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JUAN ELIAS

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A NOTE ON THE ONE-DIMENSIONAL SYSTEMS OF FORMAL EQUATIONS

by Juan ELIAS

To Joan

0. Introduction.

Let $(X, 0)$ be an algebroid singularity defined by the ideal $I \subset \mathbf{k}[[X_1, \dots, X_N]]$. J. Nash in [N] proposed to study $(X, 0)$ using the set of arcs A_X , i.e. the set of $\alpha \in \mathbf{k}[[T]]^N$ such that $\alpha(0) = 0$, and $f(\alpha) = 0$ for all $f \in I$. Let A_X^n be the set of n -th truncations of A_X : $\gamma \in \mathbf{k}[[T]]^N$ belongs to A_X^n if and only if $\deg(\gamma_i) \leq n$ for all $i = 1, \dots, N$ and there exists $\alpha \in A_X$ such that $\alpha - \gamma \in (T)^{n+1}\mathbf{k}[[T]]^N$. Denote by $\pi_n : A_X^n \rightarrow A_X^{n-1}$ the truncation map $\pi_n((\sum_{j=0}^n \gamma_j^i T^j)_{i=1, \dots, N}) = (\sum_{j=0}^{n-1} \gamma_j^i T^j)_{i=1, \dots, N}$, so we have a projective system of sets $\{A_X^n, \pi_n\}_{n \geq 0}$ and an isomorphism of sets $A_X \cong \varprojlim A_X^n$. Hence a way to study A_X is look into A_X^n . In the complex case from the existence of a non-singular model of $(X, 0)$ J. Nash deduces that A_X^n is constructible for all n (see [N], [Le]), on the other hand J.C. Tougeron (see [Le]) proves that A_X^n is constructible from the formal version of the approximation theorem of M. Artin ([A]) due to J. Wavrik ([W1]). In particular from this result one can deduce that there exists a numerical function $\beta : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ such that: $\gamma \in A_X^n$ if and only if there exists $\tilde{\gamma} \in \mathbf{k}[[T]]^N$ such that $f(\tilde{\gamma}) \in (T)^{\beta(n)}\mathbf{k}[[T]]^N$ for all $f \in I$ and $\gamma - \tilde{\gamma} \in (T)^{n+1}$.

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As far as we know only in a few cases we have an explicit determination of β : first case is due to J. Wavrik for X reduced plane curve taking non-singular arcs ([W2]), the second one is due to M. Lejeune for hypersurface singularities taking general arcs ([Le]).

In this paper we determine the function β in the case of one-dimensional singularities X , taking non-singular arcs, in terms of the Milnor number associated to X_{red} . See [La] for other results on β .

The paper is divided in two sections, in the first we give some preliminaries results about contact between curves. In the second one we define the numerical function β and we prove the main result of this paper (Theorem 2.1).

Throughout this paper R will be the power series ring $\mathbf{k}[[X_1, \dots, X_N]]$, where \mathbf{k} is an infinite field. We denote by \mathfrak{M} the maximal ideal of R .

A curve of $(\mathbf{k}^N, 0) = \text{Spec}(R)$ is a one-dimensional, Cohen-Macaulay closed subscheme X of $(\mathbf{k}^N, 0)$, i.e. $X = \text{Spec}(R/I)$ where $I = I(X)$ is a perfect height $N-1$ ideal of R ; we put $\mathcal{O}_X = R/I$. A branch is an integral curve.

2. Contact of curves.

If X is a reduced curve of $(\mathbf{k}^N, 0)$ then we denote by $\delta(X)$ the dimension over \mathbf{k} of the quotient $\tilde{\mathcal{O}}_X/\mathcal{O}_X$ where $\tilde{\mathcal{O}}_X$ is the integral closure of \mathcal{O}_X . If r is the number of branches of X then we define the Milnor number of X by $\mu(X) = 2\delta - r + 1$.

