \ldots, r$ span

$$\theta(f)/\text{Im } T(f) + \bigoplus_{\ell \in L} f_{\ell u \in 0} m(p_{v_0}) \theta(f_{\ell}).$$

Theorem 0.3.3 (Theorem 5.2.1 [N1]). - Two stable $r$-parameter unfoldings $F, G$ of $f$ are equivalent as unfoldings.

Theorem 0.3.4 (Proposition 2.1.1 [N2]). - The condition $r < C(I_0(f))$ is an algebraic condition on the $e(r) (= e(G, P + r) + 1)$ jet of $f$, which defines an algebraic set $\Sigma^{e(r)} \subset J^{e(r)}(G, P = \prod_{\ell \in L} J^{e(\ell)}(p_{\ell}), p_{\ell})$ such that

$$\pi_{e(r) - s}(\Sigma^{e(r)}) \subset \Sigma^{e(s)}$$

for any $s < r$. If $C(I_0(f)) \leq r, f$ is $e(r) - I_0$ determined, i.e., any $g$ with the same $e(r)$ jet as $f$ is $I_0$ equivalent to $f$. (This is a consequence of Theorem 0.3.1 and 0.3.3.)

We say finite $I_0$ determinacy holds in general in $\mathcal{E}(G, P)$ if $\text{codim } \Sigma^{e(r)} \to \infty$ as $r \to \infty$.

Here we state our transversality theorem.

Theorem 0.3.5. - Let $G = (V, L, \Lambda)$ be a finite oriented graph, $M = (M_\ell)$ a collection of smooth manifolds and $S \subset J^k(G, M)$ a submanifold. Then the set $\mathcal{F}_S$ of diagrams $f \in C^\infty(G, M)$ for which the $k$ jet sections $J^k f = (J^k f_{\ell}) : \prod_{\ell \in L} M_{\ell} \to J^k(G, M)$ are transversal to $S$ is a countable intersection of open dense subsets.

From now on we apply all concepts for convergent graphs and convergent diagrams of map germs to trees of diagrams $f \in C^\infty(G, M)$.

Let $f_x, x_r \in M_{\ell}$ be a (finite) tree of a diagram $f \in C^\infty(G, M)$. The prolongation of $f_x$ is the tree $f_x^-$ defined by the set $X^- = X \cup f(X) \setminus M_{v_0}$.

The critical point set $C_{\ell}(f) \subset M_{\ell}$, $v \neq v_0$ of $f$ is the set of roots $x$ of (finite connected) trees $f_x$, for which the prolongation $f_x^- = f_x \cup \{f_{\ell x} : x \to f_{\ell}(x)\}$ ($X^- = X \cup f(X) = X \cup x$) is not trivial. The critical value set $D_{\ell}(f) \subset M_{\ell}$ is the set of roots $f_{\ell}(x)$ of those prolongations: $D_{\ell}(f) = \bigcup_{\ell \neq v(\ell', f) f_{\ell'}(C_{\ell'}(f))}$. 
A tree $f_X$ is indecomposable if $X_\ell \subset C_{\mathcal{U}(\ell)}(f)$ for all $\ell \in L$, and maximal if furthermore any tree $f_X$, with $X \neq X'$ is not indecomposable. We see easily that any point $x \in C_u(f) \cup D_v(f)$ is contained in unique maximal tree (possibly infinite) called the maximal tree of $x$ and denoted $f_{X_x}$, and its branch on $x \in X_x \cup f(X_x)$ is called the maximal branch on $x$ and denoted $f_{X_x}^{br}$. Note that $X_x^{br} = \bigcup_{\nu' < u} C_{\nu'}(f) \cap f_{\nu'}^{-1}(x)$. Conventionally we define $X_x = x$ if $x \notin C_v(f) \cup D_v(f)$.

We call a diagram $f \in C^\infty(G,M)$ a good representative of a tree $f_X$ if the following conditions are satisfied:

1. $f_X$ is maximal if $f_X$ is indecomposable,
2. The function $C_{\mathcal{U}(\ell)}(f_X)$ is upper semi continuous with $x \in M_v$, $v \in V$.
3. $C_v(f)$, $D_v(f) \subset M_v$ are closed and the restrictions $f_{\mathcal{U}(\ell)}: C_{\mathcal{U}(\ell)}(f) \to D_{\mathcal{U}(\ell)}(f)$ are proper and locally uniformly finite-to-one.
4. For any subgraph $G'$, the restriction $f_{G'}$ is a good representative of the subtree $f_{X'}$, $X' = \bigcup_{\ell \in L'} X_\ell$.

**Proposition 0.3.6 (Proposition 1.6.1 [N2]).** Any finitely $I_0$ determined convergent diagram of smooth map germs admits a good representative.

**Chapter 1**

**Critical Value Stratification (CVS)**

1.1. A canonical construction of CVS.

Let $G = (V,L,A)$ be a convergent diagram of height 1 with root $v_0$: $V = \{v_0,v_1,\ldots,v_k\}$, $L = \{\ell_1,\ldots,\ell_k\}$, $\ell_i: v_i \to v_0$. Let $M = (M_0,\ldots,M_k)$ be a collection of smooth manifolds and $f = (f_i)_{i=1,\ldots,k}$, $f_i \in C^\infty(M_i,M_{v_0})$.

We suppose Whitney $(B)$ regular stratifications $S_i$ of $M_i$ are given. Let $\Sigma(f_i|A)$, $A \in S_i$ denote the set of points $x \in A$ where $f_i|A$ is not a $C^\infty$ submersion. Let $\Sigma_{S_i}(f_i) = \bigcup_{A \in S_i} \Sigma(f_i|A)$, $D_{S_i}(f_i) = f_i(\Sigma_{S_i}(f_i))$ and $D_S(f) = \bigcup_{i=1}^k D_{S_i}(f_i)$, where $S$ stands for the $k$-tuple $(S_i)_{i=1,\ldots,k}$. By the $A$ regularity condition for $S_i$, $\Sigma_{S_i}(f_i)$ is closed in $M_{v_i}$.
A critical value stratification (CVS) **S'** of **D_s(f)** is a Whitney regular stratification of **D_s(f)** which possesses the following properties: for any \(i\) and strata **A**, **B** \(\in S_i\), **U**, **V** \(\in S'\) (where we allow \(A=B\) and \(U=V\)),

1. \(\Sigma(f_i|A) \cap f_i^{-1}(U)\) is a smooth submanifold of **A**,
2. \(f_i: \Sigma(f_i|A) \cap f_i^{-1}(U) \rightarrow U\) is locally diffeomorphic,
3. \(\Sigma(f_i|B) \cap f_i^{-1}(V)\) is Whitney regular over \(\Sigma(f_i|A) \cap f_i^{-1}(U)\),
4. \(B \cap f_i^{-1}(U) - \Sigma(f_i|B), B \cap f_i^{-1}(M_{v_0} - D_s(f))\) are Whitney regular over \(\Sigma(f_i|A) \cap f_i^{-1}(U)\).

If **D_s(f)** is a closed subset, the mapping \(f\) admits the Thom regular stratification \((S'_i, S')\) called the stratification of \(f\) associated with the critical value stratification \(S'\), defined by

\[
S'_i = \{\Sigma(f_i|A) \cap f_i^{-1}(U), (A - \Sigma(f_i|A)) \cap f_i^{-1}(U), A \cap f_i^{-1}(M_{v_0} - D_s(f)) | A \in S_i, U \in S'\}
\]

for \(i = 1, \ldots, k\); (Thom’s \(A_f\) regularity and Whitney \(B\) regularity of \(S'_i\) follow immediately from the properties (2) and (3), (4) respectively. For the definitions of Whitney \(B\) regularity and Thom’s \(A_f\) condition, see [Gi]).

In general let \(G = (V, L, \Lambda)\) be a convergent graph with root \(v_0\), \(M = (M_v)\) a collection of smooth manifolds, and \(f \in C^\infty(G, M)\). Let \(v^+\) denote the set of vertices \(v' \in V\) with \(\beta(v') = v\) (\(\beta: v' \rightarrow v\)) and let \(f_{v^+} = (f_i)_{\beta(v') = v}\). Now suppose that we have stratifications \(\mathcal{C}_v(f)\) of \(M_v\), for which the union of strata with positive codimension gives CVS of the critical value set \(D_v(f)\) of \(f_{v^+}\):

\[
D_v(f) = D_{\mathcal{C}_v}(f_{v^+}), \quad \mathcal{C}_{v^+} = (\mathcal{C}_{v'})_{v' \in v^+},
\]

for any \(v \in V\) \((D_v(f) = \emptyset\) for the source \(v\) of \(G\)).

Then \(f\) admits the stratification \(\mathcal{S}(f) = (\mathcal{S}_v(f))_{v \in V}\) associated with \(\mathcal{C}(f)\) defined by \(\mathcal{S}_{v_0}(f) = \mathcal{C}_{v_0}(f)\) and

\[
\mathcal{S}_{v(\ell)}(f) = \{\Sigma(f_i|A) \cap f_i^{-1}(U), (A - \Sigma(f_i|A)) \cap f_i^{-1}(U) | A \in \mathcal{C}_{v(\ell)}(f), U \in \mathcal{S}_{\beta(\ell)}(f)\}.
\]

Now we construct a CVS for the following mappings. Let \(p_0, \ldots, p_k\) be positive integers and \(U_i\) a semialgebraic open neighbourhood of \(0 \in \mathbb{R}^{p_i}\) for \(i = 0, \ldots, k\). Let \(S_i, i = 1, \ldots, k\) be Whitney regular semialgebraic stratifications of \(U_i\) and \(f_i: U_i, 0 \rightarrow U_0, 0\) polynomial
mappings. If the $f_i|\Sigma_{S_i}(f)$ are finite to one, then by shrinking $U_i$ if necessary, we may assume $f_i^{-1}(0) \cap \Sigma_{S_i}(f) = 0$ and $f_i|\Sigma_{S_i}(f)$ are proper hence $D_S(f), S = (S_i)$ is closed in $U_0$. In this situation the germ of $D_S(f)$ at $0$ is well defined by the germs of $f_i$ at $0$, and we have:

PROPOSITION 1.1.1. — $D_S(f)$ admits the CVS $\mathcal{C}_s(f)$ so-called canonical critical value stratification, which possesses the following properties: The germ of $\mathcal{C}_s(f)$ at $0$ is well defined by the germs of the $f_i$. Let $g_i : V_i \to V_0$ be polynomial mappings and $S'_i$ stratifications of $V_i$ satisfying the above conditions. If there are germs of diffeomorphisms $\phi_i$ of $\mathbb{R}^p_i$ such that $\phi_i \circ f_i = g_i \circ \phi_i$ and $\phi_i(S_i) = S'_i$, then $\phi_0(\mathcal{C}_s(f)) = \mathcal{C}_{s'}(g)$.

Proof. — We construct a filtration $Y_d \supset Y_{d-1} \supset \cdots \supset Y_0$ of the critical value set $Y_d = D_S(f)$ ($d = \dim D_S(f)$) by semialgebraic subsets $Y_i$ of dimension $\leq i$ inductively, so that $M_i = Y_i - Y_{i-1}$ and $M_0 = Y_0$ are Whitney regular submanifolds, $\dim M_i = i$ and possess the properties $(1) - (4)$ of CVS.

As induction hypothesis we assume that we have constructed a filtration $Y_d \supset \cdots \supset Y_i$ with the conditions $(1) - (4)$ for $A, B \in S_j$, $j = 1, \ldots, k$ and $U = M_m, V = M_n, i + 1 \leq m, n \leq d$. Then we define $Y_{i-1}$ in the following way: Let

(i) $Y_i^1 = Y_i - \Sigma(Y_i)$

(ii) $Y_i^2 = Y_i^1 - \bigcup_{\ell = i+1, \ldots, d} B(Y_i^1, M_\ell)$

(iii) $X_A^1 = A \cap f_j^{-1}(Y_i^2) \cap \Sigma(f_j|A)$ for each $A \in S_j$

(iv) $X_A^2 = X_A^1 - \Sigma(X_A^1)$

(v) $X_A^3 = X_A^2 \bigcap \bigcup_{B \in S_j} B(X_A^2, X_A^3)$

(vi) $X_A^3 = X_A^2 - \bigcup_{B \in S_j} B(X_A^2, f_j^{-1}(Y_i^2) \cap B - \Sigma(f_j|B))$

(vi) $X_A^3 = X_A^2 - \bigcup_{B \in S_j} B(X_A^2, f_j^{-1}(M_\ell) \cap B - \Sigma(f_j|B))$

(vi) $X_A^4 = X_A^3 - \text{Sing } (f_j|X_A^3)$.
Here \( \Sigma(Y_i) \) denotes the set of points \( x \in Y_i \) where \( Y_i \) is not a smooth manifold of dimension \( i \), \( B(X, Y) \) denotes the set of points \( x \in X \) where \( Y \) is not Whitney regular over \( X \) and \( \text{sing}(h: X \to Y) \) is the set of points \( x \in X \) where \( h: X \to Y \) is not of maximal rank. Then \( Y^1_i, Y^2_i, X^1_A, X^2_A \) and \( X^1_A \) are all semialgebraic, \( X^1_A \) is open in \( X^1_A \) and \( \dim(X^1_A-X^2_A) < i \) (for the properties of semialgebraic sets, see [Gi]), so we see that \( \dim f_j(X^1_A-X^2_A) < i \) and \( f_j: X^1_A \to f_j(X^1_A) \subset Y^2_i \) is locally isomorphic by the assumption that \( f_j|\Sigma S_j(f_i) \) is finite-to-one and the refining process (vi). Let

\[
Y^3_i = Y^2_i - \bigcup_{A \in S_i, j=1, \ldots, k} f_j(X^1_A-X^2_A).
\]

Then \( Y^3_i \) is smooth of dimension \( i \). Now we define \( Y_{i-1} = Y_i - Y^3_i \). We claim that \( Y_{i-1} \) possesses the required property. The properties (1) – (4) of CVS for \( M_i = Y_i - Y_{i-1} \) involving the other strata \( M_{i'}, i' = i+1, \ldots, d \) and \( A, B \in S_j \) follow respectively from the refining process (iv), (vi) and (vii), the 4-th term in RHS of (v), and the 5-th and 6-th terms in RHS of (vi). If \( 0 \neq n = \dim Y_{i-1} < i - 1 \), then we define \( Y_{i-2} = Y_{i-2} = \cdots = Y_n \) and go on to the next step of refining \( Y_n \) to define \( Y_{n-1} \) so that \( M_n = Y_n - Y_{n-1} \) is smooth of dimension \( n \) and satisfies the required properties. If \( n = 0 \), we complete the induction.

