FRANÇOISE LUST-PIQUARD

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MEANS ON $CV_p(G)$-SUBSPACES OF $CV_p(G)$ WITH RNP AND SCHUR PROPERTY

by Françoise LUST-PIQUARD

Introduction.

Let $G$ be a lca group and $1 \leq p \leq 2$. We generalize to the space $CV_p(G)$ of bounded convolution operators: $L^p(G) \to L^p(G)$ ($1 < p < 2$) some results which are obvious for $p = 1$ and were obtained for $p = 2$ by L. H. Loomis, G. S. Woodward, P. Glowacki and the author. We also generalize some results of N. Lohoué on convolution operators. Our motivation was a question raised by E. Granirer: is there a generalization of Loomis theorem [Loo] for convolution operators? A positive answer is given in theorem 2.8: Let $E \subset G$ be compact and scattered. Then $CV_p(E)$, the space of convolution operators on $L^p(G)$ which are supported on $E$, is the norm closure of finitely supported measures on $E$, and this space has Radon-Nikodym property. We also prove (theorem 2.14) that under the same assumptions $CV_p(E)$ has the Schur property.

The natural predual of $CV_p(G)$ is $A_p(G)$, which by C. Herz fundamental result is an algebra for pointwise multiplication and has some properties similar to those of $A_2(G)$ (we recall that $A_2(G)$ is isometric to $L^1(\hat{G})$ and $CV_2(G)$ is isometric to $L^\infty(\hat{G})$). But the proofs of Loomis theorem for $p = 2$ actually use the fact that every $\chi \in \hat{G}$ defines an isometric multiplier : $CV_2(G) \to CV_2(G)$ and that if $S \subset CV_2(G)$ has a compact support

$$\|S\|_{CV_2(G)} = \sup_{\chi \in \hat{G}} |\langle S, \chi \rangle|$$

where $\hat{G}$ is a group (the dual group of $G$).

Key-words: Invariant means - Convolution operators - Schur property - Radon-Nikodym property.

One of the ingredients in this paper is to provide $CV_p(G)$ with an equivalent norm such that

$$\|\|S\|\|_p = \sup_{f \in \mathcal{S}_p(G)} |\langle S, f \rangle|.$$  

where $\mathcal{S}_p(G)$ is a semi-group of functions of $A_p(G)$. This is done by using numerical ranges. We can thus adapt to $CV_p(G)$ a theory of means which is the usual one on $CV_s(G)$ or rather on $L^\infty(\hat{G})$ [Gr], and which fits Eberlein’s theory ([Eb1] Part. I). Topological means on $CV_p(G)$ were already defined in [G]. This is done in part 1 where we also give notation, definitions and recall the properties of $CV_p(G)$ and $A_p(G)$ that we need.

In part 2 we prove our main results theorems 2.8 and 2.14. The crucial lemma 2.2 allows to adapt the techniques of [Loo] [W1] [W2] [L-P1] [L-P2] [Gl]. In part 3 we show how theorems 2.8 and 2.14 also imply results on some $CV_p(\Lambda)$ where $\Lambda$ is discrete. The main result is theorem 3.3, which is a generalization of a result of [L-P1] and [L-P3].

In part 3, 4 we give some transfer theorems between $CV_p(G)$ and $CV_p(G_d)$ ($G_d$ is $G$ provided with the discrete topology) and we prove an Eberlein decomposition (theorem 4.2) for elements of $CV_p(G)$ which are totally topologically p-ergodic (see definition 1.7) and we precise it for (weak) p-almost periodic elements of $CV_p(G)$ (see definition 4.5). This generalizes results of [Eb2] [W2] [L-P2] [Gra] [Lohl].

We take this opportunity to thank Ed. Granirer for nice and useful discussions.

1. Notation, definitions, states and means on $CV_p(G)$.

We consider Banach spaces over the field $\mathbb{C}$ of complex numbers. We denote by $X^*$ the dual space of a Banach space $X$.

For $\varepsilon > 0$ $D_\varepsilon$ is the open disc in $\mathbb{C}$ centered at $\{0\}$ with radius $\varepsilon$.

$G$ denotes a lca group, $G_d$ is the same group provided with the discrete topology, $\hat{G}$ is the dual group of $G$.

For $1 \leq p < \infty$ $L^p(G)$ is the space of equivalence classes of $p$-integrable functions with respect to the Haar measure on $G$; $L^\infty(G)$ is the dual space of $L^1(G)$. For $1 \leq p \leq 2$ $p'$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.
the duality between $L^p(G)$ and $L^{p'}(G)$ is defined by

$$\langle f, g \rangle = \int_G f(x) g(x) \, dx.$$  

$C_0(G)$ is the space of continuous functions on $G$ which tends to 0 at infinity. $M(G)$ is the space of bounded Borel measures on $G$, i.e. the dual space of $C_0(G)$. For $1 \leq p \leq 2$ $CV_p(G)$ denotes the space of bounded convolution operators: $L^p(G) \to L^p(G)$, i.e. operators which commute with translation by elements of $G$, provided with the operator norm. We recall that $CV_1(G) = M(G)$ and $CV_2(G)$ is the space of Fourier transforms of the functions in $L^\infty(\hat{G})$.

$CV_p(G)$ is also the space of bounded convolution operators: $L^{p'}(G) \to L^{p'}(G)$ ($1 < p \leq 2$) hence, by Riesz interpolation theorem, identity is continuous with norm 1

$$CV_{p_1}(G) \to CV_{p_2}(G), \quad 1 \leq p_1 \leq p_2 \leq 2.$$  

For $1 \leq p \leq \infty$ and $f \in L^p(G)$ we denote $\hat{f}(x) = f(-x)$.

For $1 < p \leq 2$ $A_p(G)$ denotes the space of functions $f$ on $G$ which can be represented as

$$f = \sum_{n \geq 1} u_n \ast \hat{v}_n$$

where $\sum_{n \geq 1} \|u_n\|_{L^p(G)} \|v_n\|_{L^{p'}(G)} < +\infty$ and the norm of $f$ is the infimum of these sums over all such representations of $f$.

Hence $A_2(G)$ is the space of Fourier transforms of the elements of $L^1(\hat{G})$.

For $p = 1$ we replace $L^{p'}(G)$ by $C_0(G)$ in the definition above, hence $A_1(G) = C_0(G)$.

The duality between $CV_p(G)$ and $A_p(G)$ is defined by

$$\langle S, u \ast \hat{v} \rangle = \langle S(u), v \rangle.$$  

$CV_p(G)$ is clearly the dual space of $A_p(G)$. In particular

$$A_{p_1}(G) \leftrightarrow A_{p_2}(G), \quad 1 \leq p_1 \leq p_2 \leq 2.$$  

As functions which are continuous on $G$ with a compact support are dense in $L^p(G)$ ($1 \leq p \leq 2$) $A_{p_2}(G)$ is dense in $A_{p_1}(G)$, hence identity: $CV_{p_1}(G) \to CV_{p_2}(G)$ is one to one.
For \( x \in G \) and \( f \in L^p(G) \) \((1 \leq p < \infty)\) or \( A_p(G) \) \((1 \leq p \leq 2)\) we denote by \( f_x \) the translate of \( f \) by \( x \) i.e. \( f_x(t) = f(t - x) \). For \( S \in CV_p(G) \) \((1 \leq p \leq 2)\) the translate \( S_x \) is defined by \( S_x(f) = (S(f))_x \) for \( f \in L^p(G) \). Translation in \( A_p^{**}(G) \) is defined by duality, i.e. \( \langle S, F_x \rangle = \langle S_x, F \rangle \) for \( F \in A_p^{**}(G) \); when restricted to \( A_p(G) \) this definition coincides with the first one. The support of \( S \in CV_p(G) \) is the (closed) set of \( x's \in G \) such that for every neighborhood \( V(x) \) there exists \( f \in A_p(G) \) such that \( f \) is supported on \( V(x) \) and \( \langle S, f \rangle \neq 0 \).

Let \( E \subseteq G \) be a closed subset; we denote by \( CV_p(E) \) the closed subspace of \( CV_p(G) \) whose elements are supported on a subset of \( E \). We denote by \( \ell^1(E) \|_{\ell^\infty} \) the closed subspace of \( CV_p(E) \subseteq CV_p(G) \) spanned by measures whose support is finite and lies in \( E \). We denote by \( CV_p(E_d) \) the closed subspace of \( CV_p(G_d) \) whose elements are supported on a subset of \( E \). We recall Herz’s fundamental results ([P] proposition 10.2, 19.8): \( A_p(G) \) is a Banach algebra for pointwise multiplication \((1 \leq p \leq 2)\). Let \( B_p(G) \) denote the algebra of pointwise multipliers of \( A_p(G) \). Then for \( f \in A_p(G) \)

\[
\|f\|_{A_p(G)} = \|f\|_{B_p(G)}.
\]

More generally let \( H \) be a lca group such that \( G_d \) is a subgroup of \( H_d \), the embedding \( G \to H \) is continuous and \( G \) is dense in \( H \) (hence \( H \) continuously embeds in \( \hat{G} \) the Bohr compactification of \( G \) i.e. the dual group of \( G_d \)). Then ([Ey] théorème 1, [Lohl] chap. IV, théorème IV.1, p. 108)

\[
\forall f \in B_p(H), \quad \|f\|_{B_p(H)} = \|f\|_{B_p(G)}.
\]

In the sequel we will write only \( G \to H \) and this will mean that the above assumptions on \( G \) and \( H \) are satisfied. Actually we will only use the particular cases \( G \to G \), \( G_d \to G \), \( G \to \hat{G} \).