Let X be a reduced curve and let Q be an infinitely near point of X , see [ECh], [VdW]. It is known that there exists a unique sequence $\{Q_i\}_{i=0, \dots, s}$ of infinitely near points of X such that $Q_0 = 0, \dots, Q_s = Q$, and that Q_{i+1} belongs to the first neighbourhood of Q_i for $i = 0, \dots, s-1$. We denote by (X, Q) the union of the irreducible components through Q of the proper transform of X by the composition of the blowing-up centered at Q_i for $i = 0, \dots, s-1$. We denote by $p_{(X, Q)}(T) = e(X, Q)T - \rho(X, Q)$ the Hilbert polynomial of the local ring $\mathcal{O}_{(X, Q)}$.

For the readers convenience we will summarize some properties of $e(X, Q)$ and $\rho(X, Q)$ that we will use in the paper:

- (1) $e(X, Q) - 1 \leq \rho(X, Q)$, ([M] Proposition 12.14),
- (2) $e(X, Q) = 1$ if and only if $\rho(X, Q) = 0$, ([M] Proposition 12.16),

- (3) $e(X, Q) = 2$ if and only if $\rho(X, Q) = 1$, ([M] Proposition 12.17),
- (4) $\dim_{\mathbf{k}}(R/I + M^n) = p_{(X, Q)}(n)$ for all $n \geq e(X, Q) - 1$, ([K] Theorem 6, or [M] Proposition 12.11).

Let $T(X)$ be the set of infinitely near point Q of X such that its multiplicity $e(X, Q)$ is greater than one. From [Ca] we obtain that

$$\delta(X) = \sum_{Q \in T(X)} \rho(X, Q).$$

Let X, Y be curves of $(\mathbf{k}^N, 0)$, without components in common, we denote by $(X.Y)$ the number $\dim_{\mathbf{k}}(R/I(X) + I(Y))$ ([H]).

Let Z_1 be a branch, for every reduced curve Z_2 , such that Z_1 is not a component of Z_2 , we define $f(Z_1, Z_2)$ as the number of non-singular points shared by Z_1 and Z_2 .

PROPOSITION 1.1. — *If Z_1 is a non-singular branch then*

$$(Z_1.Z_2) \leq \mu(Z_2) + f(Z_1, Z_2) + 1.$$

Proof. — From [C] and [M], Proposition 12.16, we deduce

$$(Z_1.Z_2) \leq \sum_{Q \in K} (\rho(Z_1 \cup Z_2, Q) - \rho(Z_2, Q))$$

where K is the set of infinitely near points shared by Z_1 and Z_2 .

Since (Z_i, Q) is a curve of $(\mathbf{k}^N, Q) \cong (\mathbf{k}^N, 0)$, we put $(Z_i, Q) = \text{Spec}(R/I_{i, Q})$ for $i = 1, 2$. Consider the projection

$$\frac{R}{(I_{1, Q} \cap I_{2, Q}) + M^n} \rightarrow \frac{R}{I_{2, Q} + M^n}$$

for all $n \geq e(Z_2, Q)$; from this and [K], Corollary 6, we get

$$(e(Z_2, Q) + 1)n - \rho(Z_1 \cup Z_2, Q) \geq e(Z_2, Q)n - \rho(Z_2, Q).$$

Therefore $\rho(Z_1 \cup Z_2, Q) - \rho(Z_2, Q) \leq e(Z_2, Q)$, and hence

$$(1) \quad (Z_1.Z_2) \leq \sum_{Q \in K} e(Z_2, Q) .$$

Assume that Z_2 is singular. Since $e(Z_2, 0) \leq \rho(Z_2, 0) + 1$, [M] Proposition 12.14, and $r \leq e(Z_2, 0)$ we deduce

$$(2) \quad e(Z_2, 0) \leq (2\rho(Z_2, 0) + 1 - r) + 1 .$$

Let K^* be the set of points belonging to K such that $e(Z_2, 0) \geq 2$. From [M], Proposition 12.17, we obtain that for all $Q \in K^*$

$$(3) \quad e(Z_2, Q) \leq 2\rho(Z_2, Q).$$

By (2) and (3) we get

$$\sum_{Q \in K^*} e(Z_2, Q) \leq \left(2 \sum_{Q \in K^*} \rho(Z_2, Q) + 1 - r \right) + 1,$$

since $\rho(Z_2, Q) = 0$ if and only if $e(Z_2, Q) = 1$ we have

$$\sum_{Q \in K^*} e(Z_2, Q) \leq \mu(Z_2) + 1.$$

Recall that $e(Z_2, Q) = 1$ for $Q \in K - K^*$, from (1) we obtain the claim.