By construction the filtration \( Y_d \supset \cdots \supset Y_0 \) is determined by the germs of \( f_j \) at \( \Sigma S_j(f_j) \subset U_j \). Since \( f_j|\Sigma S_j(f_j) \) are proper and \( f_j^{-1}(0) \cap \Sigma S_j(f_j) = 0 \), the germ of \( Y_i \) at \( 0 \in U_0 \) is determined by the germs of \( f_j \). The naturality of the germ of \( \mathcal{C}_s(f) \) at 0 with respect to coordinate transformations is clear.

### 1.2. Some properties of CVS.

**Proposition 1.2.1.** — Any stable convergent diagram of smooth map germs \( f \in \mathcal{E}(G,M) \) admits a representative \( \hat{f} = (\hat{f}_i) \), \( \hat{f}_i: U_{\alpha(i)} \to U_{\beta(i)} \) defined on open neighbourhoods \( U_{\alpha} \) of \( 0 \in \mathbb{R}^{p_0} \), with a CVS \( \mathcal{C}(\hat{f}) = (\mathcal{C}_s(\hat{f})) \) such that the restrictions \( f_i: \Sigma S_{j}(\hat{f}_i) \to U_{\beta(i)} \) are proper and finite-to-one. The germs of \( \mathcal{C}_s(\hat{f}) \) at 0 are well defined by \( f \), and called the canonical CVS and denoted \( \mathcal{C}(f) = (\mathcal{C}_s(f)) \). If \( \hat{f} \) is a diagram of polynomial map germs, \( U_\alpha \), \( \mathcal{C}_s(\hat{f}) \) and \( \mathcal{C}_s(f) \) are semialgebraic. Let \( g \in \mathcal{E}(G,P) \) and assume there are germs of diffeomorphisms \( \phi_\ell \) of \( (\mathbb{R}^{p_0},0) \) with \( \phi_\ell^{-1} \circ g_\lambda \circ \phi_\ell(0) = f_\ell \) for \( \ell \in L \). Then \( \phi_\ell(\mathcal{C}_s(f)) = \mathcal{C}_s(g) \) for \( v \in V \).
Proof. — By the determinacy theorem (Theorem 0.3.1), stable diagrams are equivalent to diagrams of polynomial map germs. So it suffices to prove the statement for polynomials \( f, g \). We construct \( \mathcal{G}_v(f) \) by descending induction on the height of vertices \( v \in V \).

Let \( f_{v^+} \) denote the restriction of \( f \) to the edges \( \ell_v : v' \rightarrow v \). We assume that there are semialgebraic open neighbourhoods \( U_v \) of \( 0 \in \mathbb{R}^p_v \), \( f_v(U_{a(\ell)}) = U_{b(\ell)} \) such that the restrictions of \( f \) to the branches \( G_{v'} \), on \( v' \), \( \beta(v') = v \), admit the canonical CVS \( \mathcal{G}_{v''} \) on \( U_{v''} \), \( v'' \leq v' \). We will construct a stratification \( \mathcal{G}_v(f) \) of \( U_v \) so that the union of strata with positive codimension gives the CVS of the critical value set \( D_{\mathcal{G}_{v''}}(f_{v^+}) \) of \( f_{v^+} \) with respect to \( \mathcal{G}_{v''}(f) = (\mathcal{G}_{v''}(f))_{\beta(v'') = v} \).

By Proposition 0.3.6, we may assume that \( (f_\ell | U_{a(\ell)}) \) is a good representative of \( f \). By the definition of the critical sets in Section 0.3, a tree \( f_x \) with its root in \( U_v \) and vertices of height 1 off the critical sets \( \mathcal{G}_{v''}(f) \), \( \beta(v') = v \) is trivial. Therefore we see \( \Sigma_{\mathcal{G}_{a(\ell)}(f)(\ell)} = \mathcal{G}_{a(\ell)}(f) \) for \( \ell \) with \( \beta(\ell) = v \). By the properties of good representatives, the restrictions \( f_{\ell} \mid \Sigma_{\mathcal{G}_{a(\ell)}(f)}(\ell) \) are also proper and finite-to-one, for \( \Sigma_{\mathcal{G}_{a(\ell)}(f)}(\ell) \) is closed in \( U_{a(\ell)} \).

By Proposition 1.1.1, there are semialgebraic open neighbourhoods \( U_{v'} \) of \( 0 \in U_{v'} \) for \( v' \), \( \beta(v') = v \) such that the restriction \( (f_{\ell} | U_{a(\ell)})(\ell) = v \), \( \beta(v') = v \) admits the canonical CVS \( \mathcal{G}_{v''}(f) \) of the critical value set \( \bigcup_{\beta(\ell) = v} f_{\ell}(\Sigma_{\mathcal{G}_{a(\ell)}(f)}(\ell)) \). We put \( U_{v''} = U_{v''} \cap f_{\ell}^{-1}(U_{v'}) \), \( \beta(v') = v \) for \( v'' < v \). Then the restrictions \( \mathcal{G}_{v''}(f) \cap U_{v''} \), \( v'' \leq v \) gives the CVS of \( (f_{\ell} | U_{a(\ell)})(\ell) = v \). This completes the construction of the canonical CVS of \( f \) by induction. The final property of CVS in the proposition follows from the naturality of the canonical CVS in Proposition 1.1.1.

Now we state some properties of the above CVS.

Proposition 1.2.2. — Let \( G = (V, L, \Lambda) \), \( G' = (V', L', \Lambda') \) be convergent diagrams with a common root \( v_0 \), and let \( P = (p_\ell)_{\ell \in V} \), \( P' = (p'_\ell)_{\ell \in V} \) be tuples of positive integers with \( p_{v_0} = p'_{v_0} \). Let \( f \lor f' \) denote the union of \( f \in \mathcal{E}(G, P) \) and \( f' \in \mathcal{E}(G', P') \). If \( f \lor f' \) is stable, so are \( f \) and \( f' \), and \( \mathcal{G}_v(f) \), \( \mathcal{G}_v(f') \) meet in a general position at \( 0 \in \mathbb{R}^{p_{v_0}} \) and \( \mathcal{G}_v(f \lor f') = \mathcal{G}_v(f) \cap \mathcal{G}_v(f') \).

Proof. — This follows from the construction of the canonical CVS and its naturality with respect to coordinate transformations.
For a stable convergent diagram \( f \in \mathcal{E}(G, P) \) we define \( \text{codim}(f) = (\text{codim}_v(f))_{v \in V} \) by \( \text{codim}(X)_{v \in V} \), where \( X_v \) is the stratum of the canonical CVS containing the origin \( 0 \in \mathbb{R}^P \) (we put \( \text{codim} X_0 = 0 \) for the sources \( v \) of \( G \)). The uniqueness of stable unfolding (Theorem 0.3.3) enables us to define \( \text{codim}(f) \) for finitely \( I_0 \) determined \( f \in \mathcal{E}(G, P) \) to be the codimension \( \text{codim}(F) \) of its stable unfolding \( F \). By definition of the equivalence relation \( I_0 \) in this paper (section 0.3), \( \text{codim}(f) \) is determined by the \( I_0 \) equivalence class of \( f \).

From now on, we say that a finitely \( I_0 \) determined diagram \( f \) is \textit{topologically trivial} if \( \text{codim}_{I_0}(f) = 0 \) (it seems that \( f \) is topologically trivial if and only if \( C^\infty \) is trivial. For the definition, see Section 0.3), and we call \( f \) \textit{is topologically indecomposable} if all prolongations \( f_{v_0} \) of branches \( f_{v_0} \) on \( v \in V \) are topologically non-trivial, in other words, for some stable unfoldings, \( F, \Sigma_{v \in (\mathcal{E})}(F) = \emptyset \) for all \( \ell \in L \).

We call a sub tree \( f_X \) of the maximal tree \( f_{X_\chi} \) of \( x \) of \( f \in C^\infty(G, M) \) with \( C_{f_0}(f_{X_\chi}) < \infty \), the \textit{topologically maximal tree} of \( x \) if \( f_X \) contains \( x \) as a vertex, \( f_X \) is topologically indecomposable and its complement \( f_{X_\chi-x} \) is topologically trivial. We denote this tree by \( f_{\text{top}
olimits_X} \) and its branch on \( x \) by \( f_{\text{top}
olimits_X} \text{br} \), i.e.

\[
\text{top}
olimits_X \text{br}_x = \bigcup_{v' < v} (f_{v'v}^{-1}(x) \cap \text{top}
olimits_X), \quad x \in M_0.
\]

The \textit{topologically characterizing tree} \( f_{\text{top}
olimits_X} \) of \( x \) is the union of the above \( f_{\text{top}
olimits_X} \), the sequence \( f: x \to f(x) \to f^2(x) \cdots \to f^h(x) \in M_{v_0} \) and the tree

\[
f_{\text{top}
olimits_X \text{ch}(x)}: \text{top}
olimits_X \cup \{x, f(x), \ldots, f^{h-1}(x)\} \cup \text{top}
olimits_X \text{ch}(x).
\]

The next proposition follows directly from the construction of the canonical CVS.

**Proposition 1.2.3.** Let \( f \in \mathcal{E}(G, P) \) be a stable diagram and let \( \hat{f} = (\hat{f}_v: U \rightarrow U'_v) \) be a good representative as in Proposition 1.2.1. Then the canonical CVS \( \mathcal{E}_h(\hat{f}) \) and the associated stratification \( \mathcal{P}_h(\hat{f}) \) coincide with the partitions of \( U_v \) by \( \text{codim}(f_{\text{top}
olimits_X} \text{br}), \text{codim}(f_{\text{top}
olimits_X} \text{ch}) \), respectively. And the germs \( \mathcal{E}_v(\hat{f})_x, \mathcal{P}_v(\hat{f})_x \) at \( x \) coincide with \( \mathcal{E}_x(f_{\text{top}
olimits_X} \text{br}), \mathcal{P}_x(f_{\text{top}
olimits_X} \text{ch}) \) respectively, where \( x \)'s are regarded as vertices of the underlying oriented trees \( f_{\text{top}
olimits_X} \text{br}, f_{\text{top}
olimits_X} \text{ch} \).
1.3. Construction of the stratification of the jet space of convergent diagrams.

Let $e(r)(= e(G,P+r)+1)$ be the increasing function in Theorem 0.3.4 and $\Sigma^{e(r)} \subset J^{e(r)}(G,P)$ be the well-defined set of $e(r)$ jets of diagrams $f \in \mathcal{E}(G,P)$ such that $C_{i_0}(f) > r$ (in other words, $f$ does not admit a stable $r$-parameter unfolding). By Theorem 0.3.1, 0.3.3 and 0.3.4, a stable unfolding of $f$ with $C_{i_0}(f) \leq r$ is uniquely determined up to the equivalence of diagrams by the $e(r)$ jet of $f$. So we can define the set $S^{e(r)}(G,P) \subset J^{e(r)}(G,P) - \Sigma^{e(r)}$ by

$$S^{e(r)}(G,P) = \{ z \in J^{e(r)}(G,P) - \Sigma^{e(r)} : \text{codim } f = I, \pi^{e(r)}(f) = z \},$$

for a tuple $I = (e_v)_{v \in V}$ of non-negative integers ($C_v=0$ for the sources $v$ of $G$). Then $S^{e(r)}(G,P)$ defines a partition of the complement of $\Sigma^{e(r)}$, denoted $S^{e(r)}(G,P)$. Again by the finite determinacy of stable diagrams, $S^{e(r)}(G,P)$, $r = 0, 1, \ldots$ defines a partition of $\mathcal{E}(G,P) - \Sigma$ by pro-sets, where $\Sigma$ is the set of non finitely $I_0$ determined diagrams.

**Proposition 1.3.1.** Let $f \in \mathcal{E}(G,P)$ be a stable diagram and $\hat{f}$, $\mathcal{E}(\hat{f})$ and $\mathcal{F}(\hat{f})$ as in Proposition 1.2.1. Let $X_v$, $X'_v$ ($X_v = U_v$ for sources $v \in X'_a(\ell) = \Sigma(\hat{f}_\ell|X_a(\ell)) \cap \hat{f}_\ell^{-1}(X'_b(\ell))$ be the strata of $\mathcal{E}(\hat{f})$, $\mathcal{F}(\hat{f})$ containing the origin in $U_v$, respectively. Let $I = \text{codim } f = (\text{codim } X_v)_{v \in V}$ and assume that codim $Y < \text{codim } X_v$ for all other strata $Y \in \mathcal{E}(\hat{f})$ and $v \in V$. Then $J^{e(0)}(x_{a(\ell)}(\ell) \in L) \subset S^{e(0)}(G,P) \times \Delta_G$ if and only if $x_{a(\ell)} \in X'_a(\ell)$, $x_{b(\ell)} \in X'_b(\ell)$ and $\hat{f}(x_{a(\ell)}) = x_{b(\ell)}$ for all $\ell \in L$, (where $e(0) = e(G,P) + 1$. See Section 0.3).