Let \( \varphi \in B_p(G) \); we will consider the pointwise multiplication operator associated to \( \varphi \) and the adjoint operators

\[
A_p(G) \to A_p(G)
\]
\[
f \mapsto \varphi f
\]
\[
CV_p(G) \leftarrow CV_p(G)
\]
\[
\varphi S \leftarrow S
\]
\[
A_p^{**}(G) \to A_p^{**}(G)
\]
\[
F \mapsto \varphi F.
\]
Let $E \subset G$ be a closed subset; $I_p(E)$ is the closed ideal of functions of $A_p(G)$ which are zero on $E$. We denote the quotient algebra $A_p(G)/I_p(E)$ by $A_p(E)$. We recall that every $x \in G$ is a set of synthesis for $A_p(G)$ ([H1] theorem B, [P] proposition 19.19) which means that if $f \in A_p(G)$ and $f(x) = 0$, $f$ is the norm limit of a sequence of functions in $A_p(G)$ which are zero on a neighborhood of $x$ in $G$.

Let $W \subset G$ be a set of positive finite Haar measure. We denote

$$\varphi_w = |W|^{-1} 1_w \ast \tilde{1}_w.$$  

$$\|\varphi_w\|_{A_p(G)} = 1 = \varphi(0) \ (1 \leq p \leq 2).$$

The group $G$ satisfies Fölner-condition ([Gre] theorem 3.6.2) : for every $\varepsilon > 0$ and every compact $K \subset G$ there is a compact set $W = W(K) \subset G$ with finite positive Haar measure such that

$$\forall x \in K, \quad \frac{1}{|W|} |W \setminus W \Delta W| \leq \varepsilon.$$  

Hence

$$\forall x \in K, \quad \left\| \frac{(1_w)_x}{|W|^{1/p}} - \frac{1_w}{|W|^{1/p}} \right\|_{L^p(G)} \leq \frac{1}{p}.$$  

By [H2] 9. lemma 5, the family $(\varphi_{W(K)})_K$ is an approximate identity for $A_p(G)$ i.e. for every $\varepsilon > 0$ and $f \in A_p(G)$ there exists a compact set $K \subset G$ such that $\|f - f \varphi_{W(K)}\|_{A_p(G)} \leq \varepsilon$. Obviously every $\varphi_{W(K)}$ has a compact support.

If $G$ is provided with its discrete topology and if $F \subset G$ is a finite set (i.e. $F$ is a compact set in $G_d$) we denote $P_F = |F|^{-1} 1_F \ast \tilde{1}_F$ (convolution is taken in $G_d$) instead of $\varphi_F$. Let $\mathcal{F}$ be the net of finite subsets of $G$. For every $x \in G$ $P_F(x) \xrightarrow{\mathcal{F}} 1$.

We recall that a Banach space $X$ has the Schur property if every sequence $(x_n)_{n \geq 1}$ in $X$ such that $x_n \to 0$ $\sigma(X,X^*)$ is norm convergent. A Banach space $X$ has the Radon-Nikodym property (RNP in short) if every bounded linear operator $T : L^1[0 \ 1] \to X$ is representable i.e. there exists a bounded strongly measurable function $F : [0 \ 1] \to X$ s.t.

$$\forall \varphi \in L^1[0 \ 1], \quad T(\varphi) = \int_{[0 \ 1]} F(t)\varphi(t) \ dt.$$
We recall that if every separable subspace of $X$ has RNP so has $X$ and that every separable dual space has RNP.

**States on $CV_p(G)$.**

$CV_p(G) (1 \leq p \leq 2)$ is a convolution algebra with unit $\delta_0$.

Following the theory of numerical ranges [BD], we denote by $\mathcal{S}_p(G)$ the following set of states on $CV_p(G)$:

$$\mathcal{S}_p(G) = \{ f \in A_p(G) | \| f \|_{A_p} = 1 = f(0) \}. $$

Let

$$\pi_p(G) = \{ f \in A_p(G) | f = g * h, \| g \|_{L^p(G)} = \| h \|_{L^p(G')} = \int g(x)h(x)dx = 1 \}. $$

Obviously $\pi_p \subseteq \mathcal{S}_p$.

**Lemma 1.1.** - (i) $\mathcal{S}_p(G)$ is the norm closure of the convex hull of $\pi_p(G)$.

(ii) $\mathcal{S}_p^{00}(G) = \{ F \in A_p^*(G) | \| F \|_{A_p^*(G)} = 1 = \langle F, \delta_0 \rangle \}$.

**Proof.** - Let us denote the last set by $\mathcal{D}_p$.

Obviously $\mathcal{D}_p$ is norm closed and convex, and

$$\overline{\mathcal{C}_0 \pi_p} \subseteq \mathcal{S}_p \subseteq \mathcal{S}_p^{00} \subseteq \mathcal{D}_p.$$  

By [BD] chap. 1, §2, definition 1 and chap. 3, §9, theorem 3:

$$\forall S \in CV_p(G), \quad \mathcal{C}_0 \{ \langle S, f \rangle \}_{f \in \pi_p} = \{ \langle S, F \rangle \}_{F \in \mathcal{D}_p} \subset \mathbb{C}. $$

As

$$\mathcal{C}_0 \{ \langle S, f \rangle \}_{f \in \pi_p} \subset \{ \langle S, f \rangle \}_{f \in \mathcal{S}_p} = \{ \langle S, F \rangle \}_{F \in \mathcal{S}_p^{00}} \subset \{ \langle S, F \rangle \}_{F \in \mathcal{D}_p}$$

these sets are the same and Hahn-Banach theorem implies (i) and (ii).

By the fundamental theorem on numerical ranges [BD] chap. 1, §4, theorem 1,

$$\| S \|_{CV_p(G)} \geq \sup_{F \in \mathcal{D}_p} | \langle S, F \rangle | \geq e^{-1} \| S \|_{CV_p(G)}.$$

hence by lemma 1.1

(1) $\forall S \in CV_p(G) \| S \|_{CV_p(G)} \geq \sup_{F \in \mathcal{S}_p(G)} | \langle S, f \rangle | \geq e^{-1} \| S \|_{CV_p(G)}.$
As we are investigating geometric properties of subspaces of \( CV_p(G) \) we can as well provide \( CV_p(G) \) with the equivalent norm \( \sup_{f \in \mathcal{S}_p(G)} |\langle S, f \rangle| \). The set \( \mathcal{S}_2(G) \) is the set of functions in the unit sphere of \( A_2(G) \) such that \( f^2 \geq 0 \) on \( \hat{G} \). Hence \( \mathcal{S}_p(G) \) \((1 \leq p < 2)\) will replace the face of positive elements in the unit sphere of \( L^1(\hat{G}) \).

**Remark 1.2.** Let us mention ([BD] chap. 6, § 31, theorem 1) that the mappings

\[
S \mapsto (\langle S, f \rangle)
\]

\( CV_p(G) \rightarrow C(\mathcal{S}_p) \) or \( CV_p(G) \rightarrow C(\mathcal{S}_p^0) \)

are isometries of \( CV_p(G) \) provided with its new norm into a closed subspace of the continuous functions on \( \mathcal{S}_p \) or \( \mathcal{S}_p^0 \) provided with the \((A_p(G)^{**}, CV_p(G))\) topology. \( \mathcal{S}_p^0 \) is compact for this topology and the closure of \( \mathcal{S}_p \). Every \( F \in A_p^{**}(G) \) can be written as

\[
F = \alpha_1 F_1 - \alpha_2 F_2 + i \alpha_3 F_3 - i \alpha_4 F_4
\]

where \( F_i \in \mathcal{S}_p^0(G) \), \( \alpha_i \geq 0 \) \((1 \leq i \leq 4)\) and \( \sum_{i=1}^{4} \alpha_i \leq \sqrt{2} \sup_{S \in CV_p(G)} |\langle S, f \rangle| \) where the supremum is taken on

\[
\{S \in CV_p(G) | \forall f \in \mathcal{S}_p(G) \quad |\langle S, f \rangle| \leq 1 \}.
\]

As \( A_p(G) \) is an algebra for pointwise multiplication \( \mathcal{S}_p(G) \) is an abelian semi-group. Multiplication by \( f \in \mathcal{S}_p(G) \) is continuous on \( \mathcal{S}_p(G) \) provided with \( \sigma(A_p(G)^{**}, CV_p(G)) \), i.e. \( \mathcal{S}_p(G) \) is a semi-topological semi-group. In this setting the measures \( \alpha \delta_\alpha(\alpha \in \mathbb{C}) \) are constant functions on \( \mathcal{S}_p(G) \) and if \( S \in CV_p(G) \), \( f \in \mathcal{S}_p(G) \) \( fS \) is the translate of \( S \) (considered as a function on \( \mathcal{S}_p(G) \)) by \( f \). The set \( \{fS\}_{f \in \mathcal{S}_p(G)} \) is the orbit of \( S \) under the action of \( \mathcal{S}_p(G) \). We denote by \( K_S \) its pointwise closure (for pointwise convergence on \( \mathcal{S}_p(G) \)) by remark 1.2 \( K_S \) can be also identified with the closure of \( \{fS\}_{f \in \mathcal{S}_p(G)} \) for \( \sigma(CV_p(G), A_p(G)) \). \( \mathcal{S}_p(G) \) is convex (as a subset of functions on \( G \)) and \( S \) defines an affine function on \( \mathcal{S}_p(G) \).

**Means on \( CV_p(G) \).**

**Definition 1.3.** Let \( G \) be a lca group and let \( G \to H \). Let \( 1 \leq p \leq 2 \). A \( H \)-mean on \( CV_p(G) \) is an element \( \hat{m} \in \mathcal{S}_p^0(G) \) such that

\[
\forall \varphi \in \mathcal{S}_p(H) \quad \varphi \hat{m} = \hat{m}.
\]
This definition is consistent because $\mathcal{S}_p(H) \subset B_p(G)$. The set of $H$-means is compact for $\sigma(A_p(G)^{**}, CV_p(G))$.

If $H = G$ a $H$-mean is called a topological mean [Gra].

If $H = \bar{G}$ a $H$-mean is called a mean. If $p = 2$ means and topological means on $CV_2(G)$ are Fourier transforms of usual means and topological means on $L^\infty(\bar{G})$. If $G$ is discrete the only topological mean on $CV_p(G)$ is $1_{[0]}$ $(1 \leq p \leq 2)$. If $p = 1$ and $G$ is any lca group the only mean on $CV_1(G) = M(G)$ is $1_{[0]}$.