PROPOSITION 1.2. — *Let $Z_i = \text{Spec}(R/I_i)$ be curves, $i = 1, 2$. Assume that Z_1 is non-singular and $I_2 + M^{\mu(Z_2)+n+1} \subset I_1 + M^{\mu(Z_2)+n+1}$. Then we have*

$$n \leq f(Z_1, Z_2).$$

Proof. — From the hypothesis we deduce that

$$I_1 + I_2 \subset I_1 + I_2 + M^{\mu(Z_2)+n+1} = I_1 + M^{\mu(Z_2)+n+1},$$

so that

$$\mu(Z_2) + n + 1 \leq \dim_{\mathbf{k}}(R/I_1 + M^{\mu(Z_2)+n+1}) \leq (Z_1, Z_2).$$

The claim follows from Proposition 1.1.

COROLLARY 1.3. — *If $n \geq 2$ then there exists a non-singular branch Y of Z_2 such that $n \leq f(Z_1, Y)$.*

Proof. — By Proposition 1.2 we get $f(Z_1, Z_2) \geq n \geq 2$, so there exists a branch Y of Z_2 such that Z_1 and Y share n non-singular infinitely near points. Since a non-singular branch and a singular branch cannot share two non-singular near points, we get that Y is non-singular.

The following result is well known :

PROPOSITION 1.4. — *Let Z_1, Z_2 be non-singular branches, for all n the following inequalities are equivalent :*

$$(1) \quad (Z_1, Z_2) \geq n,$$

- (2) Z_1 and Z_2 share n infinitely near points,
- (3) for all parametrization of Z_1 :

$$Z_1 : \begin{cases} X_1 = t \\ X_i = X_i(t) \text{ for all } i = 2, \dots, N, \end{cases}$$

there exists a parametrization of Z_2 :

$$Z_2 : \begin{cases} X_1 = t \\ X_i = \tilde{X}_i(t) \text{ for all } i = 2, \dots, N, \end{cases}$$

such that

$$X_i(t) - \tilde{X}_i(t) \equiv 0 \text{ modulo } (t)^n,$$

for all $i = 2, \dots, N$.

2. The function β .

DEFINITION. — We say that a system of formal equations $\{F = 0\} = \{F_1 = 0, \dots, F_s = 0\}$, $F_i \in R$, is one-dimensional if and only if $(F) = (F_1, \dots, F_s)$ is a height $N - 1$ ideal of R . We denote by \mathcal{F} the set of one-dimensional systems of formal equations.

Let $\{F = 0\}$ be a one-dimensional system of formal equations, we define the curve $Z_F = \text{Spec}(R/\text{rad}(F))$, and the numbers $\mu(\{F = 0\}) = \mu(Z_F)$ and $m(\{F = 0\}) = \text{Min}\{n \in \mathbf{N} \mid \text{rad}((F))^n \subset (F)\}$.

DEFINITION. — Let $\beta : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ be the numerical function:

$$\beta(n, \{F = 0\}) = m(\{F = 0\})(2\mu(\{F = 0\}) + n + 1).$$

THEOREM 2.1. — Given a one-dimensional system of formal equations $\{F = 0\}$, and a non-negative integer $n \geq 0$ if Z_F is singular and $n \geq 1$ if Z_F is non-singular. Let $X_i(X_1, \dots, X_r) \in \mathbf{k}[[X_1, \dots, X_r]]$, $1 \leq r \leq N$, $i = r + 1, \dots, N$ be a set of formal power series such that for every $G \in (F)$:

$$G(X_1, \dots, X_r, X_{r+1}(X_1, \dots, X_r), \dots, X_N(X_1, \dots, X_r)) \equiv 0 \text{ modulo } (X_1, \dots, X_r)^{\beta(n, \{F=0\})}.$$

Then there exist $\tilde{X}_i(X_1, \dots, X_r) \in \mathbf{k}[[X_1, \dots, X_r]]$, $i = r + 1, \dots, N$, such that:

- (1) $G(X_1, \dots, X_r, \tilde{X}_{r+1}, \dots, \tilde{X}_N) = 0$ for all $G \in (F)$,
- (2) $X_i(X_1, \dots, X_r) - \tilde{X}_i(X_1, \dots, X_r) \equiv 0$ modulo $(X_1, \dots, X_r)^n$ for all $i = r + 1, \dots, N$.