**Proof.** It suffices to prove the statement for indecomposable stable diagrams. First we prove the «if» part. Let $X = (x_{a(\ell)})_{\ell \in L}$ be as above and let $\hat{f}_X$ denote the maximal tree of the good representative $\hat{f}$ including the tree $\hat{f}_X$. By the properties (1), (2) of CVS, $\Sigma(\hat{f}_\ell|Y) \cap \hat{f}_\ell^{-1}(X'_b(\ell))$ is a smooth submanifold on which $\hat{f}_\ell$ restricts to a locally isomorphic covering map onto $X'_b(\ell)$ for any $Y \subset \mathcal{E}_{a(\ell)}(\hat{f})$. Since $\Sigma_{a(\ell)}(\hat{f}_\ell)(\hat{f}_\ell) \cap \hat{f}_\ell^{-1}(0) = 0$, we see

$$\Sigma_{a(\ell)}(\hat{f}_\ell)(\hat{f}_\ell) \cap \hat{f}_\ell^{-1}(X'_b(\ell)) = \Sigma(\hat{f}_\ell,X_{a(\ell)}) \cap \hat{f}_\ell^{-1}(X'_b(\ell)) = X'_a(\ell)$$

and the restriction $\hat{f}_\ell : X'_a(\ell) \to X'_b(\ell)$ is isomorphic. So we see by Proposition 1.2.2, the CVS of $\hat{f}_X$ is trivial at its roots $x_{b(\ell)}$ in those
strata \( X_{\beta(\ell)} \), in other words, \( \hat{f}_{X_X} \) is topologically trivial, thus
\( \mathcal{C}_{\beta(\ell)}(\hat{f}_{X_X}) = \mathcal{C}_{\beta(\ell)}(f) \) and the germ of \( \mathcal{C}_{\beta(\ell)}(f) \) at the vertices \( x_{\beta(\ell)} \) coincide with these germs. In particular we have

\[
\text{codim } \hat{f}_{X_X} = (\text{codimensions of the strata of } \mathcal{C}_{\nu}(\hat{f}_{X_X}) \text{ containing } x_{\nu})
\]

\[
= (\text{codim } X_{\nu})_{\nu \in \mathbb{V}} = \text{codim } f.
\]

Conversely we assume a connected tree \( f_{X_X} \), \( X = (x_{\alpha(\ell)}) \) has a vertex \( f_{\ell}(x_{\alpha(\ell)}) \) off the stratum \( X_{\beta(\ell)} \) for some \( \ell \in \mathbb{L} \). Let \( \hat{f}_{X_X} \) be the maximal tree with \( X \subseteq \hat{X}_{\nu} \), and \( x' \in U_{\nu} \) one of the highest vertices of \( \hat{f}_{X_X} \) where
\( \hat{f}_{X_X} \) is branching off \( f_{X_X} \). Let \( \hat{f}_{X_X} \) denote the branch of \( \hat{f}_{X_X} \) on \( x' \) and \( \hat{f}_{X_X} \) the sub-graph of \( \hat{f}_{X_X} \) branching off \( \hat{f}_{X_X} \) at \( x' : X_{X} = X_{X} - X_{X} \) (a connected component of \( \hat{f}_{X_X} - f_{X_X} \)). By Proposition 1.2.2, we have
\( \mathcal{C}_{\nu}(\hat{f}_{X_X}) = \mathcal{C}_{\nu}(\hat{f}_{X_X}) \cap \mathcal{C}_{\nu}(\hat{f}_{X_X}) \). By the condition of the proposition, the stratum of \( \mathcal{C}_{\nu}(\hat{f}_{X_X}) \) containing \( x' \) has codimension smaller than \( \text{codim } X_{\nu} = \text{codim}_{\nu} (f) \). So we have \( \text{codim } (\hat{f}_{X_X}) \neq \text{codim } f = I \). This completes the proof.

**Proposition 1.3.2.** — Let \( I = (a_{\nu})_{\nu \in \mathbb{V}} \) be a tuple of positive integers \( (a_{\nu} = 0 \text{ for sources } \nu) \). Then the set \( S^{(r)}_{p}(G,P) \subset J^{(r)}(G,P) \) is a semialgebraic submanifold of codimension
\[
\sum_{\nu \neq \nu_0} p_{\nu} + a_{\nu_0} - \sum_{\nu \in \mathbb{V}} b_{\nu} p_{\nu}, \text{ where } b_{\nu}
\]
denotes the number of edges \( \ell \in \mathbb{L} \) with \( (\beta(\ell) = \nu) \).

**Proof.** — Let \( z \in S^{(r)}_{p}(G,P) \subset J^{(r)}(G,P) \) and let
\[
F = (F_{\ell})_{\ell \in \mathbb{L}} \in E(G,P) + s, \quad F_{\ell} : (\mathbb{R}^{p_{\nu} + s}, 0) \rightarrow (\mathbb{R}^{p_{\nu} + s}, 0),
\]
\( F_{\ell}(x,u) = (f_{\ell, \nu}(x),u), x \in \mathbb{R}^{p_{\nu}}, u \in \mathbb{R}^{s} \) be a stable sequence of polynomial map germs unfolding the polynomial map germ \( f \), such that the \( r \)-jet section \( F_{\ell}^{(r)} F \) : \( \prod_{\nu \neq \nu_0} \mathbb{R}^{p_{\nu} + s} \rightarrow J^{(r)}(G,P) = \prod_{\ell \in \mathbb{L}} J^{(r)}(\mathbb{R}^{p_{\nu}}, \mathbb{R}^{p_{\nu}}) \) is locally diffeomorphic at the origin, and by Theorem 0.3.2, \( F \) is stable. Let \( \mathcal{S}(F) = (\mathcal{S}_{\nu}(f))_{\nu \in \mathbb{V}} \) be the canonical stratification of \( F \) and let \( \mathcal{S}_{\nu} \) be the strata of \( \mathcal{S}_{\nu}(F) \) containing the origin in \( \mathbb{R}^{p_{\nu} + s} \). By Proposition 1.2.1, each \( \mathcal{S}_{\nu} \) is semialgebraic and \( F_{\ell}(\mathcal{S}_{\nu}) : \mathcal{S}_{\nu}(G) \rightarrow \mathcal{S}_{\nu}(G) \) is isomorphic. Let
\[
X = (x_{\alpha(\ell)})_{\ell \in \mathbb{L}} \in \prod_{\ell \in \mathbb{L}} \mathcal{S}_{\alpha(\ell)} \subset \prod_{\ell \in \mathbb{L}} \mathbb{R}^{p_{\nu} + s}, \quad f_{\ell, \nu}(x_{\alpha(\ell)}) = x_{\beta(\ell)}
\]
and let $X = ((x_\alpha, u)_\ell \in \prod_{\ell \in \mathcal{L}} \mathbb{R}^{p_\ell} \times \mathbb{R}^s$ with $(x_\alpha, u)_\ell \in \mathcal{S}_\alpha(\ell)$ and $F_\ell(x_\alpha, u) = (x_\beta(\ell), u)$. Then both $X$ and $\bar{X}$ are semialgebraic submanifolds. By Proposition 1.3.1, we have

$$X = \mathcal{J}^r F^{-1}(S_\ell'(G, P + r) \times \Delta_G), \quad \Delta_G = \prod_{\ell \in \mathcal{L}} \mathbb{R}^{p_\ell(r) + r} \times \mathbb{R}^{p_\ell(r) + r},$$

and by the definitions of $S_\ell'(G, P)$ and $S_\ell'(G, P + r)$, we have

$$\bar{X} = \mathcal{J}^r F^{-1}(S_\ell'(G, P) \times \Delta_G), \quad \Delta_G = \prod_{\ell \in \mathcal{L}} \mathbb{R}^{p_\ell(r)} \times \mathbb{R}^{p_\ell(r)}.$$

Since $F$ is a sequence of polynomial map germs, $\mathcal{J}^r F$ is also a polynomial map germ, and since $\mathcal{J}^r F$ is a diffeo-germ, the image $S_\ell'(G, P) \times \Delta_G$ of $\bar{X}$ is a semialgebraic submanifold. Now we have the following equality,

$$\dim S_\ell'(G, P) \times \Delta_G = \dim \bar{X} = \dim \mathcal{S}_v,$$

from which we have

$$\operatorname{codim} S_\ell'(G, P) \times \Delta_G = \sum_{v \neq v_0} p_v + r - \dim S_v,$$

$$= \sum_{v \neq v_0} p_v - p_{v_0} + \operatorname{codim} S_{v_0} \text{ in } \mathbb{R}^{p_{v_0}},$$

$$\operatorname{codim} S_\ell'(G, P) = \sum_{v \neq v_0} p_v - p_{v_0} + \operatorname{codim} S_{v_0} - \operatorname{codim} \Delta_G$$

$$= \sum_{v \neq v_0} p_v + a_{v_0} - \sum_{v \in \mathcal{V}} b_v, p_v.$$

This completes the proof.

**Proposition 1.3.3.** Let $\mathcal{F} = (f_\ell)_{\ell \in \mathcal{L}} \in \mathcal{S}(G, P)$ be a finitely determined diagram: $C_0(f) \leq r$. Let $F \in \mathcal{S}(G, P + s)$, $F_\ell: (\mathbb{R}^{p_\ell(\ell) + s}, 0) \to (\mathbb{R}^{p_\ell(\ell) + s}, 0)$, $F_\ell(x, u) = (f_\ell(x), u)$, $f_\ell = f_\ell$ be a stable unfolding of $f$, and let $\mathcal{C} = (\mathcal{C}_v(F))_{v \in \mathcal{V}}$ be the canonical CVS of $F$ ($\mathcal{C}_v(F)$ is trivial for sources $v$). Then $\mathcal{C}$ is transversal to $\mathcal{C}_v$ for all $v \in \mathcal{V}$, if and only if $J^{r(v)} f$ is transversal to $S^{r(v)}(G, P) \times \Delta_G$ at $(0)_{v \neq v_0} \in \prod_{v \neq v_0} \mathbb{R}^{p_v}$.

**Proof.** By Theorem 0.3.3, the transversality of $\mathcal{C}$ to $\mathcal{S}_v$ is independent of the choice of the stable unfolding $F$. So we assume that the $(r)$ jet section $J^{r(v)} F: \prod_{v \neq v_0} \mathbb{R}^{p_v} \times \mathbb{R}^s \to J^{r(v)}(G, \mathbb{R}^p)$ defined by $J^{r(v)} F((x_v)_{v \neq v_0}, u) = J^{r(v)} f_u((x_v)_{v \neq v_0})$ is the germ of a diffeomorphism,
where \( f_u = (f_{(u)})_{u \in L} \in \mathcal{E}(G, P) \). Since \( J^{e(r)}f = J^{e(r)}F \circ \iota : \prod_{v \neq v_0} \mathbb{R}^p_v \to \prod_{v \neq v_0} \mathbb{R}^p_v \times \mathbb{R}^r \rightarrow J^{e(r)}(G, \mathbb{R}^p_v) \) we see \( J^{e(r)}f \mapsto S^{e(r)}_i(G, P) \times \Delta_0 \) if and only if \( \iota \mapsto J^{e(r)}F^{-1}(S^{e(r)}_i(G, P) \times \Delta_0) \), where \( \iota \) is the natural inclusion. As we have seen in the proof of Proposition 1.3.2, \( J^{e(r)}F^{-1}(S^{e(r)}_i(G, P) \times \Delta_0) \) is the set of points \( ((x_v), u) \in \prod_{v \neq v_0} \mathbb{R}^p_v \times \mathbb{R}^r \) such that \( \text{codim} \ F_x = 1 \),

\[
X = ((x_v), u)_{u \neq v_0},
\]
in other words, by Proposition 1.3.1,

\[
(x_{u(\ell)}, u) \in \Sigma(F_{\ell}, X_{u(\ell)}(F)) \cap F^{-1}_{\ell}(X'_{u(\ell)}(F)) = X_{u(\ell)}(F), \quad \ell \in L.
\]

Let \( Pr : \prod_{v \neq v_0} \mathbb{R}^p_v \times \mathbb{R}^r \to \mathbb{R}^r \) by the natural projection. The image of the inclusion \( \iota \) is the fibre of \( Pr \) on \( 0 \in \mathbb{R}^r \), so the above transversality holds if and only if the restriction \( Pr : J^{e(r)}f^{-1}(S_i(G, P) \times \Delta_0) \to \mathbb{R}^r \) is a submersion. Since the \( F_{\ell} : X_{u(\ell)}(F) \to X'_{u(\ell)}(F) \) are isomorphisms, this holds if and only if the second projections \( P_{\ell} : X'_{u(\ell)}(F) \to \mathbb{R}^r \) are submersions, if and only if the inclusions \( \iota_{\ell} : \mathbb{R}^p_v \to \mathbb{R}^{p_v + s} \) are transversal to \( \mathcal{S}_v(F) \) for \( v \in V \), and if and only if \( \iota_{\ell} \) is transversal to \( C_{\ell}(F) \) for \( \ell \in L \).

From now we say a finitely \( I_0 \) determined diagram \( f \in \mathcal{E}(G, P) \) is **topologically transversal** if the condition in the above proposition is satisfied.