**Lemma 1.4.** Let $G$ be a lca group, $G \rightarrow H$, $1 \leq p \leq 2$.

(i) Let $\hat{m}$ be a $H$-mean. Let $\varphi \in B_p(H)$ be such that $\|\varphi\|_{B_p(H)} = 1 = \varphi(0)$. Then $\hat{m} = \varphi \hat{m}$.

(ii) A topological mean on $CV_p(G)$ is a $H$-mean.

**Proof.** (i) Let $\varphi_0 \in \mathcal{S}_p(H)$. By definition $\varphi_0 \hat{m} = \hat{m}$ hence $\varphi_0 \hat{m} = \varphi \hat{m}$. As $\varphi \varphi_0 \in \mathcal{S}_p(H)$ $\varphi \varphi_0 \hat{m} = \hat{m}$.

(ii) Let $\hat{m}$ be a topological mean and $\varphi \in \mathcal{S}_p(H)$. As $\varphi \in B_p(G)$ $\varphi \hat{m} = \hat{m}$ by (i).

This proof is similar to [Gre] proposition 2.1.3.

**Lemma 1.5.** Let $G$ be a lca group, $G \rightarrow H$, $1 \leq p \leq 2$.

(i) Let $(W_\alpha)_{\alpha \in A}$ be a basis of open neighborhoods of $\{0\}$ in $H$. Let $(f_\alpha)_{\alpha \in A}$ be a net in $\mathcal{S}_p(G)$ such that $f_\alpha$ is supported on $W_\alpha$ for every $\alpha$. Then every cluster point of $(f_\alpha)_{\alpha \in A}$ for $\sigma(A_p^{**}(G), CV_p(G))$ is a $H$-mean.

(ii) Conversely let $\hat{m}$ be a $H$-mean on $CV_p(G)$. There exists a net $(f_\alpha)_{\alpha \in A}$ in $\mathcal{S}_p(G)$ such that (a) $f_\alpha \rightarrow \hat{m}$, $\sigma(A_p^{**}(G), CV_p(G))$; (b) for every open neighborhood $W$ of $\{0\}$ in $H$ there exists $\alpha_0 \in A$ such that for every $\alpha > \alpha_0$ $f_\alpha$ is supported on $W \cap G$.

**Proof.** (i) Let $F \in \mathcal{S}_p^{00}(G)$ be a cluster point of $(f_\alpha)_{\alpha \in A}$. Let $\varphi \in \mathcal{S}_p(H)$. As $\{0\}$ is a set of synthesis for $A_p(H)$, for every $\varepsilon > 0$ there exists $\varphi_\varepsilon$ such that $\|\varphi - \varphi_\varepsilon\|_{A_p(H)} \leq \varepsilon$ and $\varphi = 1$ in a neighborhood $W$ of $\{0\}$ in $H$. As soon as $W_\alpha \subset W \varphi \varphi_\varepsilon f_\alpha = f_\alpha$ hence $\varphi \varphi F = F$ and $\|\varphi F - \varphi_\varepsilon F\|_{A_p^{**}(G)} \leq \|\varphi - \varphi_\varepsilon\|_{B_p(G)} \leq \varepsilon$. This implies $F = \varphi F$.

(ii) Let $\hat{m}$ be a $H$-mean on $CV_p(G)$. For every neighborhood $W$ of $\{0\}$ in $H$ let $W'$ be a neighborhood of $\{0\}$ in $H$ such that $W' \cap W \subset W$. For every
As \( \varphi_w \) is a multiplier of \( \mathcal{S}_p(G) \), \( \hat{m} = \varphi_w \hat{m} \) lies in \( \{ \mathcal{S}_p(G) \cap I_p(W^c \cap G) \}^{00} \). Hence

\[
\hat{m} \in \bigcap_w \{ \mathcal{S}_p(G) \cap I_p(W^c \cap G) \}^{00}
\]

where \( W \) runs through a basis of neighborhoods of \( \{0\} \) in \( H \), and this proves the claim. \( \square \)

Let \( G \) be a lca group and \( G \to H \). For \( 1 \leq p \leq 2 \) and \( S \in CV_p(G) \) let us define

\[
M^H_p(S) = \{ \langle S, \hat{m} \rangle | \hat{m} \mbox{ is a } H\text{-mean on } CV_p(G) \}.
\]

If \( H = G \) we will write \( M^G_p(S) = M^S_p \).

\( M^H_p(S) \) is a compact subset of \( \mathbb{C} \) and \( M^H_p(S) \supset M^G_p(S) \) \( (1 \leq p \leq 2) \).

If \( \varphi \in \mathcal{S}_p(G) \) \( M^H_p(\varphi S) = M^H_p(S) \).

**Lemma 1.6.** — Let \( G \) be a lca group and \( G \to H \). Let \( S \in CV_p(G) \) \( (1 \leq p \leq 2) \). Then for every \( \varepsilon > 0 \) there exists an open neighborhood \( W(0) \) in \( H \) such that \( M^H_p(S) \subseteq \{ \langle S, f \rangle | f \in \mathcal{S}_p(G), f \mbox{ is supported on } W \cap G \} \subseteq M^H_p(S) + D_\varepsilon \).

**Proof.** — The left inclusion is obvious by lemma 1.5 (ii). If the right one does not hold there exists \( \varepsilon > 0 \) such that for every \( W(0) \) in \( H \) there exists \( f_{(w)} \in \mathcal{S}_p(G) \), supported on \( W(0) \) such that \( d(\langle S, f_{(w)} \rangle, M^H_p(S)) \geq \varepsilon \). By lemma 1.5 (i) any cluster point of \( (f_{(w)}) \) for \( \sigma(A^*_p(G), CV_p(G)) \) (when \( W \) runs through a basis of neighborhoods of \( \{0\} \) in \( H \)) is a \( H \)-mean \( \hat{m} \), and the distance from \( \langle S, \hat{m} \rangle \) to \( M^H_p(S) \) would be greater than \( \varepsilon \), which is a contradiction.

**Definition 1.7.** — Let \( G \) be a lca group, \( G \to H \), \( 1 \leq p \leq 2 \). An element \( S \in CV_p(G) \) is \( H \)-ergodic at \( 0 \) if \( M^H_p(S) \) is a point. \( S \) is \( H \)-ergodic at \( x \in G \) if \( S_x \) is \( H \)-ergodic at \( 0 \) and \( S \) is \( H \)-totally ergodic if it is \( H \)-ergodic at every point \( x \in G \). If \( H = G \) we say that \( S \) is topologically \( p \)-ergodic at \( x \) instead of \( G \)-\( p \)-ergodic at \( x \).

This definition is apparently weaker than [Eb1] definition 3.1. Hence our next lemma is stronger than [Eb1] theorem 3.1 applied to this setting.

For \( p = 2 \) it was proved in [W1] corollary 3, under the assumption that \( \hat{S} \) is uniformly continuous and in full generality in [L-P2] proposition 1.
Lemma 1.8. – Let $G$ be a lca group, $G 	o H$, $1 \leq p \leq 2$. The following assertions on $S \in CV_p(G)$ are equivalent:

(i) $S$ is $H$-p-ergodic at $0$.

(ii) There exists $M \in \mathbb{C}$ such that

$$\forall \varepsilon > 0, \exists \varphi \in \mathcal{S}_p(H), \|\varphi S - M \delta_0\|_{CV_p(G)} \leq \varepsilon.$$

(iii) There exists $M \in \mathbb{C}$ such that for every $\varepsilon > 0$ there exists $\psi \in A_p(H)$ whose support is disjoint from $\{0\}$ and

$$\|S - M \delta_0 - \psi S\|_{CV_p(G)} \leq \varepsilon.$$

Proof. – (iii) $\Rightarrow$ (i) by lemma 1.5 (ii), and $M_p^H(S) = \{M\}$.

(i) $\Rightarrow$ (ii): let us put $\{M\} = M^H_p(S)$ hence $M_p^H(S - M \delta_0) = \{0\}$. For every $\varepsilon > 0$ we choose $W$ as in lemma 1.6. Hence if $W' - W' \subseteq W$ and $W'$ is an open neighborhood of $\{0\}$ in $H$

$$\forall f \in \mathcal{S}_p(G), \|S - M \delta_0, f \varphi_w\|_H \leq \varepsilon$$

which implies by (1)

$$\|\varphi_w, S - M \delta_0\|_{CV_p(G)} \leq \varepsilon.$$

(ii) $\Rightarrow$ (iii) For every $\varepsilon > 0$ let $\varphi$ be as in (ii). As $\{0\}$ is a set of synthesis for $A_p(H)$ there exists $\varphi_\varepsilon \in A_p(H)$ such that $\|\varphi - \varphi_\varepsilon\|_{A_p(H)} \leq \varepsilon$ and $\varphi_\varepsilon = 1$ in a neighborhood of $\{0\}$ in $H$. For $\psi = 1 - \varphi_\varepsilon$

$$\|S - M \delta_0 - \psi S\|_{CV_p(G)} = \|\varphi_\varepsilon S - M \delta_0\|_{CV_p(G)} \leq \varepsilon + \varepsilon \|S\|_{CV_p(G)}.$$

Definition 1.9. – Let $G$ be a lca group, $1 \leq p \leq 2$. $UC_p(G)$ is the closed subspace of $CV_p(G)$ spanned by compactly supported elements.

Obviously $UC_p(G)$ is the norm closure in $CV_p(G)$ of

$$\{fS|f \in A_p(G), S \in CV_p(G)\}.$$

It is a norm closed unitary subalgebra of $CV_p(G)$ ([Gra], proposition 12). $UC_2(G)$ is the space of Fourier transforms of uniformly continuous functions on $\hat{G}$. $B_p(G)$ can be identified with a subspace of $UC_p(G)^*$ in the following way: let $(\varphi_a)_{a \in A} \in \mathcal{S}_p(G)$ be an approximate identity for $A_p(G)$ and $F \in B_p(G)$. For every $S \in CV_p(G)$ and $f \in A_p(G)$

$$\langle fS, F \varphi_a \rangle = \langle S, f F \varphi_a \rangle \rightarrow \langle S, fF \rangle.$$
hence the net \((F\varphi_a)_{a \in A}\) which is bounded in \(A_p(G)\) (hence in \(UC_p^*(G)\)) converges for \(\sigma(UC_p(G)^*, UC_p(G))\), its limit can be identified with \(F\).