Proof. — First of all we will prove that $r = 1$. From now on we put $\mu(\{F = 0\}) = \mu(Z_F) = \mu$, $\rho(Z_F, 0) = \rho$ and $e(Z_F, 0) = e$.

Let J be the ideal of R generated by $X_i - \tilde{X}_i(X_1, \dots, X_r)$ for $i = r + 1, \dots, N$. Notice that J is the kernel of the map $\varphi : R \rightarrow \mathbf{k}[[X_1, \dots, X_r]]$ defined by

$$\varphi(G) = G(X_1, \dots, X_r, X_{r+1}(X_1, \dots, X_r), \dots, X_N(X_1, \dots, X_r)) .$$

From the hypothesis we deduce that

$$(F) \subset J \text{ modulo } (X_1, \dots, X_r)^{\beta(n, \{F=0\})},$$

so

$$(1) \quad \text{rad}((F)) \subset J \text{ modulo } (X_1, \dots, X_r)^{2\mu+1+n} .$$

Recall [C] that

$$\delta(Z_F) = \sum_{Q \in T(Z_F)} \rho(Z_F, Q),$$

by [M], Proposition 12.14, we obtain that $\delta(Z_F) + 1 \geq e$; from this we deduce $\mu \geq \delta(Z_F)$, so $\mu \geq \rho$.

From [M], Proposition 12.11, we get

$$\dim_{\mathbf{k}} \left(\frac{R}{\text{rad}((F)) + M^{2\mu+n+1}} \right) = e(2\mu + n + 1) - \rho .$$

Since $\text{Spec}(R/J)$ is non-singular, from (1) we have

$$e(2\mu + n + 1) - \rho \geq \binom{2\mu + n + r}{r} .$$

Assume that $r \geq 2$, then $(2\mu + n + 1)(e - (\mu + 1) - n/2) \geq \rho$. Since $\mu \geq \rho \geq e - 1$ ([M], Proposition 12.14) we obtain: $\rho \leq (2\mu + n + 1)(-n/2)$. If Z_F is singular then we get $\rho \leq 0$, but from [M], Propositions 12.14 and 12.17, we have that $\rho \geq 1$, so $r=1$. If Z_F is non-singular we get that $\rho < 0$, since ρ is a non-negative integer ([M], Propositions 12.14) we deduce $r = 1$.

Consider the non-singular branch Z_1 which admits the parametrization:

$$Z_1 : \begin{cases} X_1 = t \\ X_i = X_i(t) \text{ for all } i = 2, \dots, N . \end{cases}$$

Notice that the series $H_i = X_i - X_i X_1$, $i = 2, \dots, N$, form a system of generators of the ideal I_1 defining the curve Z_1 . If $G \in \text{rad}(F)$ then

$$G(X_1, X_2(X_1), \dots, X_N(X_1)) \equiv 0 \text{ modulo } (X_1)^{\mu+1+n},$$

thus

$$\text{rad}((F)) \subset I_1 \text{ modulo } (X_1)^{\mu+1+n}.$$

From Propositions 1.2, 1.3 and 1.4 we deduce the claim.

Remark. — (1) From the proof of the theorem it is easy to prove that for the systems of formal equations with $r = 1$ one can take

$$\beta(n, \{F = 0\}) = m(\{F = 0\})(\mu(\{F = 0\}) + n + 1).$$

(2) If we consider reduced systems of formal equations, i.e. $\text{rad}((F)) = (F)$, then we have

$$\beta(n, \{F = 0\}) = 2\mu(\{F = 0\}) + n + 1.$$

Notice that the number $2\mu(\{F = 0\}) + 1$ has the following property ([E]): the analytic type of Z_F is determined by any of its truncations: $(Z_F)_n = \text{Spec}(R/(F) + M^n)$ for all $n \geq 2\mu(\{F = 0\}) + 1$.

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Juan ELIAS,
Universitat de Barcelona
Facultat de Matemàtiques
Departament d'Àlgebra i Geometria
Gran Via 585
08007 BARCELONA
(Espanya).