**Corollary 1.3.4.** — Let \( f, F \) be as above. Then there is a good representative \( \hat{F}, \hat{F}_{\ell} : U_{u(\ell)} \to U_{\ell(\ell)} \) of \( F \) defined on open neighbourhoods \( U_v \) of \( C \in \mathbb{R}^{p_v + s} \) which admits the canonical CVS \( \mathcal{C}(F) = (\mathcal{C}_v(F))_{v \in V} \) and the natural inclusions \( \iota_v : \mathbb{R}^p_v \to \mathbb{R}^{p_v + s} \) are transversal to \( \mathcal{C}_v(\hat{F}) \) (for the definition of good representatives, see Section 0.3). In this situation, the restriction \( \hat{f}(\hat{F}_{\ell}) | U_{u(\ell)} \times \mathbb{R}^{p_v(\ell) \times 0} \) is a good representative of \( f \), which admits the canonical CVS \( \mathcal{C}(\hat{f}) = (\mathcal{C}_v(\hat{F})) | \mathbb{R}^{p_v(\ell) \times 0} \) and all connected trees of \( \hat{f} \) are topologically transversal. The germs of \( \mathcal{C}_v(\hat{f}) \) at \( 0 \) are independent of the choice of \( \hat{F} \) and denoted \( \mathcal{C}_v(f) \). Let \( \mathcal{S}(\hat{f}) \) denote the stratification of \( \hat{f} \) associated with \( \mathcal{C}(\hat{f}) \). Then \( \mathcal{C}_v(\hat{f}), \mathcal{S}_v(\hat{f}) \) coincide respectively with the partition of \( U_v \times \mathbb{R}^{p_v} \times 0 \) by the numbers \( \text{codim}_x \hat{f}_{\text{top}X_{x}} = \text{codim}_x \hat{f}_{X_{x}} \), \( \text{codim}_x \hat{f}_{\text{top}X_{x}} = \text{codim}_x \hat{f}_{X_{x}} \) associated with those points \( x \in U_v \cap \mathbb{R}^{p_v} \times 0 \), where \( x \) is regarded as vertices of those trees of \( \hat{f} \). Consequently the germs \( \mathcal{C}_v(\hat{f})_x, \mathcal{S}_v(\hat{f})_x \) at \( x \) coincide with the germs \( \mathcal{C}_x(\hat{f}_{\text{top}X_{x}}) = \mathcal{C}_x(\hat{f}_{X_{x}}), \mathcal{S}_x(\hat{f}_{\text{top}X_{x}}) = \mathcal{S}_x(\hat{f}_{X_{x}}) \), respectively. (For the definition of the above trees, see Section 1.2.)
2.1. Some generic properties.

Let $G = (V, L, A)$ be a convergent graph, $Q = (q_v)$ a tuple of integers $0 \leq q_v \leq \infty$. We call $G$ a $Q$ graph if each fibre $\beta^{-1}(v)$ of $\lambda : L \to V$ consists of at most $q_v$ edges (finite if $q_v = \infty$). Let $i : \Gamma \to G$, $i = (i_v, i_L) : (V_\Gamma, L_\Gamma) \to (V_G, L_G)$ be a morphism of oriented graphs. We call $i$ a $Q$ morphism if $\Gamma$ is a $i^*Q$-graph.

Let $P = (p_v)_{v \in V}$ be a tuple of integers $0 < p_v < \infty$. We say $P$ satisfies Condition $G_q(G)$ if finite $I_0$ determinacy holds in general in $\mathcal{E}(\Gamma, i^*P)$ for any $Q$-morphism (finite morphism) $i : \Gamma \to G$.

We call a diagram (tree) $f_X$ of $f$ of embedding type $i : \Gamma \to G$ (defined by the inclusions of germs) a $Q$-diagram ($Q$-tree), if $i$ is a $Q$-morphism.

Let $0 \leq r \leq \infty$ be an integer and $U \subset M_{v_0}$ a subset. We denote by $A_{Q,U}$ the set of smooth diagrams $f \in C^\infty(G, M)$ such that for any $Q$ tree $f_X$ of the restriction $f_U = (f_\Gamma|f_\alpha_0(U))$, the $I_0$ codimension $C_{I_0}(f_X)$ is at most $r$ (finite if $r = \infty$), and we denote $A^r_{Q,U} = A_{Q,U}$, $A_{Q,M_{v_0}} = A_{Q}$.

**Proposition 2.1.1.** Let $0 < r < \infty$, $P + r + 1 = (\dim M_v + r + 1)_{v \in V}$ and let $U \subset M_{v_0}$ be a subset. Then

$$A_{Q,U}^r = A_{P+r+1,U}^r, \quad \infty = (\infty)_{v \in V}.$$

**Proof.** From the definition it follows immediately that $A_{Q,U}^r = A_{P+r+1,U}^r$. Conversely let $f \in A_{P+r+1,U}^r$ and for simplicity of notations assume $U = M_{v_0}$. Then we prove that any connected and indecomposable tree of $f$ with root in $M_v$ is a $P + r$ tree, by the descending introduction on the height $h(v)$ of the vertices $v \in V$. It then follows that all finite trees of $f$ admit stable unfoldings of $\dim \leq r$ hence $f \in A_{Q,U}^r$.

We may assume inductively all indecomposable trees of $f$ with roots in $M_v$, $h(v) \geq h$ are $(P+r)$-trees of $f$. Let $f_X$ be an arbitrary finite indecomposable tree of $f$ with root $x \in M_v$ of height $h(v) = h$. Suppose that $f_X$ is a union of the prolongations $f_{x_i}$ of the branches $f_{x_i}$ of $f_X$ on $x_i \in X \cap M_{\beta(\ell_i)}$, $\beta(\ell_i) = v$, $f_{\ell_i}(x_i) = x$ and that $q \geq p_v + r + 1$.
By the induction hypothesis these prolongations are all \((P+r)\)-trees. Let \(f_Y\) be a \(p_v + r + 1\) union of these prolongations. Then, by Corollary 1.1.4, at least one of these \(p_v + r + 1\) branches must be trivial. This contradicts the assumption that \(f_X\) is indecomposable. Therefore \(f_X\) is a union of at most \(p_v + r\) prolongations, and in particular is a \((P+r)\)-tree. This completes the proof.

Our purpose in this chapter is to prove

**Theorem 2.1.2.** — Let \(G = (V,L,\Lambda)\) be a convergent diagram with root \(v_0\) and \(M = (M_v)_{v \in V}\) a collection of smooth manifolds. Then the set \(A_\infty \cap C^\infty(G,M)\) is open in \(C^\infty(G,M)\) with the Whitney topology for any closed subset \(K \subseteq M_{v_0}\), and if \(K\) is compact the set is open in the weak \(C^\infty\) topology. If \(P = (\dim M_v)_{v \in V}\) satisfies the condition \(G_Q\) then \(A^r_\infty \cap C^\infty(G,M)\) is dense in \(C^\infty(G,M)\) for any sufficiently large \(r\) and the complement of \(A^\infty\) has infinite codimension: any smooth family \(u_v\), \(u \in \mathbb{R}^s\) of arbitrary dimension \(s\) can be approximated by a smooth family \(f_u^r\) in \(A^\infty\).

**Remark.** — It seems that if finite \(I_\emptyset\) determinacy holds in general in \(\mathcal{E}(G,P)\) then \(P\) satisfies Condition \(G_Q\) for any \(Q\). So although we state everything for general \(Q\) in this and the next chapters, we will prove them only for the case \(Q = \infty = (\infty)_{u \in V}\), and restrict ourselves to reminding here that the topological stability theorem in Section 0.1 can be proved under the Condition \(G_{P+1}\). For the case of \(Q = P + 1\), there is only one point of the proof that does not go the same in those proofs, that is, the maximal trees may not be finite. However, if we define topologically maximal trees by substituting \(C^0\) triviality for \(C^\infty\) triviality in the definition, then those trees are finite, and the rest of the proof remains valid.

### 2.2. Some properties of critical sets and maximal trees and branches.

To generalize the notions of \(C_e(f)\) and \(D_\emptyset(f)\) of diagrams \(f \in C^\infty(G,M)\), let \(Q = (q_v)_{v \in V}\) be a tuple of positive integers. The set \(C_{eq}(f)\) is defined to be the set of roots of \(Q\) trees of \(f\) in \(M_v\), whose prolongation is not trivial and \(D_{v,Q}(f) = \bigcup_{\beta'(t) = v} f_t(C_{u(t)}Q(f))\). Clearly, \(C_{\infty}(f), D_{\emptyset}(f), \infty = (\infty)_{v \in V}\) coincide with the sets previously defined in Section 0.3.
THEOREM 2.2.1. Let $G = (V, L, A)$ be a convergent diagram with root $v_0$ and $M = (M_v)_{v \in V}$ be a collection of smooth manifolds. Let $K \subseteq M_{v_0}$ be a subset, $0 < r \leq \infty$ an integer and $f = (f_v) \in A_{\infty}^r \cap C_{p,v}^r(G, M)$ (resp. $A_{\infty}^r \cap C_{p,v}^r(G, M)$), $Q = (q_v)_{v \in V}$, $0 < q_v < \infty$ integers. Then there is an open neighbourhood $U$ of $K$ in $M_{v_0}$ such that the following properties are satisfied for any integers $k = 0, 1, \ldots, h(G)$:

1. $C_{v_0}^{-1}(f) \cap f_{v_0}^{-1}(U)$ (resp. $C_{v_0}^{-1}(f) \cap f_{v_0}^{-1}(U)$) is closed in $f_{v_0}^{-1}(U)$ for any $v \in V$, $h(v) = k$.

2. $D_{v_0}^{-1}(f) \cap f_{v_0}^{-1}(U)$ (resp. $D_{v_0}^{-1}(f) \cap f_{v_0}^{-1}(U)$) is closed in $f_{v_0}^{-1}(U)$ for any $v \in V$, $h(v) = k$ and the restriction $f_v : C_v \cap f_v^{-1}(U) \to D_{v} \cap f_v^{-1}(U)$ (resp. $f_v : C_v \cap f_v^{-1}(U) \to D_{v} \cap f_v^{-1}(U)$) is proper and locally uniformly finite-to-one for any $v \in L$, $h(v) = k$.

3. For any $v \in V$, $h(v) = k$, the number of vertices of the maximal branch $f_x^{br}$ on $x$ is locally bounded at any point $x \in f_{v_0}^{-1}(U)$ and if $x_i \in f_{v_0}^{-1}(U)$ is convergent to a point $x \in f_{v_0}^{-1}(U)$ as $i \to \infty$ then $X_x^{br} \to X_x^{br}$ i.e., $X_x^{br}$ is the cluster point set of $\bigcup_{i=1}^{\infty} X_x^{br}$. (The coherence of maximal branches.)

4. For any connected tree (resp. Q tree) $f_x$ of $f$ with root in $f_{v_0}^{-1}(U)$, $h(v) = k$, the $I_0$ codimension $C_{I_0}^r(f_x) < r + 1$.

5. $f \in A_{\infty}^r \cap C_{p,v}^r(G, M)$ (resp. $A_{Q}^r \cap C_{p,v}^r(G, M)$).

Proof. We consider only the case $f \in A_{\infty}^r$. The other case can be proved similarly.

We prove the statements by descending induction on the height $k$ of vertices. The outline is as in the diagram:
First we assume that \( C_v(f) \cap f_{uv_0}(U) \) is closed for any \( v \in V, h(v) = k + 1 \). Since \( f_{\ell} \) are proper, the restrictions \( f_{\ell} : C_{a(\ell)}(f) \cap f_{uv_0}^{-1}(U) \to f_{uv_0}^{-1}(U), h(\alpha(\ell)) = k \) are also proper and the union of the images

\[
D_v(f) \cap f_{uv_0}^{-1}(U) = \bigcup_{\lambda(\ell) - v} f_{\ell}((C_{a(\ell)}(f) \cap f_{uv_0}^{-1}(U))
\]

is closed in \( f_{uv_0}^{-1}(U) \) for any \( v \in V, h(v) = k \).

Next we assume (2), for \( k < i \) and (3), for \( k + 1 \). Then for any point \( x \in f_{uv_0}^{-1}(K) \), \( h(\alpha(\ell)) = k \), the prolongation of the maximal branch \( f_{x(K)}^{-1} \) is finite and \( C_{a(\ell)}(f) < r + 1 \), and by Proposition 0.3.6 there are disjoint open neighbourhoods \( U_y \) of the vertices \( y \in M_v, v \in V \) of \( f_{x(K)}^{-1} \) such that the restrictions \( f_x = (f_{\ell} : U_y \to U_{f_{\ell}(y)}) \), \( y \in X^k_x^{-1} \cap M_v, \ell \in L \), is a good representative of the tree \( f_{x(K)}^{-1} \). By (2), \( k < i \), we may assume that the maximal branch \( f_{x(K)}^{-1} \) on \( x' \in U_x \) is a tree of the restriction \( f_x \) for all \( x \in f_{uv_0}^{-1}(K) \). Then by the properties of good representatives we see that \( C_{a(\ell)}(f) \cap U_x = C_x(f_x) \) and \( f_{\ell} : C_{a(\ell)}(f_x) \cap U_x \to U_{f_{\ell}(x)} \) are uniformly finite-to-one for any \( x \in f_{uv_0}^{-1}(K) \), where \( x \) is regarded as a vertex of the underlying oriented graph of \( f_x \). Since \( f_{\ell} \) are proper, we may assume that, by shrinking the neighbourhood \( U \ni K \),

\[
f_{uv_0}^{-1}(U) = \bigcup_{x \in f_{uv_0}^{-1}(K)} U_x,
\]

from which the statement (2), follows.