**Lemma 1.10.** - Let \(G\) be a lca group, \(G \to H\), \(1 \leq p \leq 2\).

(i) Let \(\hat{m}\) be a \(H\)-mean on \(CV_p(G)\). For every \(\varphi \in \mathcal{S}_p(G)\) \(\varphi \hat{m}\) is a topological mean.

(ii) A topological mean is uniquely determined by its restriction to \(UC_p(G)\).

(iii) Let \(S \in UC_p(G)\). Then \(M_p^H(S) = M_p(S)\).

**Proof.** - Let \(K \subset G\) be a compact set. The topologies on \(K\) induced by \(G\) and \(H\) are the same. For every neighborhood \(V\) of \(\{0\}\) in \(G\) there exists a neighborhood \(W\) of \(\{0\}\) in \(H\) such that \(V \cap K \supseteq W \cap K\).

(i) Let \((f_a)_{a \in A} \in \mathcal{S}_p(G)\), \(f_a \to \hat{m}\) as in lemma 1.5 (ii). Hence if \(\varphi \in \mathcal{S}_p(G)\) \(\varphi f_a \to \varphi \hat{m}\), \(\sigma(A_p^*(G), CV_p(G))\) and if \(\varphi\) has a compact support \(K\) the above remark and lemma 1.5 (i) imply that \(\varphi \hat{m}\) is a topological mean. Every \(\varphi \in \mathcal{S}_p(G)\) is the norm limit in \(A_p(G)\) of \((\varphi_n)_{n \geq 1} \in \mathcal{S}_p(G)\) where \(\varphi_n\) has a compact support \((n \geq 1)\). Hence \(\varphi_n \hat{m}(n \geq 1)\) and \(\varphi \hat{m}\) are topological means.

(ii) Let \(\hat{m}\) be a topological mean on \(CV_p(G)\). Then

\[\forall S \in CV_p(G), \forall \varphi \in \mathcal{S}_p(G), \langle S, \hat{m} \rangle = \langle S, \varphi \hat{m} \rangle = \langle \varphi S, \hat{m} \rangle\]

hence if \(\hat{m}\) and \(\hat{m}'\) are topological means which coincide on \(UC_p(G)\) they coincide on \(CV_p(G)\).

(iii) Let us first assume that \(S\) has a compact support and let \(K \subset G\) be a compact set whose interior contains the support of \(S\). Let \(\varphi \in \mathcal{S}_p(G)\). As \(\{0\}\) is a set of synthesis for \(A_p(G)\), for every \(\varepsilon > 0\) there exists \(\varphi_\varepsilon\) such that \(\|\varphi - \varphi_\varepsilon\|_{A_p(G)} \leq \varepsilon\) and \(\varphi_\varepsilon = 1\) in a neighborhood of \(\{0\}\) in \(G\) which we denote by \(V\). Let \(W \subset H\) be such that \(W \cap K \subset V \cap K\). Hence for every \(f \in A_p(G)\) which is supported on \(W(1 - \varphi)f \in 1_p(K)\) and \(\langle S, (1 - \varphi_\varepsilon)f \rangle = 0\). For every \(H\)-mean \(\hat{m}\) lemma 1.5 (ii) now implies

\[\langle S, \hat{m} \rangle = \langle S, \varphi \hat{m} \rangle\]

hence \(\langle S, \hat{m} \rangle = \langle S, \varphi \hat{m} \rangle\). The same is true if \(S\) is a norm limit of \(S_\eta\)'s with compact supports. By (i) \(\varphi \hat{m}\) is a topological mean, hence \(M_p(S) = M_p^H(S)\).

Lemma 1.10 (iii) generalizes the fact that there is no need to distinguish means and topological means on uniformly continuous functions of \(\hat{G}\) ([Gre], lemma 2.2.2).
Though we won’t use the next results in the next parts of this paper we think they are worth being noticed.

**Lemma 1.11.** — Let $G$ be a lca group, $G 	o H$, $1 \leq p \leq 2$. Let $(V_\beta)_{\beta \in B}$ be a basis of neighborhoods of $\{0\}$ in $H$ and $S \in CV_p(G)$. The following assertions are equivalent:

(i) $S$ is $H$-p-ergodic.

(ii) For every net $(f_\beta)_{\beta \in A}$ in $\mathcal{S}_p(G)$ such that for every $V_\beta$ there exists $\alpha(\beta)$ such that $f_\alpha$ is supported on $V_\beta$ for every $\alpha > \alpha(\beta)$, the net $(\langle S, f_\beta \rangle)_{\beta \in A}$ converges.

(iii) For every net $(f_\alpha)_{\alpha \in A}$ as in (ii) $(f_\alpha S)_{\alpha \in A}$ is norm converging in $CV_p(G)$.

**Proof.** — (i) $\Rightarrow$ (ii): by lemma 1.5 (i) every cluster point of $(f_\alpha)_{\alpha \in A}$ for $\sigma(A_p^{**}(G), CV_p(G))$ is a $7^-$-mean.

(ii) $\Rightarrow$ (iii) if $(f_\alpha)_{\alpha \in A}$ is a net as in (ii) such that $(f_\alpha S)_{\alpha \in A}$ is not a Cauchy filter for the norm there exists $\varepsilon > 0$ such that for every $\alpha \in A$ there exist $\alpha'' > \alpha' > \alpha$ and

$$\|f_{\alpha''}S - f_{\alpha'}S\|_{CV_p(G)} \geq \varepsilon,$$

hence by (1) there exists $g_\alpha \in \mathcal{S}_p(G)$ such that

$$|\langle f_{\alpha''}S, g_\alpha \rangle - \langle f_{\alpha'}S, g_\alpha \rangle| \geq \varepsilon \varepsilon^{-1}.$$

The net $(h_\gamma)_{\gamma \in C}$ defined by $h_{\gamma, 1} = f_{\alpha''}g_\alpha$ $h_{\gamma, 2} = f_{\alpha'}g_\alpha$ i.e. $C = (A, \{1, 2\})$ satisfies the assumptions of (ii), yet $(\langle S, h_\gamma \rangle)_{\gamma \in C}$ does not converge.

(iii) $\Rightarrow$ (i): let $(f_\alpha)_{\alpha \in A}$ be a net as in (ii). The norm limit of $(f_\alpha S)_{\alpha \in A}$ must be $M\delta_0$ where $M \in \mathbb{C}$ might depend on $(f_\alpha)_{\alpha \in A}$. Hence $M\delta_0$ belongs to the norm closure of $\mathcal{S}_p(G)S$. Let $\tilde{m}$ be a topological mean on $CV_p(G)$. Then $\langle S, \tilde{m} \rangle = \langle M\delta_0, \tilde{m} \rangle = M$ hence $M$ does not depend on the net $(f_\alpha)_{\alpha \in A}$: In particular for every net $(f_\alpha)_{\alpha \in A}$ as in (ii)

$$f_\alpha S \to M\delta_0, \quad \sigma(CV_p(G), \ A_p^{**}(G))$$

hence

$$f_\alpha S \to M\delta_0, \quad \sigma(UC_p(G), \ UC_p^*(G)).$$

As the constant function 1 belongs to $B_p(G)$ hence to $UC_p^*(G)$

$$\langle S, f_\alpha \rangle = \langle f_\alpha S, 1 \rangle \to M.$$

By lemma 1.5 (ii) this implies $\langle S, \tilde{m} \rangle = M$ for every $H$-mean $\tilde{m}$ on $CV_p(G)$.
Lemma 1.11 generalizes [L-P2], theorem 1.

Actually \((f_\alpha)_{\alpha \in A}\) in lemma 1.11 can be taken in \(\mathcal{S}_p(G)\); hence if \(S \in CV_p(G)\) is \(H\)-\(p\)-ergodic there is a scalar multiple of \(\delta_o\) in the norm closure of \(\mathcal{S}_p(G)S\) in \(CV_p(G)\).

Let \(S \in CV_p(G)\). We recall that \(K_S\) is the closure of the convex set \(\mathcal{S}_p(G)S\) for \(\sigma(CV_p(G), A_p(G))\). \(K_S\) is compact for this topology. For every \(F \in B_p(G)\) such that \(\|F\|_{B_p(G)} = F(0) = 1\) \(FS\) belongs to \(K_S\) as a limit of \((\varphi FS)_{\alpha \in A}\) where \((\varphi_\alpha)_{\alpha \in A} \in \mathcal{S}_p(G)\) is an approximate identity for \(A_p(G)\). But this does not give the whole of \(K_S\) in general (especially if \(G\) is compact). Let \(\varphi'' \in \mathcal{S}_p(G)^{00}\). We define \(\varphi''S\) as an element of \(CV_p(G)\) as follows: let \((\varphi_\alpha)_{\alpha \in A}\) be a bounded net in \(\mathcal{S}_p(G)\) converging to \(\varphi''\) for \(\sigma(A_p^{**}(G),CV_p(G))\); \(\varphi''S\) is the limit of \((\varphi_\alpha S)_{\alpha \in A}\) for \(\sigma(CV_p(G), A_p(G))\). Clearly

\[
K_S = \{\varphi''S| \varphi'' \in \mathcal{S}_p^{00}(G)\}
\]

and actually we only have to consider the restriction of \(\varphi''\)'s to \(UC_p(G)\). If \(G\) is discrete \(UC_p(G)\) is the norm closure in \(CV_p(G)\) of finitely supported measures. In this case \(UC_p(G) = B_p(G)\) by [Loh], chap. IV, theorem 1, p. 79, [H2], theorem 2, [P], proposition 19.11.