We assume the statements (2), for \( k \leq i \):

\[
f_{\ell} : C_{a(\ell)} \cap f_{uv_0}^{-1}(U) \to D_{\ell}(f) \cap f_{uv_0}^{-1}(U)
\]

is proper and locally uniformly finite-to-one for any \( \ell \in L, h(\beta(\ell)) > k \).

Let \( v \in V \) be a vertex of height \( k \). Then \( X^k_x = C_v(f) \cap f_{x^{-1}}(x) \) is a finite set of which the number of elements is locally bounded at any point \( x \in f_{uv}^{-1}(U) \) and \( v' < v \) by the assumption above, and the union \( X^k_x = \bigcup_{v' < v} X^k_x \) gives the maximal branch of \( f \) on \( x \). The coherence of the maximal branches follows from the properness of \( f_{\ell}|C_{a(\ell)} \cap f_{uv_0}^{-1}(U), \lambda(\ell) \leq v \).
(3) \Rightarrow (1). We assume (3). Let \( v \in V \) be a vertex of height \( k \) and let \( x_t \in C_v(f) \cap f_{v_0}^{-1}(U) \) be a sequence convergent to a point \( x \in f_{v_0}^{-1}(U) \). By Proposition 0.3.6, there are open neighbourhoods \( U_y \) of vertices \( y \) of the prolongation \( f_{x_t} \) of the tree \( f_{x_t} \) on \( x \) such that the restriction

\[
f_x = (f_\ell : U_y \to U_{f(0)}, \ y \in X-x \cap M_{x(\ell)}, \ \ell \in L)
\]

is a good representative of \( f_{x_t} \). By the coherence of maximal branches, the prolongation \( f_{x_t} \) is a tree of \( f_x \) hence \( x_t \in C_x(f_x) \) for any sufficiently large \( i \). By the property of good representatives, the critical point set \( C_x(f_x) \) is closed so it follows that \( x \in C_x(f_x) \subset C_v(f) \). Therefore \( C_v(f) \cap f_{v_0}^{-1}(U) \) is closed in \( f_{v_0}^{-1}(U) \).

(3) \Rightarrow (4). Let \( v \in V \) be a vertex of height \( k \). By the same argument as the implication of (2), any maximal branch \( f_{x_t} \) on \( x' \in f_{v_0}^{-1}(U) \) is a tree of a good representative of some branch \( f_{x_t'} \) on \( x \in f_{v_0}^{-1}(K) \).

By the assumption, we have \( C_{f_0}(f_{x_t'}) < r \) and by the property of good representatives, we have \( C_{f_0}(f_{x_t'}) \leq r \).

(4) \Rightarrow (5). Trivial.

This completes the proof of Theorem 2.2.1.

### 2.3. \( C^\infty \) stability and infinitesimal stability.

In this section, we prove a theorem on \( C^\infty \) stability of diagrams as an application of our theory of maximal trees and branches (Theorem 2.3.1). This theorem was proved already by Baas and Dufour [B1, Du], however the part of implication (3) \Rightarrow (2) is not clear in their papers. The reader may appreciate our theory in proving this part.

Let \( f \in C^\infty(G,M) \) be a convergent diagram of smooth mappings. Let \( Q = (q_v)_{v \in V} \) be a tuple of integers \( 0 \leq q_v < \infty \). We say \( f \) is multi (resp. \( Q \)-) infinitesimally stable if any finite (resp. \( Q \)-) tree \( f_x \) of \( f \) is infinitesimally stable.
Our theorem is

**Theorem 2.3.1.** — Let \( G = (V, L, \Lambda) \) be a finite convergent tree with root \( v_0 \) and let \( M = (M_v)_{v \in V} \) be a collection of smooth manifolds and \( f = (f_\ell)_{\ell \in L} \in C^\infty_{pr}(G, M) \). Then the following conditions are equivalent:

1. \( f \) is \( C^\infty \) stable,
2. \( f \) is infinitesimally stable,
3. \( f \) is multi infinitesimally stable,
4. \( f \) is \((P+1)\)-infinitesimally stable, where \( P + 1 = (\dim M_v + 1)_{v \in V} \).

The part (2) \( \Leftrightarrow \) (1) is a generalization of Mather’s theory of adequate homomorphisms [M1], and can be found in the papers [Ba1, Bu, Du1]. The implication (2) \( \Rightarrow \) (3) is obvious.

By Proposition 2.1.1, the conditions (3), (4) are equivalent.

**Proof of the implication** (3) \( \Rightarrow \) (2). — We fix an element \( \nu = \bigoplus_{\ell \in L} \nu_\ell \in \Theta(f) = \bigoplus_{\ell \in L} \Theta(f_\ell) \). In the remainder of this section, we construct a \( \omega = \bigoplus_{v \in V} \omega_v \in \Theta(M) = \bigoplus_{v \in V} \Theta(M_v) \), such that \( T(f)(\omega) = \nu \) by induction on the height of vertices \( v \in V \).

Let \( C^h_v(f) \subset M_v \) denote the set of points \( x \in M_v \) whose maximal trees \( f_{\mathcal{X}_x} \) have their roots in \( M_{v'} \), \( h(v') \leq h \), for any \( v \in V \) and integer \( h \geq h(v) \). It is easy to see \( C^h_v(f) = \bigcap_{h(v') > h} f^\leftarrow_{v'}(C^h_{v'}(f)) \).

By Theorem 2.2.1, the critical point sets \( C^h_v(f) \), \( h(v) > h \) are closed and the restrictions \( f_\ell : C^h_{a(\ell)}(f) \rightarrow C^h_{b(\ell)}(f) \), \( h(\beta(\ell)) \leq h \) are proper and locally uniformly finite-to-one.

Let \( 0 < h < h(G) \) be an integer. We assume that for each \( v \in V \), there is a vector field \( \omega_v^{h-1} \) defined on an open neighbourhood \( U_v^{h-1} \) of \( C_v^{h-1}(f) \) in \( M_v(U_v^{h-1} = M_v \) for \( v \in V \), \( h(v) \leq h - 1 \) \) such that \( f_\ell(U_v^{h-1}) \subset U_{b(\ell)}^{h-1} \), \( \ell \in L \) and the restriction \( f|U_v^{h-1} \) of \( f \) to the open neighbourhoods \( U_v^{h-1}, v \in V \) satisfies

\[
F(f|U_v^{h-1}) \left( \bigoplus_{\ell \in L} \omega_v^{h-1} \right) = \bigoplus_{\ell \in L} \omega_\ell|U_v^{h-1}.
\]
We then extend $\alpha_v^{h^{-1}}$ to vector fields $\alpha_v^h$ defined on open neighbourhoods $U_v^h$ respectively for $v \in V$ so that the restriction $f|U_v^h$ satisfies the equality above. The final step $h = h(G)$ of the extension of vector fields completes the construction of a vector field $\alpha = \bigoplus_{v \in V} \alpha_v = \bigoplus_{v \in V} \alpha_v^{h(G)}$ with the required property $T(f)(\alpha) = \nu$.

Let $v \in V$ be a vertex of height $h$ and $x \in M_v - C_v^{h^{-1}}(f)$. Then the prolongation $f_{X^v_{x^v}}$ of the maximal tree $f_{X^v_{x}}$ on $x$ is trivial:

$$T(f_{X^v_{x^v}}) = (\theta(M_{X^v_{x^v}})) = \theta(f_{X^v_{x}}),$$

where $\theta(M_{X^v_{x^v}}) = \bigoplus_{x' \in X^v_{x^v} \cap M_0} \theta(M_0)_{x'}$. So there is a vector field $\alpha_x^h = \bigoplus_{x' \in M_0} \alpha_{x'} \in \Theta(M_{X^v_{x^v}})$ such that

$$T(f_{X^v_{x^v}})(\alpha_x^h \bigoplus \alpha_v^{h^{-1}}_{(\nu')_{f(x)}}) = \nu_{X^v_{x^v}} = \bigoplus_{x' \in X^v_{x^v} \cap M_0} \alpha_{x'} \nu',$$

Let $\alpha_x^{x'}$ be representatives of $\alpha_x^{x'}$ defined on disjoint open neighbourhoods $U_{x'}$ of vertices $x' \in X^v_{x^v} \cap M_0$, $v' \in V$ in $M_v - C_v^{h^{-1}}(f)$ such that

1. $f_{x'}(U_{x'}) \subset U_{f_{x'}}(x')$, for $x' \in X^v_{x^v} \cap M_0$, $\ell \in L$,

2. $C_{x(\ell)}^h(f) \cap f_{x(\ell)'}^{-1}(U_{x}) \subset \bigcap_{x' \in X^v_{x^v} \cap M_0(\ell)} U_{x'}$ for $\ell \in L$,

and

3. $T(f|U_x^{x})\left(\bigoplus_{x' \in X^v_{x^v} \cap M_0} \alpha_x^{x'} \bigoplus \alpha_v^{h^{-1}}_{(\nu')_{f(x)}}\right) = \nu|U_x^{x}$,

where $f|U_x^{x}$, $\nu|U_x^{x}$ denote respectively the sets of restrictions $f_{x'}|U_{x'}$, $\nu_{x'}|U_{x'}$, $x' \in X^v_{x^v} \cap M_0(\ell)$, $\ell \in L$ (the existence of such representatives is proved by Theorem 2.2.1). Let $x_i \in M_v - C_v^{h^{-1}}(f)$, $i = 1, \ldots$ be a countable family of points such that $C_v^h(f) - U_v^{-1} \subset \bigcup_i U_{x_i}^{x_i}$. Then by the property (ii), we have

$$C_v^h f - U_v^{-1} \subset \bigcup_{x' \in X^v_{x^v}} U_{x_i}^{x_i},$$

for all $\nu' \leq \nu$. 

By shrinking the open neighbourhoods $U^x_{x_i}$, $x' \in X^x_{x_i}$, $i = 1, 2, \ldots$, we may assume that $\{U^x_{x_i}\}$ is locally finite. Then we can take a partition of unity $h_i: U^x_{x_i} \to \mathbb{R}$, $h_v: U^h_{v-1}(f) \to \mathbb{R}$ subordinate to the covering $\{U^x_{x_i}, i = 1, 2, \ldots, U^h_{v-1}(f)\}$ of $M_v$. Now let

$$U^h_v = U^h_{v-1} \cup \bigcup_{x' \in X^x_{x_i}} U^x_{x'}, \quad v' \leq v$$

and define the vector field $\omega^h_v$ on $U^h_v$ by $\omega^h_v = \omega^h_{v-1}$ for $v \in V$, $h(v) \leq h - 1$, and

$$\omega^h_v = f^*_{\omega} h_v \cdot \omega^h_{v-1} + \sum_{x' \in X^x_{x_i}} f^*_{\omega} h_v \cdot \omega^x_{x'}$$

for $v' \leq v$. Then $U^h_v$ and $\omega^h_v$ have the required properties.

This completes the construction of the vector field $\omega$ hence the proof of the implication $(3) \Rightarrow (2)$.

2.4. Proof of Theorem 2.1.2.

We prove the openness of $A^\infty_{\infty}$. The openness of the other sets follows the same way.

First we prove that $A^\infty_{\infty} \cap C_{pr}(G,M)$ is a neighbourhood of $A^\infty_{\infty} \cap C_{pr}(G,M)$ in the weak $C^\infty$ topology if $K$ is compact. Since the weak $C^\infty$ topology has countable open basis it suffices to prove that for any sequence $f_i \in C^\infty_{pr}(G,M)$ convergent to an $f \in A^\infty_{\infty} \cap C_{pr}(G,M)$, $f_i \in A^\infty_{\infty}$ for any sufficiently large $i$. Then $f_i$ can be imbedded in a smooth one parameter family $f_i \in C^\infty_{pr}(G,M)$ so that $f_{i,t} = f_i$ with a sequence $t_i \in \mathbb{R}$ convergent to 0 (see the book [Gi], p. 146). Let $F \in C^\infty(G,M \times \mathbb{R})$, $F_t : M_{u_{\beta}(\ell)} \times \mathbb{R} \to M_{\beta_{\ell}(\ell)} \times \mathbb{R}$, $F_t(x,t) = (f_t(x),t)$. In general for an unfolding $H \in \delta^\infty_{\ell}(G,P + s)$ of $h \in \delta^\infty_{\ell}(G,P)$, we see $C_{i_0}(h) - s \leq C_{i_0}(H) \leq C_{i_0}(h)$ by definition of the $I_0$ codimension. So we see $F \in A^\infty_{\infty} \cap C^\infty_{pr}(G,M \times \mathbb{R})$, $K \subset M_{v_0} \times 0$ and then Theorem 2.2.1 applies to $F$ and shows that there is an open neighbourhood $U$ of $K \times 0$ in $M_{v_0} \times \mathbb{R}$, such that $F \in A^\infty_{\infty} \cap C^\infty_{pr}(G,M \times \mathbb{R})$. Since $K$ is compact, $K \times t_i \subset U$ holds for any sufficiently large $i$, and for such $i$ we see that $F \in A^\infty_{\infty} \times t_i$, from
which we have \( f_{t_j} \in A_{\infty,K}^{r+1} \subset A_{\infty,K}^\infty \). (By a more detailed argument, we can prove that \( f_{t_j} \in A_{\infty,K}^{r+1} \).) This argument shows that \( A_{\infty,K}^\infty \cap C_{pr}^\infty(G,M) \) is open.

Secondly, we prove the openness of \( A_{\infty,M_0}^\infty \cap C_{pr}^\infty(G,M) \) in the Whitney topology. Let \( K_i, i = 1, 2, \ldots \) be a locally finite covering of \( M_0 \) by compact subsets and \( f \in A_r^\infty \cap C_{pr}^\infty(G,M), r < \infty \). Naturally we then expect that the countable intersection \( \bigcap_i A_{\infty,K_i}^{r+1} \cap C_{pr}^\infty(G,M) = A_{\infty,K_i}^{r+1} \cap C_{pr}^\infty(G,M) \) of the open neighbourhood \( A_{\infty,K_i}^{r+1} \cap C_{pr}^\infty(G,M) \) of \( f \) is again an open neighbourhood in the Whitney topology. This argument has already appeared in the book [Gi] to prove the topological stability theorem for single-mappings \( f \in C_{pr}^\infty(N,P) \). Unfortunately we cannot find a satisfactory reference for this argument in the generality needed here. So we present a sketch of a proof to cover this point.

Since \( A_{\infty,K_i}^{r+1} \cap C_{pr}^\infty(G,M) \) is open in the weak \( C^\infty \) topology there exist a positive integer \( r_i \) and an open neighbourhood \( U_i \subset J^{r_i}(G,M) \) of \( J^r f(M), M = \prod_{\ell \in L} M_{\alpha(\ell)} \) with the property: if \( J^{r_i+1} g(M) \subset U_i \) then \( g \in A_{\infty,K_i}^{r_i+1} \cap C_{pr}^\infty(G,M) \) (for the definition of the weak \( C^\infty \) and Whitney topologies, see [M2]). We claim that \( r_i \) can be chosen independently of \( i \). Then the openness of the intersection \( \bigcap_i A_{\infty,K_i}^{r+1} \cap C_{pr}^\infty(G,M) \) is easily seen.

By Proposition 2.1.1, \( A_{\infty,K_i}^{r+1} = A_{P^{r+1}+1}^{r+1} \) for \( P = (\dim M_0)_{\alpha \in \nu} \). By Theorem 0.3.1, there is a positive integer \( e = e(G,P+r+2) + 1 < \infty \) with the following property: let \( f_X \) be a connected \((P+r+2)\)-tree of a diagram \( f \in C^\infty(G,M) \) of \( I_0 \) codimension \( C_{I_0}(f_X) \leq r + 1 \). If \( g \in C^\infty(G,M) \), and \( g \) has the same \( e \)-jet as \( f \) at \( X = \cup X_\ell \) then \( C_{I_0}(g_X) \leq r + 1 \).

Now we use the following lemma which is proved in Appendix 1.

**Lemma 2.4.1.** — Let \( f : N \rightarrow P \) be a smooth mapping of manifolds \( N, P \) and \( U \) an open neighbourhood of \( J^s f(N) \) in \( J^s(N,P) \) and let \( 0 < s, q < \infty \) be integers. Then there is an open neighbourhood \( U' \subset J^{q(s+1)}(N,P) \) of \( J^{q(s+1)} f(N) \) with the following property: for any \( g \in C^\infty(N,P) \) with \( J^{q(s+1)} g(N) \subset U' \) and any \( q \) distinct points \( x_1, \ldots, x_q \in N \), there is a \( g' \) such that \( J^k g'(N) \subset U \) and \( J^k g'(x_i) = J^k g(x_i) \), \( i = 1, \ldots, q \).
We apply the lemma to our problem in the setting: \( U = U_i, k = r, s = e = e(G, P + r + 2) + 1 \) and \( q = \prod_{\ell \in L} \dim M_{\beta(\ell)} + r + 2 \). Then we get an open neighbourhood \( U'_i \subset J^{q(s+1)}(G, M) \) of \( J^{q(s+1)}f(M) \) with the following property: for any \( g \in C^\infty(G, M) \) with \( J^{q(s+1)}g(M) \subset U'_i \) and any \((P + r + 2)\)-tree \( g_x \) of \( g \), there is a \( g' \in C^\infty(G, M) \) such that

(i) \( J^e g'(M) \subset U_i \)

(ii) \( g'_\ell \) has the same \( e \)-jet as \( g_\ell \) at \( X_\ell = X \cap M_{\alpha(\ell)} \), for all \( \ell \in L \).

From (i) and the property of \( U_i \), it follows \( g' \in A_{r+1}^r \subset C^\infty(G, M) \) and in particular \( C_{t_0}(g'_x) < r + 2 \), and from (ii) and the property of the number \( e \), it follows that \( g \in A_{r+1}^{r+1} = A_{r}^{r+1} \).

This completes the proof of our claim.

Now to complete the proof of Theorem 2.1.2, we prove that if \( P = (\dim M_{\alpha})_{\ell \in V} \) satisfies Condition \( G_Q \), then

(1) \( A_Q' \) is dense in \( C^\infty(G, M) \) with the Whitney topology for any sufficiently large \( r \),

(2) the complement of \( A_Q' \) has infinite codimension.

Let \( G' = (V', L', \Lambda') \) be a finite union of convergent trees and \((i_{\Lambda'}, i_{L'}) : G' \to G(i_{\Lambda'} : V' \to V, i_{L'} : L \to L)\) be a morphism and assume \( G' \) is a union of \( i_{\Lambda'} \) trees: these are strictly less than \( q_{i_{\Lambda'}(v')} + 1 \) edges \( \ell' \in L' \) with \( \beta'(\ell') = v' \) at each vertex \( v' \in V' \). The set \( V' \) is naturally indexed by the set \( V \) as \( v' \in V' \to i_{\Lambda'}(v') \in V \). We denote by \( \Gamma_Q \) the set of these triplets \((G', i_{\Lambda'}, i_{L'})\) as above.

Let \( q = \prod_{\ell \in L} q_{\beta(\ell)}, q^{J^k(G, M)} = J^k(G, M)^q \) and

\[
\pi : q^{J^k(G, M)} \to \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q
\]

the natural projection. Let \( \Delta \subset \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q \) denote the set of \((x_{\alpha(\ell)}, y_{\beta(\ell)})_{\ell \in L}, i = 1, \ldots, q \) such that \( x_{\alpha(\ell)} = x_{\alpha(\ell)} \) for an \( \ell \in L \), \( i \neq j \). A point \((X, Y) : X = (x_{\alpha(\ell)}), Y = (y_{\beta(\ell)})\) is naturally regarded as an oriented graph \( G_{XY} = (V_{XY}, L_{XY}) : V_{XY} = \{x_{\alpha(\ell)}, y_{\alpha(\ell)}\}, L_{XY} = \{\ell : x_{\alpha(\ell)} \to y_{\beta(\ell)}\}\). The set \( V_{XY} \) is naturally indexed by \( V \): to \( x_{\alpha(\ell)}, y_{\beta(\ell)} \in V_{XY} \) it assigns the vertices \( \alpha(\ell) \), \( \beta(\ell) \in V \), respectively. For each triple \((G', i_{\Lambda'}, i_{L'}) \in \Gamma_Q\), we denote by \( \Delta_{G'} \) the set of points \((X, Y) \subset \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q - \Delta \) for which the
associated graph $G_{XY}$ is equivalent to $G'$ as oriented graphs indexed by
the set $V$. It is easy to see that the $\Delta_{G'}$, $G' \in \Gamma_q$ are smooth submanifolds
and Whitney $B$ regular over each other.

Let $J^k f : \prod_{\ell \in L} M^q_{\Delta(\ell)} \to J^k(G, M)$ be the multi $k$ jet section of
$f = (f_\ell) \in C^\infty(G, M)$. Then we see $\pi \circ J^k f(x) \in \Delta$, if and only if the
tree $f_x$ (regarded as an oriented graph with index $\nu \in V$ for each
vertex $x \in (X \cup f(X)) \cap M_q$) is equivalent to $G'$, where

$$X = \bigcup_{\ell \in L} X_\ell, \quad X_\ell \in M_{\Delta(\ell)}, \quad f(x) = \bigcup_{\ell \in L} f_\ell(X_\ell).$$

We regard the fibre $J^k(G, P)$ of the projection $\pi$ on $(X, Y) \in \Delta_{G'}$ as the
jet space $J^k(G', e^*_Y P)$.

By theorem 0.3.4, the set $\Sigma \subset \delta(G', e^*_Y P)$ of $f$ with finite codimension
is a pro algebraic set defined by algebraic subsets

$$\Sigma^{(r)} \subset J^{\epsilon(r)}(G', e^*_Y P)$$

$(\pi^{(r)}(f) \notin \Sigma^{(r)} \Leftrightarrow C_{t_0}(f) \leq r)$, and by the Condition $G_q$, codim $\Sigma^{(r)} \to \infty$ as $r \to \infty$. Choose an $r$ so that codim $\Sigma^{(r)} > q \cdot \sum_{\ell \in L} P_{\Delta(\ell)}$ for all
$G' \in \Gamma_q$.

Let $S_{G'}$ be a stratification of $\Sigma^{(r)}$ invariant under diffeomorphisms
$J^{\epsilon(r)}(G', e^*_Y P)$ induces from coordinate transformations of the germs
$R^p_{\nu v}$ associated with vertices $v' \in V'$, and let $S'_{G'} \subset \pi^{-1}(\Delta_{G'})$ be
the stratified set with fibre $S_{G'}$ over each $(X, Y) \in \Delta_{G'}$ and finally let $S$
be the union of $S_{G'}$ for $G' \in \Gamma_q$. Then the set $S$ possesses the following
property: if $J^{\epsilon(r)} f$ is transversal to $S$ at $X \in \prod_{\ell \in L} M^q_{\Delta(\ell)}$ then $J^{\epsilon(r)} f(x) \notin S$
(for codim $S > \dim \prod_{\ell \in L} M^q_{\Delta(\ell)}$) hence $C_{t_0}(f_x) \leq r$. Conversely any
connected $Q$-tree of $f_{x'}$ of $f$ is realised as a connected component of
diagrams $f_X$ defined above for $X \in \prod_{\ell \in L} M^q_{\Delta(\ell)}$. From this property of $S$
and the transversality theorem (Theorem 0.3.5), the density of $A_q$ in
$C^\infty(G, M)$ follows.

The infiniteness of codimension of $A_q$ follows from the same
argument using transversality and unboundedness of the codimension
of $\Sigma^{(r)}$ as $r \to \infty$. 
CHAPTER 3

PROOF OF THE TOPOLOGICAL STABILITY THEOREM

3.1. Topological multi transversality, topologically maximal trees and branches.

Let $f \in C^\infty(G, M)$ be a convergent diagram of smooth mappings, $Q = (q_v)_{v \in V}$ a tuple of integers $0 \leq q_v < \infty$, $S$ a Whitney regular stratification of $M_{r_0}$ and $K \subseteq M_{r_0}$ a subset. We say $f$ is topologically $Q$-transversal relative to $S$ on $K$ if: (i) any connected $Q$-tree $f_X$ of $f$ is topologically transversal and (ii) if $f_X$ has root $x_0 \in M_{q_0}$ then the canonical CVS $C_{x_0}(f_X)$ of $f_X$ at $x_0$ is transversal to $S$.

For the case $q_v = \infty$, $v \in V$, we say simply $f$ is topologically transversal relative to $S$ (for the definition of topological transversality of trees $f_X$, see Proposition 1.3.3).

Let $F = (F_\ell) \in C^\infty(G, M \times \mathbb{R}^r)$, $M \times \mathbb{R}^r = (M_{q_\ell})_{v \in V}$, $F_\ell(x, t) = (f_\ell(x), t), x \in M_{q_\ell}$, $t \in \mathbb{R}^r$, $f_\ell = f_\ell$ be an unfolding of $f$ such that the trees $F_X, X \subseteq \bigcup_{\ell} M_{q_\ell} \times 0$ are infinitesimally stable unfoldings of $f_X$ and let $\mathcal{C}(F_X) = (\mathcal{C}_x(F_X))_{x \in X \cup F(X)}$ be the canonical critical value stratification of $F_X$ constructed in Chapter 1.1. By Proposition 1.3.3, the tree $f_X$ is topologically transversal if and only if the inclusions $\iota_v : M_v \hookrightarrow M_v \times \mathbb{R}^r, v \in V$ are transversal to $\mathcal{C}_x(F_X)$ at the vertices $x \in X \cup F(X)$.

Let $\chi_x(F_X)$ denote the stratum of $\mathcal{C}_x(F_X)$ containing the vertex $x \in X \cup F(X)$. Let $x_1, \ldots, x_q \in X$ be the vertices of $f_X$ such that $f(x_i) = x$, and let $f_{X_{x_i}}$, $F_{X_{x_i}}$ be the branches of $f_X$, $F_X$ on $x_i$, and $f_{X_{x_i}}$, $F_{X_{x_i}}$ their prolongations (with root $x$). By Proposition 1.2.2, the canonical CVS's $\mathcal{C}_x(F_{X_{x_i}})$ meet in general position at $x$ and $\mathcal{C}_x(F_X) = \bigcap_{i=1, \ldots, q} \mathcal{C}_x(F_{X_{x_i}})$, and in particular $\chi_x(F_X) = \bigcap_{i=1, \ldots, q} \chi_x(F_{X_{x_i}})$. Therefore if $\iota_v : M_v \hookrightarrow M_v \times \mathbb{R}^r$ is transversal to $\mathcal{C}_x(F_X)$ at $x \in M_v$, the number of indices $i$ for which $\mathcal{C}_x(F_{X_{x_i}})$ is topologically non-trivial is at most $\dim M_v$.

From the above fact and the same argument as in the proof of Proposition 2.1.1, we have
**Proposition 3.1.1.** — A convergent diagram \( f \in A^\infty \) is topologically transversal if and only if \( f \) is topologically \((P+1)\)-transversal. \( P + 1 = (\dim M_v + 1)_{v \in V} \). Any topologically maximal tree of \( f \) is a \( P \) tree.