We now consider the following questions: when is a \(H\)-mean constant on \(K_S\)? when is it a Baire \(-1\) function on \(K_S\) (provided with its \(\sigma(CV_p(G), A_p(G))\) topology)?

**Lemma 1.12.** - Let \(G\) be a lea group, \(G \to H\), \(1 \leq p \leq 2\). Let \(S \in CV_p(G)\). Let \(\dot{m}\) be a \(H\)-mean which is constant on \(K_S\). Then \(\dot{m}\) coincide on \(K_S\) with a topological mean and \(S\) is topologically \(p\)-ergodic.

**Proof.** - By assumption for every \(\varphi'' \in \mathcal{S}_p^{00}(G)\) \(\langle \varphi''S, \dot{m}\rangle = M\). For every \(\varphi \in \mathcal{S}_p(G)\) \(\varphi\varphi'' S \in K_S\) hence

\[
\langle \varphi''S, \varphi\dot{m}\rangle = \langle \varphi\varphi''S, \dot{m}\rangle = M
\]

and \(\varphi\dot{m}\) is a topological mean by lemma 1.10.

Let \((f_\alpha)_{\alpha \in A}\) be a net in \(\mathcal{S}_p(G)\) converging to \(\dot{m}\) for \(\sigma(A_p^{**}(G), CV_p(G))\):

\[
\forall \varphi'' \in \mathcal{S}_p(G)^{00}, \langle f_\alpha S, \varphi'' \rangle = \langle \varphi'' S, f_\alpha \rangle \to \langle \varphi'' S, \dot{m}\rangle = M = \langle M\delta_0, \varphi'' \rangle.
\]

By Remark 1.2 it implies that \(M\delta_0\) belongs to the weak closure of \(\mathcal{S}_p(G)S\), hence to the norm closure of \(\mathcal{S}_p(G)S\) which implies the claim by lemma 1.5.
LEMMA 1.13. — Let $G$ be a lca group, $G \to H$, $1 \leq p \leq 2$. Let $S \in CV_p(G)$. The following assertions are equivalent:

(i) $S$ is $H$-p-ergodic

(ii) every $H$-mean on $CV_p(G)$ is constant on $K_S$

(iii) all $H$-means on $CV_p(G)$ are constant and equal on $K_S$.

If $H = G$ these assertions are equivalent to

(iv) there exists a topological mean which is constant on $K_S$.

Proof. — (i) $\Rightarrow$ (ii): By lemma 1.8 there exists $M \in C$ such that for every $\varepsilon > 0$ there exists $\psi \in \mathcal{F}_p(H)$ with $\|\psi S - M \delta_0\| \leq \varepsilon$ hence for every $H$-mean $\tilde{m}$ and $\varphi'' \in \mathcal{F}_p^0(G)$

$$\langle \varphi'' S, \tilde{m} \rangle = \langle \psi \varphi'' S, \tilde{m} \rangle$$

and $\|\psi S - M \delta_0\| \leq \varepsilon$

which implies $\langle \varphi'' S, \tilde{m} \rangle = M$.

(ii) $\Rightarrow$ (iii): By lemma 1.12.

(iii) $\Rightarrow$ (i): we saw that $S \in K_S$ hence the claim is obvious.

If $H = G$ (iii) $\Rightarrow$ (iv) is obvious and (iv) $\Rightarrow$ (i) by lemma 1.12.

$S \in CV_p(G)$ may be topologically p-ergodic without $K_S$ being the norm closure of $\mathcal{F}_p(G)S$: for example if $G$ is discrete, if $S$ does not belong to the norm closure of finitely supported measures, $S$ belongs to $K_S$ and not to $UC_p(G)$ hence not to $\mathcal{F}_p(G)\mathcal{S}'$, though $S$ is topologically $p$-ergodic.

LEMMA 1.14. — Let $G$ be a lca group, $G \to H$, $1 \leq p \leq 2$. Let $S \in CV_p(G)$. Then $\mathcal{F}_p(H)S$ is dense in $K_S$ for $\sigma(CV_p(G), A_p(G))$.

Proof. — As $\mathcal{F}_p(H)$ lies in $B_p(G)$ we saw that $\mathcal{F}_p(H)S$ lies in $K_S$.

By [Loh1], chap. II, theorem 1.2 or [Loh2], theorem 1, if $T \in CV_p(G)$ has a compact support it determines $\tilde{T} \in CV_p(H)$ such that $\|T\|_{CV_p(G)} = \|\tilde{T}\|_{CV_p(H)}$ and

$$\forall F \in A_p(H), \quad \langle \tilde{T}, F \rangle = \lim_{\alpha} \langle FT, \varphi_\alpha \rangle$$

where $(\varphi_\alpha)_{\alpha \in A}$ is an approximate identity (in $\mathcal{F}_p(G)$) for $A_p(G)$.

Hence there is a canonical isometry from $UC_p(G)$ to a closed unitary subalgebra $E_p$ of $UC_p(H) \subset CV_p(H)$. 
Every \( \varphi \in \mathcal{S}_p(G) \) defines a state on \( UC_p(G) \) hence it can be identified with the restriction to \( E_p \) of an element \( \tilde{\varphi} \in \mathcal{S}^0_p(H) \). Hence there exists a net \((\varphi_\beta)_{\beta \in \mathcal{B}} \) in \( \mathcal{S}_p(H) \) such that

\[
\forall f \in A_p(G), \quad \langle \varphi_\beta S, f \rangle = \langle f S, \varphi_\beta \rangle \to \int S \tilde{\varphi} = \langle \varphi S, f \rangle
\]

which proves the claim.

Lohoué's theorem is obvious if \( p = 2 \) and easy if \( G \) is discrete (see lemma 13.2 below).

Lemma 1.14 implies that a \( H \)-mean which is continuous on \( K_S \) is constant on \( K_S \).

**Proposition 1.15.** — Let \( G \) be a metric lca group, \( G \rightarrow H, 1 \leq p \leq 2 \). Let \( S \in CV_p(G) \) and let \( \check{m} \) be a \( H \)-mean on \( CV_p(G) \). If \( \langle S, \check{m} \rangle \notin M_p(S) \) \( \check{m} \) is not a Baire 1-function on \( K_S \).

**Proof.** — If \( \check{m} \) is a Baire 1-function on \( K_S \) there is an open set \( 0 \subset K_S \) such that

\[
\text{diam} \{\langle 0, \check{m} \rangle\} \leq \frac{1}{2} d(\langle S, \check{m} \rangle, M_p(S)).
\]

As \( \mathcal{S}_p(G)S \) and \( \mathcal{S}_p(H)S \) are dense in \( K_S \) by definition and lemma 1.14 there exist \( \psi \in \mathcal{S}_p(G) \) and \( \varphi \in \mathcal{S}_p(H) \) such that

\[
\text{diam} \{\langle 0, \check{m} \rangle\} \geq |\langle \psi S, \check{m} \rangle - \langle \varphi S, \check{m} \rangle | = |\langle \psi S, \check{m} \rangle - \langle S, \check{m} \rangle |.
\]

By lemma 1.10 \( \psi \check{m} \) is a topological mean, hence

\[
|\langle \psi S, \check{m} \rangle - \langle S, \check{m} \rangle | \geq d(\langle S, \check{m} \rangle, M_p(S))
\]

which is a contradiction.

If \( G \) is discrete every \( S \in CV_p(G) \) has a countable support hence \( K_S \) is metrizable and the conclusion of proposition 1.15 holds true:

If \( \check{m} \) is a \( H \)-mean and if \( \langle S, \check{m} \rangle \neq \langle S, 1_{[0]} \rangle \) \( \check{m} \) is not a Baire 1-function on \( K_S \).

For general lca group \( G \) we do not know if there exist \( H \)-means on \( CV_p(G) \) which are Baire 1-functions on \( K_S \) without being constant on \( K_S \).
2. Some subspaces of $CV_p(G)$ with Radon-Nikodym and Schur property.

A generalization of Loomis theorem.

We first prove a lemma (lemma 2.2 (b) below) which will be a key for this paper. It is obvious when $p = 2$ and is implicitly used in [W1], [W2] for $p = 2$, in [Loh1] for $1 \leq p \leq 2$. Neither in [W1] nor in [Loh] its whole strength is used.

**Lemma 2.1.** — Let $G$ be a lca group, $G \to H$, $1 \leq p \leq 2$. Let $F \subseteq G$ be a finite set. There exists a neighborhood $W$ of $\{0\}$ in $H$ such that, for every $(k, k') \in \pi_p(G)$ supported on $W \times W$, $(k \ast k') \ast P_F$ lies in $\mathcal{S}_p(G)$, where $(k \ast k') \ast P_F$ is defined by

$$(k \ast k') \ast P_F = \sum_{P_F(x_i) \neq 0} P_F(x_i)(k \ast k')_{x_i}.$$ 

**Proof.** — We choose $W$ a neighborhood of $\{0\}$ in $H$ such that the sets $x_i + W (x_i \in F)$ are pairwise disjoint. Let $(k, k') \in \pi_p(G)$ be supported on $W \times W$. Hence

(i) $1 = \left\| |F|^{-1/p} \sum_{x_i \in F} k_{x_i} \right\|_{L^p(G)} = \left\| |F|^{-1/p'} \sum_{x_j \in F} k'_{x_j} \right\|_{L^{p'}(G)}$

(ii) $1 \geq \left( |F|^{-1} \left( \sum_{x_i \in F} k_{x_i} \right) \ast \left( \sum_{x_j \in F} (k')_{x_j} \right) \right)_{\mathcal{A}_p(G)}$

(iii) $|F|^{-1} \left( \sum_{x_i \in F} k_{x_i} \right) \ast \left( \sum_{x_j \in F} (k')_{x_j} \right)$

$= |F|^{-1} \sum_{F \times F} (k \ast k')(x_i - x_j) = (k \ast k')(0) = 1.$

**Lemma 2.2.** — Let $G$ be a lca group, $G \to H$, $1 \leq p \leq 2$. a) Let $W$ be a neighborhood of $\{0\}$ in $H$.

For every $f \in \mathcal{S}_p(G)$ $\varphi_w f$ lies in the norm closed convex hull of

$\{k \ast k' | (k, k') \in \pi_p(G), (k, k') \text{ is supported on } W \times W\}.$

b) Let $F \subseteq G$ be a finite set and $m$ be a $H$-mean on $CV_p(G)$. Then $m \ast P_F$ lies in $\mathcal{S}_p^{\text{st}}(G)$, where $m \ast P_F$ is defined by

$$(m \ast P_F)_{x_i} = \sum_{P_F(x_i) \neq 0} P_F(x_i) \cdot m_{x_i}.$$
Proof. – a) The claim is proved for \( f \in \mathcal{S}_p(G) \) as soon as it is proved for \( f = g \ast g' \) where \((g,g') \in \pi_p(G)\) owing to lemma 1.1.