Using the same idea as Theorem 2.2.1, we prove

**Theorem 3.1.2.** — Let \( f \in C_\infty^\infty(G, M) \) be a convergent diagram, \( P + 1 = (\dim M_v + 1)_{v \in V} \), \( K \subset M_{v_0} \) a closed subset and \( S \) a Whitney regular stratification of \( M_{v_0} \). If \( f \in T_{\infty KS} \subset A_{\infty K} \), then there is an open neighbourhood \( U \) of \( K \) such that \( f \in T_{\infty US} \subset A_{\infty U} \) and the restriction \( f_U = (f_\ell: f_\ell^{-1}w_0(U) \rightarrow f_\ell^{-1}w_0(U))_{\ell \in L} \) admits a critical value stratification with the following properties: \( f_\ell: \Sigma_{\alpha(\ell)}(f_U) \rightarrow f_\ell^{-1}w_0(U) \) are proper, \( \bigcup_{\beta(\ell')=v} f_\ell^{-1}(x) \cap \Sigma_{\alpha(\ell')(f_U)}(f_\ell') \) consists of at most \( \dim M_v \) points for each \( x \in f_\ell^{-1}w_0(U) \), and \( \mathcal{C}_v(f_U) \), \( S \) meet in general position in \( U \). The topologically maximal branch \( f_{\top X_{br}^v} \) on \( x \in f_\ell^{-1}w_0(U) \) is given by the set

\[
\top X_{x}^{br} = \bigcup_{\alpha(\ell') < \alpha(\ell) < \alpha(\ell') < v} \Sigma_{\alpha(\ell)}(f_U)(f_\ell') \cap f_\ell^{-1}w_0(x)
\]

and is topologically transversal (relative to \( S \) if \( v = v_0 \)) and the germ of \( \mathcal{C}_v(f_U) \) at \( x \), \( \mathcal{C}_v(f_U)x \) coincides with the canonical CVS \( \mathcal{C}_x(f_{\top X_{br}^v}) \) of the branch \( f_{\top X_{br}^v} \) given in Corollary 1.3.4.

**Proof.** — By Theorem 2.1.2, we may assume \( f \in A_{\infty U} \). We construct the CVS with the properties in the theorem by descending induction on the height of vertices \( v \in V \). So we assume that \( f_U \) admits a CVS \( \mathcal{C}_{v'}(f_U) \) for \( v' < v \) with the desired properties for the restriction of \( f_U \) to the branches \( G_{(\ell')} \) on \( v' \), \( \beta(v') = v \), and then we construct \( \mathcal{C}_v(f_U) \).

By definition the topologically maximal branch \( f_{\top X_{br}^v} \) on \( x \in f_\ell^{-1}w_0(U) \) is a union of prolongations of topologically maximal branches on some points \( x' \in f_\ell^{-1}(x) \), \( \beta(\ell) = v \). Let \( F_{\top X_{br}^v} \) be an infinitesimally stable unfolding of the prolongation \( f_{\top X_{br}^v} \) of \( \dim r \). Since \( f_{\top X_{br}^v} \) is topologically transversal by the induction hypothesis, the inclusion \( \epsilon_{x'}: M_{\alpha(\ell)} \rightarrow M_{\alpha(\ell')} \times \mathbb{R}^r \) is transversal to the canonical CVS \( \mathcal{C}_{x'}(F_{\top X_{br}^v}) \) (Corollary 1.3.4). Hence

\[
\Sigma_{\alpha(\ell)}(f_{\top X_{br}^v})(f_\ell') = \epsilon_{x'}^{-1}(\Sigma_{\alpha(\ell)(F_{\top X_{br}^v})(F_{x'})))
\]
and $f_{\text{top}X^t_x}$ is topologically trivial if and only if $\Sigma_{X'}(F_{\text{top}X^t_x})(F_x') = \emptyset$ if and only if $\Sigma_{X'}(F_{\text{top}X^t_x})(f_x') = \emptyset$. So we have

$$\top X^{\text{br}}_x = \bigcup_{a(\ell) < 0} \bigcap_{a(\ell') < a(\ell)} \Sigma_{X'}(F_{\text{top}X^t_x})(f_x') \cap f_{\text{top}X^t_x}^{-1}(x).$$

Since the $f_x$ are proper and $\Sigma_{a(\ell)}(f_x) = f_{\text{top}X^t_x}^{-1}(U)$ are closed, $f_x : \Sigma_{a(\ell)}(f_x) \to f_{\text{top}X^t_x}^{-1}(U)$ are also proper and in particular the topologically maximal branches $f_{\text{top}X^t_x}$, $x \in f_{\text{top}X^t_x}^{-1}(U)$ are coherent in the sense of (3) in Theorem 2.2.1.

Let $x \in f_{\text{top}X^t_x}^{-1}(K)$. Then the branch $f_{\text{top}X^t_x}$ is topologically transversal. Let $F^x = (F^x_\ell)_{\ell \in L}$, $F^x_\ell : M_{\alpha(\ell)} \times \mathbb{R}^r \to M_{\beta(\ell)} \times \mathbb{R}^r$, $F^x_\ell(y, u) = (f^x_{\ell u}(y), u)$ for $y \in M_{\alpha(\ell)}$, $u \in \mathbb{R}^r$, $f^x_{\ell 0} = f_x$ be a smooth unfolding of $f$ of dim $r$ such that the tree $F^x_{\text{top}X^t_x}$ on $x$ is infinitesimally stable and its restriction $\hat{F}^x_{\text{top}X^t_x} = (\hat{F}^x_\ell : U_{x'} \to U_{f(x')})$, $x' \in \top X^t_x$ to some open neighbourhoods $U_{x'}$ of the vertices $x' \in \top X^t_x \cap M_{x'}$ in $M_{x'} \times \mathbb{R}^r$, $v' \leq v$ is a good representative of $F^x_{\text{top}X^t_x}$ with the properties in Corollary 1.3.4.

Using the same notation as in Corollary 1.3.4, the transversal intersections $\mathcal{E}_x(\hat{F}^x_{\text{top}X^t_x}) \cap M_v \times 0$, $x' \in \top X^t_x \cap M_{x'}$, $v' \leq v$ give the canonical CVS denoted $\mathcal{E}_x(\hat{F}^x)$ of the restriction of $f$ to open neighbourhoods $U_{x'} \cap M_{x'} \times 0$ of $x' \in M_{x'}$. Since $f_{\text{top}X^t_x}$, $x \in f_{\text{top}X^t_x}^{-1}(U)$ are coherent, we may assume, by shrinking $U_{x'}$ that $f_{\text{top}X^t_x}$ is a tree of $\hat{F}^x$ if $x'' \in U_{x} \cap M_{x} \times 0$. Then by Corollary 1.3.4, the germ of $\mathcal{E}_x(\hat{F}^x)$ at $x''$ coincides with the canonical CVS $\mathcal{E}_x(\hat{F}^x_{\text{top}X^t_x})$ of the maximal tree $f$ on $x''$. Therefore the $\mathcal{E}_x(\hat{F}^x)$ $x \in f_{\text{top}X^t_x}^{-1}(K)$ glue up to give a stratification of $\bigcup_{x \in f_{\text{top}X^t_x}^{-1}(K)} U_x \cap M_v \times 0$.

Finally, by shrinking $U$ so that $f_{\text{top}X^t_x}^{-1}(U) \subset \bigcup_{x \in f_{\text{top}X^t_x}^{-1}(K)} U_x \cap M_v \times 0$, we complete the induction step.

In the same way as the implication of Theorem 2.2.1 to Theorem 2.1.2, the above theorem (Theorem 3.1.2) for topological transversality implies the following.
THEOREM 3.1.3. — The set $T_{\infty,K} \cap C_{pr}^\infty(G,M) \subset A_{\infty,K} \cap C_{pr}^\infty(G,M)$ of convergent diagrams $f$ topologically multi-transversal on $K$ relative to a Whitney regular stratification $S$ of $M_{v_0}$ is an open subset in the Whitney topology if $K \subset M_{v_0}$ is closed.

3.2. Proof of the theorem.

First we prove the following theorem.

THEOREM 3.2.1. — Let $f \in A_{\infty} \cap C_{pr}^\infty(G,M)$ be a convergent diagram of proper smooth mappings and let $S$ be a Whitney regular stratification of $M_{v_0}$ by relatively compact submanifolds. If $f$ is topologically $P + 1$ (hence, multi)-transversal on $M_{v_0}$ relative to $S$, then $f$ is topologically stable.

Proof. — By Theorem 3.1.3, there is an open neighbourhood $U$ of $f$ in $C_{pr}^\infty(G,M)$ such that any $g \in U$ is topologically multi-transversal relative to $S$ and joined to $f$ by a smooth path $f_t, t \in \mathbb{R}$ with $f_0 = f, f_1 = g$. Define the unfolding $F = (F_t), F_t: M_{\alpha(t)} \times \mathbb{R} \to M_{\beta(t)} \times \mathbb{R}$ by $F_t(x,t) = (f_{\alpha(t)}(x),t), x \in M_{\alpha(t)}, t \in \mathbb{R}$. Let $f_{top,x}^*, F_{top,x}^*$ be the topologically maximal branch of $f,F$ on $x \in M_{v} \subset M_{v} \times \mathbb{R}$, and let $F_{top,x}^{br}$ be an infinitesimally stable unfolding of $F_{top,x}^*$. Then the canonical CVS of $f_{top,x}^*, F_{top,x}^*$ are given by the transversal intersections of the canonical CVS $\mathcal{C}_x(F_{top,x}^*)$, $x' \in top X^x_{br}$ with $M_{v}, M_{v} \times \mathbb{R}$ respectively as described in Corollary 1.3.4. Therefore $F_{top,x}^{br}$ is also topologically transversal relative to $S \times \mathbb{R}$ and the inclusion $\iota_v: M_{v} \to M_{v} \times \mathbb{R}$ are transversal to the canonical CVS $\mathcal{C}_x(F_{top,x}^*)$ at each vertex $x' \in top X^x_{br} \cap M_{v}, v \in V$ and $\mathcal{C}_x(f_{top,x}^{br}) = \iota_v^{-1}(F_{top,x}^{br}).$ By Theorem 3.1.2, $F$ admits CVS $\mathcal{C}_v(F)$ of $M_{v} \times \mathbb{R}$ such that $\mathcal{C}_{v_0}(F)$ is transversal to $S \times \mathbb{R}$, and the germ $\mathcal{C}_v(F)_{(x,t)}$ coincides with $\mathcal{C}_{(x,t)}(F_{top,x_{(x,t)}})$ for any $(x,t) \in M_{v} \times \mathbb{R}, v \in V$. The transversal of the inclusions $\iota_{(x,t)}: M_{v} \to M_{v} \times t \subset M_{v} \times \mathbb{R}$ to the CVS's shows that the second projections $M_{v} \times \mathbb{R} \to \mathbb{R}$ are stratified submersions, i.e., submersive restricted to each stratum of $\mathcal{C}_v(F)$. Let $\mathcal{S}_v(F) = \bigcap_{v < v'} F_{v_0}^{-1}(\mathcal{C}_v(F)) \cap F_{v_0}^{-1}(S), v \in V$ be the canonical stratification associated to the CVS's $\mathcal{C}_v(F), v \in V$. Clearly the second projection $M_{v} \times \mathbb{R} \to \mathbb{R}$ are still stratified submersions.
Since \( f_t : M_{\mu(t)} \to M_{\beta(t)} \), \( t \in \mathbb{R} \), \( t \in L \) are all proper and the strata of \( S \) are relatively compact, the strata of \( S_v(F) \), \( v \in V \) are also relatively compact. Now we apply Thom's second isotopy lemma to the prolongation \((F, \text{Pr})\) with the second projection \( \text{Pr} : M_{q_0} \times \mathbb{R} \to \mathbb{R} \). Then \((F, \text{Pr})\) is topologically locally trivial and in particular the sectional mappings \( f = f_0 \), \( g = f_1 \) are topologically equivalent. This completes the proof of Theorem 3.2.1.

Now we prove the main theorem.

**Theorem 3.2.2.** - The set \( T_{\infty} S \cap C_{\text{pr}}^{\infty}(G, M) \) is open dense in \( A_{\infty} \cap C_{\text{pr}}^{\infty}(G, M) \) with the Witney topology.

**Proof.** - The openness is given by Theorem 3.1.3, and by Proposition 3.1.1, \( T_{\infty} S = T_{P+1} S \) for \( P + 1 = (\dim M_s + 1)_{s \in V} \). It remains to show \( T_{P+1} S \cap C_{\text{pr}}^{\infty}(G, M) \) is dense. We use the same relation as in the proof of the density of \( A_{P+1} \cap C_{\text{pr}}^{\infty}(G, M) \) in Theorem 2.1.2.

Let \( q = \prod_{\ell \in L} q_{\ell} \) and let \( \pi : q^{k}(G, M) \to \prod_{\ell \in L} (M_{\mu(\ell)} \times M_{\beta(\ell)})^{q} \) be the natural projection. Let \((G', \epsilon_V, \epsilon_L) \in \Gamma_q \) be a morphism of an oriented graph \( G' \) to \( G \) and let \( \Delta_{G'} \in \prod_{\ell \in L} (M_{\mu(\ell)} \times M_{\beta(\ell)})^{q} - \Delta \) be the set of points \((x_{\ell_1}, y_{\ell_1}), x_{\ell_1}, i = 1, \ldots, q \) all distinct, for which the associated graph \( G_{XY} \) is equivalent to \( G' \) as an oriented graph indexed by the set \( V \).

We regard the fibre of \( \pi \) over \( \Delta_{G'} \) as the \( k \) jet space of diagrams in \( \mathcal{E}(G', \epsilon_V, P) \). Let \( \Sigma^{(r)} \in J^{(r)}(G', \epsilon_V, P) \) be the set in Theorem 0.3.4, which defines the pro-algebraic set \( \Sigma \in \mathcal{E}(G', \epsilon_V, P) \) of non finitely \( I_q \) determined diagrams, and let \( S^{(r)}(G', \epsilon_V, P) \) be the stratification of the complement of \( \Sigma^{(r)} \) defined in Section 1.3. Since these sets are invariant under coordinate transformations of spaces, these sets and stratifications define a locally trivial partition of the fibre bundle \( \pi^{-1}(\Delta_{G'}) \to \Delta_{G'} \), denoted by \( \Sigma^{(r)}_{G'}, S^{(r)}_{G'} \), respectively. The image of the projection to the roots \( \pi' : \Delta_{G'} \to M^{q}_{q_0} \) is the complement of the diagonal set of \( M^{q}_{q_0} \), where \( q' \) is the number of connected components of \( G' \). Let \( S^{(r)}_{G'S} \) denote the refinement \( S^{(r)}_{G'} \cap (\pi' \circ \pi)^{-1}(S') \). By the transversality theorem (Theorem 0.3.5), the set \( \mathcal{F}^{(r)}_{G'} \) of diagrams \( f \in C^{\infty}(G, M) \) for which the \( e(r) \)-jet section \( J^{(r)}f : \prod_{\ell \in L} M_{\mu(\ell)} \to J^{(r)}(G, M) \) is transversal to \( \Sigma^{(r)}_{G'} \), \( S^{(r)}_{G'S} \) is a countable intersection of open dense subsets. So the countable intersection \( \mathcal{F} \) of those \( \mathcal{F}^{(r)}_{G'} \) for \( r = 0, 1, 2, \ldots \) and all morphisms
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\[ G' = (G', \epsilon_V, \epsilon_L) \in \Gamma_0 \] is still dense. By the definition of topological transversality, we see that if \( f \in \mathcal{F}_{\epsilon_V, \epsilon_L} \) then all trees \( f_X \) of \( f \) equivalent to \( G' \) with \( C_{\epsilon}(f_X) \leq r \) are topologically transversal relative to \( S \) (for the definition of the number \( e(G', \epsilon_V^{p+r}) + 1 \) see Section 0.3).

Therefore the intersection \( \mathcal{F} \cap A_\infty \) coincides with the set \( T_{\infty} \subset A_\infty \) and \( T_{\infty} \cap C^\infty_{pr}(G,M) \) is open dense in the open subset \( A_\infty \cap C^\infty_{pr}(G,M) \). This completes the proof of Theorem 3.2.2.

**Corollary 3.2.3.** — If \( \mathcal{P} = (\dim M_v)_{v \in \mathbb{V}} \) satisfies the condition \( G = G_\infty, \infty = (\infty)_{v \in \mathbb{V}} \) in Section 2.1, then the set of topologically stable convergent diagrams of proper mappings \( f \in C^\infty_{pr}(G,M) \) is open dense in \( C^\infty_{pr}(G,M) \) with Whitney topology.

**Proof.** — The statement follows immediately from Theorem 2.1.2 and Theorem 3.2.1-2.

**Theorem 3.2.4 (Thom's second isotopy lemma).** — Let \( f \in C^\infty_{pr}(G,M) \) be a convergent diagram of proper smooth mappings \( f_\ell : M_v(\ell) \to M_v(\ell) \). Assume that there are Whitney regular stratifications \( S_v(f) \) of \( M_v, \ v \in \mathbb{V} \) and \( S_v(\ell) \) are Thom \( A_\ell \) regular \( \ell \in L \), and \( S_{v_0} \) is trivial : \( S_{v_0} = \{ M_{v_0} \} \). Then \( f \) is locally topologically trivial over \( M_{v_0} \) : for any point \( p \in M_{v_0} \), there is an open neighbourhood \( U \subset M_{v_0} \) and homeomorphisms \( \phi_v : f_{v v_0}^{-1}(U) \to f_{v v_0}^{-1}(p) \times U, \ v \in \mathbb{V} \) such that the following diagram commutes

\[
\begin{array}{ccc}
f_\ell : f^{-1}_{a(\ell)}{v_0} & \to & f^{-1}_{a(\ell)}{v_0}(U) \\
\phi_{\ell(\ell)} \downarrow & & \downarrow \phi_{\ell(\ell)} \\
 f_{\ell p} \times 1 : f^{-1}_{a(\ell) {v_0}}(p) \times U & \to & f^{-1}_{a(\ell) {v_0}}(p) \times U
\end{array}
\]

for \( \ell \in L \). In particular the restriction \( f_{v'} = (f_{\ell p'})_{\ell \in L} \) is topologically equivalent to \( f_p \) for any \( p' \in U \).

**Proof.** — This is a natural generalization of Thom's second isotopy lemma. For the proof, see e.g. [Gi, M4].
APPENDIX 1.

PROOF OF LEMMA 2.4.1.

Let \( \phi_i : O_i \to \mathbb{R}^n \), \( \psi_i : O_j \to \mathbb{R}^p \), \( i, j = 1, 2, \ldots \) be coordinate systems of smooth manifolds \( N, P \) such that \( f(O_i) \subset O_j \) with some \( j(i) \) for any \( i \). For a smooth mapping \( h : O_i \to \mathbb{R}^n \) we define \( f + h : O_i \to P \) with the addition of coordinates on \( O_j \). Let \( 0 < r_0 < s_0 < r_1 < s_1 < \ldots < r_q < s_q \) be numbers such that \( \{ \phi_i^{-1}(D(r_{ij})) | i = 1, 2, \ldots \} \) is a locally finite covering of \( M \) for \( j = 0, 1, \ldots, q \), where \( D(r) \) denotes the open disk with radius \( r \) centred at \( 0 \in \mathbb{R}^n \). Let \( \phi_j : M \to \mathbb{R} \) be smooth functions with supports in \( \phi_i^{-1}(D(r_{ij+1})) \) and identically equal to 1 on \( \phi_i^{-1}(D_i(s_j)) \). Let \( Z_i \) be an open neighbourhood of 0 in the linear space \( A_n \) of polynomial functions on \( \mathbb{R}^n \) of degree at most \( \ell' \), \( \ell' = (s+1)^a \) such that

\[
J^k(f + h \cdot \phi_j^\ell \cdot \prod_{m=1}^{\ell' \cdot p} (1 - \phi_{k's_m})) (N) \subseteq U
\]

for any \( j, \ell_m = 0, \ldots, q, p \leq q \) and \( h \in Z_i \).

We apply the following Lemma A with \( Z = Z_i \subset A_n' \) and the compact neighbourhood \( D(s_{ij}) \) of \( 0 \in \mathbb{R}^n \) and let \( 0 < \varepsilon_i < \infty \) be a number with the property in the lemma.

Let \( U' \) be the set-theoretical union of jet sections \( J^{(s+1)}g(N) \) of \( g \) such that

\[
\| (f - g) \circ \phi_i^{-1} \|_{D(s+1)}^{D(s_{ij})} < \varepsilon_i
\]

for any \( i = 1, 2, \ldots \), where \( \| \phi \|_a \) denotes the sup. norm of derivatives of order \( \leq a \) on the set \( K \subset \mathbb{R}^n \). We claim that \( U' \) possesses the required property in Lemma 2.4.1.

Let \( X = \{ x_1, \ldots, x_q \} \subset N \). By an easy argument we see there is a function \( j(i) \) such that

\[
X \cap \phi_i^{-1}(D(r_{ij+1})) - D(r_{ij})) = \emptyset,
\]

for any \( i = 1, 2, \ldots, \) and by renumbering the index \( i \), we have

\[
X \subseteq \bigcup_{i=1}^{p} \phi_i^{-1}(D(r_{ij}(0))) , \quad p \leq q.
\]

We define a partition of \( X \) into the disjoint \( p \) sets

\[
X_i \subseteq \phi_i^{-1}(D(r_{ij}(0))) - \bigcup_{m=1}^{i-1} \phi_m^{-1}(D(r_{ij}(m)))
\]
i = 1, ..., p. By Lemma A, there are $h_i \in A^\prime_n$ such that the mappings $f'_i : N \to P$:

$$f'_i = f + h_i \circ \phi_i^{-1} \cdot \prod_{m=1, \ldots, i-1} (1 - \phi_{n,j(m)})$$

are well defined and satisfy

$$J^k f'_i(N) \subset U$$

and

$$J^r f'_i(x) = J^r (f + h_i)(x) = J^r g(x),$$

for $x \in X_i$, $i = 1, \ldots, p$. Now define $f' \in C^\infty(N, P)$ by $f' = f'_i$ on $\phi_i^{-1}(D(r_{i,j(0)})) - \bigcup_{m=1, \ldots, i-1} \phi_m^{-1}(D(s_{m,j(m)})$ and $f' = f$ on the complement of these subsets above. Then $f'$ possesses the required properties:

$$J^k f'(N) \subset U$$

and

$$J^r f'(x) = J^r f'_i(x) = J^r g(x)$$

for $x \in X_i$, $i = 1, \ldots, p$.

**Lemma A** (Golubitsky-Guillemin, Lemma 2.5 [GG]). Let $0 < s, 0 < q$ be integers, $K \subset \mathbb{R}^n$ a compact connected neighbourhood of the origin and let $Z \subset A^\prime_n$, $\ell = (s+1)^q$ be a neighbourhood of the constant mapping $0 \in A^\prime_n$. Then there is a positive number $\varepsilon > 0$ such that for any distinct $q$ points $p_1, \ldots, p_q \in K$ and any smooth function $g : \mathbb{R}^n \to \mathbb{R}$ with $\|g\|_{q(s+1)} < \varepsilon$ there is a polynomial function $V \in Z$ for order $\leq \ell$ such that

$$\frac{\partial^{(a)} V}{\partial x^\alpha}(p_i) = \frac{\partial^{(a)} g}{\partial x^\alpha}(p_i)$$

for $i = 1, \ldots, p$, $0 \leq |\alpha| \leq s$.

**APPENDIX 2.**

**TWO EXAMPLES DUE TO DU PLESSIS AND DUFOUR**

Example 1: due to du Plessis

$C^1$ stability is not generic in $C^\infty(M^3 \to N^4 \to P^2)$.

**Proof.** Let $(f, g) : M \to N \to P$ be a composition of proper mappings and assume the composition $g \circ f : M \to P$ is submersive at $x_i \in M - \Sigma(f)$, $i = 1, \ldots, 4$, $f(x_i) = y$ and the multi germ $f_{x_i} : (M, x_i) \to (N, y)$ is $C^\infty$ stable, i.e., $\text{Im } df_{x_i}$ are in general position. Let $(f', g')$ be a perturbation of $(f, g)$. Then by the stability of the multigerm above, there are again 4 points $x'_i$ close to $x_i$ respectively such that $f'(x'_i) = y'$ and $g' \circ f'$ is submersive at $x'_i$. The cross ratio
$C_y$ of $\text{Im } df_{s_1} \cap \ker dg_y$ in $\ker dg_u$ is clearly $C^1$ invariant of diagrams of the type or $\xrightarrow{\gamma} -$ etc., while the ratio $C_y$ can vary by a perturbation of $f, g$. Hence $(f, g)$ is not $C^1$ stable.

Example 2: due to Dufour.

All topological equivalence classes in an open dense subset of $C^\omega(M^1 \leftarrow N^2 \rightarrow P^2)$ have infinite codimension, if $W_1(N) = W_1(P) = 0$, $W_2(N) \neq 0$ and $W_2(P) = 0$.

Proof. – Let $(f, g) : M \leftarrow N \rightarrow P$ be a divergent diagram of smooth mappings. The Thom polynomial for the singularity $\Sigma^1_1(g)$ is the polynomial $W_2(\gamma) - W_1(\gamma)^2$ of Stiefel-Whitney class of the difference bundle $\gamma = TN - g*TP$. By the condition above we see the polynomial is not 0 in $H^2(N, \mathbb{Z}_2)$.

So generic mappings $g : N \rightarrow P$ have cusp singularities and in their neighbourhoods there are triples of points $x_i \notin \Sigma(f) \cup \Sigma(g)$ with $f(x_i) = y$. Dufour [D2] proved that the germs of $(f, g)$ at $x_1, x_2, x_3$ are $C^\infty$ equivalent if and only if they are topologically equivalent and $C^\infty$ equivalence classes are all of infinite codimension in the jet space $J^\infty(2, 1)^3 \times J^\infty(2, 2)^3$. From this fact the statement follows.

Furthermore, Dufour [D3] proved that $C^\infty$ classification and topological classification are the same for mappings in $C^\infty(M^1 \leftarrow N^2 \rightarrow P^2)$.

The two examples above are caused by the existence of «wild» diagrams of map germs imbedded in the global diagrams as multi germs. Now we denote them in terms of morphisms of oriented diagrams as follows:

```
example 1 4 \rightarrow 2 \hat{D}_5 \text{ wild}
3 \rightarrow
3 \rightarrow
3 \rightarrow
3 \rightarrow
```

```
example 2 1 \leftarrow 2 \rightarrow 2 \hat{E}_6 \text{ wild}
1 \leftarrow 2 \rightarrow
1 \leftarrow 2 \rightarrow
```

This explanation suggests that the stability problem is closely related with morphisms of oriented graphs and their expanded diagrams.
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Isao NAKAI,
Nagasaki University
Department of Mathematics
Faculty of Liberal Arts
1-14 Bunkyo-machi
Nagasaki 852 (Japan).