By the proof of [Ey] theorem 1, \((g \ast g')\varphi_w\) belongs to the norm closed convex hull of

\[
\frac{g \, |W|^{-1/p}1_w}{\|g \, |W|^{-1/p}1_w\|_{L^p(G)}} \ast \frac{g' \, |W|^{-1/p'}1_{w'}}{\|g' \, |W|^{-1/p'}1_{w'}\|_{L^{p'}(G)}} = k \ast k',
\]

where \( x \in G \), and

\[
k = \frac{g \, |W|^{-1/p}1_w}{\|g \, |W|^{-1/p}1_w\|_{L^p(G)}}, \quad k' = \frac{g' \, |W|^{-1/p'}1_{w'}}{\|g' \, |W|^{-1/p'}1_{w'}\|_{L^{p'}(G)}}.
\]

b) Let \((f_\alpha)_{\alpha \in A} \in \mathcal{S}_p(G)\) be such that \( f_\alpha \rightarrow \tilde{m}, \sigma(A_p^{**}(G), CV_p(G)) \). Let \( W \) be chosen as in lemma 2.1. By lemmas 2.1 and 2.2 (a) \((f_\alpha \varphi_w) \ast P_F \in \mathcal{S}_p(G)\). Obviously

\[
(f_\alpha \varphi_w) \ast P_F \rightarrow \tilde{m} \ast P_F, \quad \sigma(A_p^{**}(G), CV_p(G)). \quad \square
\]

The proof of lemma 2.2 b is much simpler for \( p = 2 \): let \((f_\alpha)_{\alpha \in A}\) be a net as in lemma 1.5 b. Then \( \hat{f}_\alpha \geq 0 \) hence \( \hat{f}_\alpha \hat{P}_F \geq 0 \). \(|f_\alpha \ast P_F|_{A_2(G)} = f_\alpha \ast P_F(0)\); moreover \( f_\alpha \ast P_F(0) = f_\alpha(0) P_F(0) = 1 \) as soon as the \( x_i + W(x_i \in F) \) are disjoint and \( f_\alpha \) is supported on \( W \).

Lemma 2.2 will be the main ingredient in the definition of the mappings \( A_m \) in part 4. It is also an ingredient in the proof of proposition 2.3 below, and it will be revisited in the proof of lemma 2.10 below. Proposition 2.3 is a generalization of [W1] theorem 9 (ii). We keep some arguments of his proof but his crucial use of properties of almost periodic functions is replaced by lemma 2.2.

**Proposition 2.3.** – Let \( G \) be a lca group, \( G \rightarrow H, \, 1 \leq p \leq 2 \). Let us assume that \( S \in CV_p(G) \) is \( H\)-\( p \)-ergodic at every \( x \neq 0, \, x \in G \). Then for every \( \varepsilon > 0 \) there exists \( \varphi \in \mathcal{S}_p(H) \) such that for every finite set \( F \subset G \)

\[
\left\| \sum_{x_i \neq 0 \atop \text{ \small P_F(x_i) \neq 0}} P_F(x_i) \varphi(x_i) M^H_p(S_{x_i}) \delta_{x_i} \right\|_{CV_p(G)} \leq \varepsilon.
\]

Let us write it in another way: let \( \tilde{m} \) be a \( H \)-mean on \( CV_p(G) \). Let

\[
\varphi^* = \sum_{x_i \neq 0} P_F(x_i) \tilde{m}_{x_i} = \tilde{m} \ast (P_F - 1_{(0)}) \in A_p^{**}(G).
\]
Then \( \varphi^*(\varphi S) \) defined as an element of \( CV_p(G) \) as in part 1 (description of \( K_S \)) satisfies

\[
\varphi^*(\varphi S) = \sum_{x_1 \neq 0} P_F(x_1) \varphi(x_1) M_p^H(S_{x_1}) \delta_{x_1}.
\]

Proposition 2.3 does not imply that \( S \) is \( H-p \)-ergodic at 0 in general. But if \( G \) is not discrete and if we apply it for \( G = G_d \) and \( H = G \) we get that for every \( \varepsilon > 0 \) there exists \( \varphi \in \mathcal{SP}_p(G) \) such that

\[
\forall F \text{ finite } F \subset G \| P_F(\varphi S - \langle S, 1_{[0]} \rangle \delta_0) \|_{CV_p(G_d)} \leq \varepsilon
\]

hence

\[
\| \varphi S - \langle S, 1_{[0]} \rangle \delta_0 \|_{CV_p(G_d)} \leq \varepsilon
\]

which means by lemma 1.8 that \( S \in CV_p(G_d) \) is \( G-p \)-ergodic at 0. For \( p = 2 \) this was noticed in [Gl].

Thus Proposition 2.3 easily implies the following corollary whose proof is the same as in [Gl] Corollary 2, where \( p = 2 \):

**Corollary 2.4.** - Let \( G \) be a lca group, \( 1 \leq p \leq 2 \). Let \( E \subset G \) be closed and scattered. Then every \( S \in CV_p(E_d) \subset CV_p(G_d) \) is \( G \)-totally \( p \)-ergodic.

**Proof.** - Let \( N = \{ x \in G | S \text{ is not } G-p \text{-ergodic at } x \} \). By lemma 1.8 \( N \subset E \) because \( E \) is closed in \( G \). Let \( \tilde{N} \) be the closure of \( N \) in \( E \). If \( N \) is not empty there exists \( x \in \tilde{N} \) which is an isolated point of \( \tilde{N} \) hence \( x \in N \). But there exists \( \varphi \in \mathcal{SP}_p(G) \) such that the support of \( \varphi_x S \) meets \( \tilde{N} \) only at \( \{ x \} \). By Proposition 2.3 and the remark above \( \varphi_x S \) is \( G-p \)-ergodic at \( x \) hence so is \( S \) and this is a contradiction.

**Proof of proposition 2.3.** - For every \( \varepsilon > 0 \) we choose \( W(0) \subset H \) as in lemma 1.6 and \( \varphi = \varphi_{w'} \in \mathcal{SP}_p(H) \) such that \( W' \) is an open neighborhood of \( \{ 0 \} \) in \( H \) and \( W' - W'' \subset W \). For every finite set \( F \subset G \), every \( H \)-mean \( \hat{m} \) on \( CV_p(G) \) and every \( g \in \mathcal{SP}_p(G) \) lemma 1.6 and lemma 2.2 (b) imply

\[
\langle g \varphi S, \sum_{P_F(x_1) \neq 0} P_F(x_1) \hat{m}_{x_1} \rangle \in M_p^H(S) + D_\varepsilon.
\]

On the other hand

\[
\langle g \varphi S, \sum_{P_F(x_1) \neq 0} P_F(x_1) \hat{m}_{x_1} \rangle = \langle S, \hat{m} \rangle + \sum_{P_F(x_1) \neq 0} P_F(x_1) g(x_1) \varphi(x_1) \langle S, \hat{m}_{x_1} \rangle.
\]
Hence for every $g \in \mathcal{S}_p(G)$, as $S$ is $H$-p-ergodic at every $x \neq 0$

$$M_p^H(S) + \left\langle \sum_{P_p(x_i) \neq 0} P_p(x_i) \varphi(x_i) M_p^H(S_{x_i}) \delta_{x_i} g \right\rangle \leq M_p^H(S) + D_c.$$ 

Hence

$$\sup_{g \in \mathcal{S}_p(G)} \left| \sum_{P_p(x_i) \neq 0} P_p(x_i) \varphi(x_i) M_p^H(S_{x_i}) \delta_{x_i} g \right| \leq \varepsilon$$

which implies by (1)

$$\left\| \sum_{P_p(x_i) \neq 0} P_p(x_i) \varphi(x_i) M_p^H(S_{x_i}) \delta_{x_i} \right\|_{CV_p(G)} \leq \varepsilon \varepsilon.$$ 

In order to prove our generalization of Loomis theorem (theorem 2.8 below) we now state the obvious generalization of a part of the original proof.

**Definition 2.5.** — Let $G$ be a lca group, and $1 \leq p \leq 2$. An element $S \in CV_p(G)$ is $p$-almost periodic if $S \in \ell^1(G)^{\hat{\ominus}}(CV_p(G))$ i.e. if $S$ lies in the norm closure in $CV_p(G)$ of finitely supported measures. $S$ is said to be $p$-almost periodic at $x \in G$ if there exists $f \in A_2(G)$ such that $f(x) \neq 0$ and $fS$ is $p$-almost periodic.

Equivalent definitions of $p$-almost periodic elements of $CV_p(G)$ are given in theorem 4.8 below.

**Lemma 2.6.** — Let $G$ be a lca group, $G \to H$, $1 \leq p \leq 2$.

a) If $S \in CV_p(G)$ is $p$-almost periodic, $S$ is totally $H$-p-ergodic and for every $\varepsilon > 0$ there exists a finite set $F \subset G$ such that for every $H$-mean $\tilde{m}$

$$\|S - (\tilde{m} \ast P_F)S\|_{CV_p(G)} \leq \varepsilon$$

and $S - (\tilde{m} \ast P_F)S \in \ell^1(G)^{\hat{\ominus}}(CV_p(G))$.

b) If $S \in CV_p(G)$ has a compact support $K$ and is $p$-almost periodic at every point of $K$, $S$ is $p$-almost periodic.

c) If $S \in CV_p(G)$ has a compact support $K$, such that $0 \in K$, is $p$-almost periodic at every $x \in K$, $x \neq 0$, and topologically p-ergodic at 0, $S$ is $p$-almost periodic.
Proof. — a) $\langle m \ast P_p \rangle S$ is defined as in part 1 (see also proposition 2.3) as a finitely supported measure. Moreover for every $S' \in CV_p(G)$
\[ \|\langle m \ast P_p \rangle S'\|_{CV_p(G)} \leq \|S'\|_{CV_p(G)} \]
by definition and lemma 2.2. Both assertions of (a) are obvious if $S$ is a finitely supported measure and verified by norm density if $S \in \ell^1(G) \cap CV_p(G)$. (These facts will be used again in lemma 3.2 and theorem 4.1.)

The proof of b) is analogue to [Loo] theorem 1: there exist $(f_j)_{1 \leq j \leq n} \in A_2(G)$ such that $f_j S$ is $p$-almost periodic and $\sum_{1 \leq j \leq n} f_j > 0$ on $K$, there exists $f \in A_2(G)$ such that $f \left( \sum_{1 \leq j \leq n} f_j \right) = 1$ in a neighborhood of $K$ hence $S = \sum_{1 \leq j \leq n} f f_j S$ is $p$-almost periodic.

c) Every $\psi S$ defined as in lemma 1.8 (iii) satisfies the assumptions of (b), hence $\psi S$ is $p$-almost periodic and so is $S$ by lemma 1.8. \[\square\]

We now prove a generalization of [Loo] theorem 2.3, but with a different proof: it will be a consequence of proposition 2.3.

**Proposition 2.7.** — Let $G$ be a lca group, $1 \leq p \leq 2$. Let $S \in CV_p(G)$ with a compact support $K$ such that $0 \in K$. If $S$ is $p$-almost periodic at every $x \in k$ except $\{0\}$ then $S$ is $p$-almost periodic.

**Proof.** — By lemma 2.6 it is enough to show that $S$ is topologically $p$-ergodic at $\{0\}$. $S$ verifies the assumptions of Proposition 2.3 for $H = G$. For every $\varepsilon > 0$ we choose $\varphi \in S_p(G)$ as in proposition 2.3 and we choose $f, g \in S_p(G)$ such that
\[ \text{diam } M_p(S) - \varepsilon = \text{diam } M_p(\varphi S) - \varepsilon = |\langle \varphi S, f - g \rangle| . \]
As $\{0\}$ is a set of synthesis for $A_p(G)$ our assumption on $S$ implies that $(f - g) \varphi S$ is $p$-almost periodic at every $x \in G$ hence $p$-almost periodic by lemma 2.6 b). By lemma 2.6 a), for every $\varepsilon > 0$ there exists a finite set $F \subset G$ such that for any mean $\hat{m}$ on $CV_p(G)$
\[ \| (f - g) \varphi S - (\hat{m} \ast P_p)(f - g) \varphi S \|_{CV_p(G)} \leq \varepsilon . \]
Let $W \subset G$ be a compact set such that
\[ \| (f - g) - (f - g) \varphi \|_{A_p(G)} \leq \varepsilon \| S \|_{CV_p(G)} . \]
Hence by our choice of $\varphi$

\[ |\langle \varphi S, f - g \rangle| \leq \varepsilon + |\langle (f - g)\varphi S, \varphi_w \rangle| \leq 2\varepsilon + |\langle (\hat{m} \ast P_{p})(f - g)\varphi S, \varphi_w \rangle| \]
\[ = 2\varepsilon + |\langle (\hat{m} \ast (P_{p} - 1_{\{0\}}))\varphi S, \varphi_w (f - g) \rangle| \leq 4\varepsilon. \]

Hence diam $M_p(S) \leq 5\varepsilon$ and $S$ is topologically $p$-ergodic at $\{0\}$.

**Theorem 2.8.** — Let $G$ be a lca group, $1 \leq p \leq 2$.

a) Let $E \subset G$ be compact and scattered. Then $CV_p(E) = M(E)^{\perp} = CV_p(E)$ and $CV_p(E)$ has Radon-Nikodym property.

b) If $E \subset G$ is compact and not scattered $CV_p(E)$ does not have Radon-Nikodym property nor Schur property.

Theorem 2.8 is obvious for $p = 1$. For $p = 2$ theorem 2.8 (a) is Loomis theorem [Loo].

**Proof.** — a) Proposition 2.7 implies that every $S \in CV_p(E)$ is $p$-almost periodic at every $x \in G$ exactly as in [Loo] proof of theorem 4. or as in the proof of corollary 2.4 above. Lemma 2.6 finishes the proof of the first assertion. Every separable subspace of $CV_p(E)$ is a subspace of $CV_p(E')$ where $E'$ is a separable closed subset of $E$. Hence $E'$ is compact and countable. By the first assertion $CV_p(E')$ is separable, and it is a dual space. Hence $CV_p(E')$ and $CV_p(E)$ have RNP.

b) The proof is the same as for $p = 2$ [L-P1] proposition 3 : By [V] chap. 4.3, $E$ has a closed perfect subset $E'$ such that

\[ M(E') = CV_2(E') = CV_p(E') \]

and $M(E')$ does not have RNP nor the Schur property. \hfill \Box

Theorem 2.8 (a) implies the following corollary exactly as Loomis theorem implies [Gl] Proposition 4:

**Corollary 2.9.** — Let $G$ be a lca group and let $F \subset G$ be closed and scattered. Then every $S \in CV_p(E)(1 \leq p \leq 2)$ is totally topologically $p$-ergodic.

**Proof.** — We prove that $S$ is topologically $p$-ergodic at $\{0\}$. Let $f \in \mathcal{L}_p(G)$ with a compact support. The support of $fS$ is compact and scattered hence by theorem 2.8 (a) and lemma 2.6 $fS$ is topologically $p$-ergodic at $\{0\}$ hence so is $S$. 
Our aim now is to prove (theorem 2.14 below) that under the assumptions of theorem 2.8 (a) $CV_p(E)$ has the Schur property. Exactly as in the case $p = 2$ [L-P1] theorem 1, we begin with the case where $E$ is a convergent sequence. The following lemma is crucial. It is a generalization of [W1], proof of theorem 9 (ii), and the proof uses the same ideas as lemma 2.2, proposition 2.3 above.

**Lemma 2.10.** Let $G$ be a lca group and $E = (e_k)_{k \geq 1} \subset G$ be a sequence such that $e_k \to 0 (k \to +\infty)$ and $e_k \neq 0 (k \geq 1)$. Let $1 \leq p \leq 2$.

a) For every $N \geq 1$ and $\varepsilon > 0$ there exists $W_{N,\varepsilon}$ a neighborhood of $\{0\}$ in $G$ such that for every $f, g \in \mathcal{S}_p(G)$ there exists $h \in \mathcal{S}_p(G)$ such that

(i) $\|g - h\|_{A_p(E^N)} \leq 2\varepsilon$

(ii) $\|f - g\|_{A_p(W_{N,\varepsilon})} \leq 2\varepsilon$

where $E^N = \{e_1, \ldots, e_N\}$.

b) Let $0$ be an open subset of the compact metric topological space $\mathcal{S}^{00}_p(G)$ provided with $\sigma(A^{**}_p(G), CV_p(E))$. There exists $W$ a neighborhood of $\{0\}$ in $G$ such that for every $S \in CV_p(E)$ which is supported on $W$ and every topological mean $\tilde{m}$ on $CV_p(G)$

(iii) $\sup_{f \in \mathcal{S}_p(G)} |\langle S - \langle S, \tilde{m}\rangle \delta_0, f \rangle| \leq 2 \sup_{h \in 0} |\langle S - \langle S, \tilde{m}\rangle \delta_0, h \rangle|,$

(iv) $\|S - \langle S, \tilde{m}\rangle \delta_0\|_{CV_p(G)} \leq 2\varepsilon \text{ diam } \{\langle S, 0 \rangle\}.$

**Proof.** a) Let $(g_i)_{i \in I_{N,\varepsilon}}$ be a finite family in $\mathcal{S}_p(G)$ such that

(v) $\forall g \in \mathcal{S}_p(G), \exists i \in I_{N,\varepsilon}, \|g - g_i\|_{A_p(E^N)} \leq \varepsilon.$

As $\{0\}$ is a set of synthesis for $A_p(G)$ there exists $V_{N,\varepsilon}$ a neighborhood of $\{0\}$ in $G$ such that

(vi) $\forall i \in I_{N,\varepsilon}, \|g_i - 1\|_{A_p(V_{N,\varepsilon})} \leq \varepsilon,$

where $V_{N,\varepsilon}$ is the closure of $V_{N,\varepsilon}$ in $G$.

There exists a finite set $F_{N,\varepsilon} \subset G$ such that

(vii) $\|1 - P_{F_{N,\varepsilon}}\|_{A_2(E^N)} \leq \sum_{k=1}^{N} |1 - P_{F_{N,\varepsilon}}(e_k)| \leq \varepsilon.$

There exists $V'_{N,\varepsilon}$ a neighborhood of $\{0\}$ in $G$ such that $V'_{N,\varepsilon} - V'_{N,\varepsilon} \subset V_{N,\varepsilon}$, and the $x_i + V'_{N,\varepsilon} - V'_{N,\varepsilon}(x_i \in F_{N,\varepsilon} \cup \{F_{N,\varepsilon} - F_{N,\varepsilon}\})$ are
pairwise disjoint. There exists $W_{N,\varepsilon}$ a neighborhood of \{0\} in $G$ such that

(viii) $\| 1 - \varphi_{V_N} \|_{\mathcal{L}(\mathcal{W}_{N,\varepsilon})} \leq \varepsilon$.

For every $f \in \mathcal{S}_p(G)$, $(\varphi_{V_N} f) \ast P_{F_N,\varepsilon} \in \mathcal{S}_p(G)$ by lemmas 2.1, 2.2 (a).
Hence by (vii)

(ix) $\forall i \in I_{N,\varepsilon} ((\varphi_{V_N} f) \ast P_{F_N,\varepsilon}) g_i \in \mathcal{S}_p(G)$.

(x) $\| g_i - (\varphi_{V_N} f) \ast P_{F_N,\varepsilon} g_i \|_{\mathcal{L}(\mathcal{W}_{N,\varepsilon})} = \| g_i \sum_{k=1}^{K} (1 - \Pi_{F_N,\varepsilon}(e_k)) \|_{\mathcal{L}(\mathcal{W}_{N,\varepsilon})} \leq \varepsilon$.

For every $g \in \mathcal{S}_p(G)$ we choose $i_0 \in I_{N,\varepsilon}$ such that $\| g - g_{i_0} \|_{\mathcal{L}(\mathcal{W}_{N,\varepsilon})} \leq \varepsilon$.
Let $h = ((\varphi_{V_N} f) \ast P_{F_N,\varepsilon}) g_{i_0}$.

Then $h \in \mathcal{S}_p(G)$ by (ix) and satisfies (i) by our choice of $g_{i_0}$ and (x). Moreover by our choice of $V''_{N,\varepsilon}$, (viii) and (vi)

$\| f - h \|_{\mathcal{L}(\mathcal{W}_{N,\varepsilon})} \leq \| f - (\varphi_{V_N} f) \ast P_{F_N,\varepsilon} \|_{\mathcal{L}(\mathcal{W}_{N,\varepsilon})} + \| (\varphi_{V_N} f) \ast P_{F_N,\varepsilon} - h \|_{\mathcal{L}(\mathcal{W}_{N,\varepsilon})} \leq 2\varepsilon$
which proves (ii).

b) Let 0 be as in the statement. By theorem 2.8 (a) there exist $h_0 \in \mathcal{S}_p(G)$, $N$ and $0 < \varepsilon < (6\varepsilon)^{-1}$ such that

$0 \Rightarrow \{ h \in \mathcal{S}_p^{00}(G) \mid \forall 1 \leq k \leq N \quad | h(e_k) - h_0(e_k) | < 2\varepsilon \}$.

Let $W = W_{N,\varepsilon}$ be chosen as in (a). Let $S \in CV_p(E)$ which is supported on $W$ and let $f \in \mathcal{S}_p(G)$ be such that

(xi) \((1 - \varepsilon) \sup_{f' \in \mathcal{S}_p(G)} | \langle S - \langle \hat{S}, \hat{m} \rangle \delta_0, f' \rangle | \leq | \langle S - \langle \hat{S}, \hat{m} \rangle \delta_0, f \rangle | \).

Let us define $h$ as in (a) for this $f$ and $g = h_0$. By (i) $h \in 0$ and (ii), (xi) imply (iii) via (1).

We now prove (iv): let $P_{F_N,\varepsilon}$ be defined as in (a) and let

$h' = (\hat{m} \ast P_{F_N,\varepsilon}) h_0$.

By lemma 2.2 (b) $h' \in \mathcal{S}_p^{00}(G)$; for $k \geq 1 \langle \delta_0, h' \rangle = P_{F_N,\varepsilon}(e_k) h_0(e_k)$ hence $h' \in 0$ by (vii). By (ii) and our choice of $W$

(xii) $| \langle S - \langle \hat{S}, \hat{m} \rangle \delta_0, f \rangle | = | \langle S, f \rangle - \langle \hat{S}, \hat{m} \rangle | = | \langle S, f \rangle - \langle S, h' \rangle | \leq 2\varepsilon | S - \langle \hat{S}, \hat{m} \rangle \delta_0 \|_{CV_p(G)} + | \langle S, h \rangle - \langle S, h' \rangle |$.

Hence (xi) and (xii) imply (iv) via (1).
PROPOSITION 2.11. – Let $G$ be a lca group and $E \subset G$ be a compact countable set with only one cluster point. Then $CV_p(E)$ $(1 \leq p \leq 2)$ has the Schur property.

Proof. – We assume that $E = (e_k)_{k \geq 1}$ as in lemma 2.10. Let $(S_n)_{n \geq 1}$ be a sequence in $CV_p(E)$ such that $S_n \to 0 \sigma(CV_p(E), A_p^{**}(G))$. By theorem 2.8 (a) and by eventually extracting a subsequence we may assume that there exists a sequence $(S'_n)_{n \geq 1}$ of measures whose finite support lies in $E \setminus \{0\}$, such that $\|S_n - S'_n\|_{CV_p(E)} \leq 2^{-n}(n \geq 1)$ and the $S'_n$ are supported on disjoint blocks $\{e_{k_n}, e_{k_n+1}, \ldots, e_{k_{n+1}-1}\}$ where $(k_n)_{n \geq 1}$ is a strictly increasing sequence of positive integers. In order to prove the claim we assume that

$$\exists \delta > 0, \forall n \geq 1, \|S'_n\|_{CV_p(G)} > \delta$$

and we will show that this is impossible.

Let $C = \sup_{n} \|S_n\|_{CV_p(G)}$; we may assume that $\|S'_n\|_{CV_p(G)} \leq 2C$. Let $\epsilon = \delta(8eC)^{-1}$. We define a subsequence $(S'_{n(j)})_{j \geq 1}$ and a decreasing sequence $(0_j)_{j \geq 1}$ in $\mathcal{P}_p^0(G)$ in the following way: $0_1 = \mathcal{P}_p^0(G)$; assume that $0_j$ and $S'_{n(j-1)}$ have been defined; by lemma 2.10 define a neighborhood $W_j$ of $\{0\}$ in $E$ such that assertion (iii) is satisfied for $0_j$ and $\epsilon$; choose $n(j) > n(j-1)$ such that $S'_{n(j)}$ is supported on $W_j$, and $0_{j-1}$ such that

$$0_{j+1} = \{h \in 0_j | \langle S_{n(j)}, h \rangle | \geq \sup_{h' \in 0_j} \langle S'_{n(j)}, h' \rangle | - e\|S'_{n(j)}\| \}.$$ 

Take $h_j$ in the closure of $0_j$ for $\sigma(A_p^{**}(G), CV_p(E))$ such that

$$|\langle S'_{n(j)}, h_j \rangle| = \sup_{h' \in 0_j} |\langle S'_{n(j)}, h'_j \rangle|, \quad j \geq 1.$$

Let $h_0 \in \mathcal{P}_p^0(G)$ be a cluster point of $(h_j)_{j \geq 1}$ for $\sigma(\mathcal{P}_p^0(G), CV_p(E))$.

Then

$$\forall j \geq 1, \ |\langle S'_{n(j)}, h_0 \rangle| \geq \frac{1}{2} \sup_{f \in \mathcal{P}_p^0(G)} |\langle S'_{n(j)}, f \rangle| - 2eC \geq \delta/4e$$

by (1). Hence $(S'_{n(j)})_{j \geq 1}$ does not converge weakly to zero, which is a contradiction.

This proof is similar to [L-P1] lemma 2. It is sufficient in order to prove theorem 2.14 below. But proposition 2.11 can be improved as follows:

...
DEFINITION 2.12. — A Banach space $X$ has the strong Schur property if there exists $C > 0$ such that for every $0 < \delta < 2$ and every sequence $(x_n)_{n \geq 1}$ in $X$ such that

(i) $\|x_n\| \leq 1$ ($n \geq 1$)

(ii) $\|x_n - x_k\| \geq \delta$ ($n \neq k$)

there exists a subsequence $(x_{n_k})_{k \geq 1}$ such that

(iii) $\forall \alpha_1, \ldots, \alpha_N \in \mathbb{C}$, $\left\| \sum_{k=1}^{N} \alpha_k x_{n_k} \right\| \geq \delta C \sum_{k=1}^{N} |\alpha_k|$.

PROPOSITION 2.13. — Let $G$ be a lca group and $E \subset G$ be a compact countable set with only one cluster point. Then $CV_p(E)$ $(1 \leq p \leq 2)$ has the strong Schur property.

Proof. — By (1) we can consider $CV_p(E)$ as a closed subspace of the continuous functions on the compact space $\mathcal{F}_p^\infty(G)$ provided with the $\sigma(A_p^{**}(G), CV_p(E))$ topology. As $CV_p(E)$ is separable by theorem 2.8 (a) this topology is metrizable. Proposition 2.13 is thus implied by theorem B of [S], if we replace lemma 1 of [S] by lemma 2.10 (b).

We do not know whether $CV_p(E)$ still has the strong Schur property when $E$ is compact countable with an infinite number of cluster points.

THEOREM 2.14. — Let $G$ be a lca group, let $E \subset G$ be compact and scattered. Then $CV_p(E)$ $(1 \leq p \leq 2)$ has the Schur property.

Proof. — As we deal with sequences of elements in $CV_p(E)$ theorem 2.8 (a) shows that we actually work in $CV_p(E_1)$ where $E_1 \subset E$ is compact and countable. We can now use the proof of [L-P1] theorem 1, writing « $CV_p(E_1)$ » instead of « $PM(E)$ ». The proof uses transfinite induction and deduces the general case from the particular case where $E_1$ has only one cluster point i.e. from proposition 2.11. □

3. A consequence of theorems 2.8 and 2.14.

Let $G$ be a lca group, $1 \leq p \leq 2$.

We denote by $X_p(G)$ the closed subspace of $CV_p(G_d)$ of those elements which are totally $G$-p-ergodic, and by $Y_p(G)$ the closed subspace of $CV_p(G)$ of those elements which are totally topologically p-ergodic.
We first show the existence of bounded linear mappings $B_\omega : CV_p(G_\omega) \to CV_p(G) (1 \leq p \leq 2)$ which are identity on finitely supported measures on $G$. They were already defined in [L-P2] for $p = 2$.

**Theorem 3.1.** Let $G$ be a lca group, $1 \leq p \leq 2$. Let $(P_F)_{F \in \mathcal{F}}$ be an approximate identity in $A_2(G_\omega)$. Let $\omega$ be a cluster point of $(P_F)_{F \in \mathcal{F}}$ for $\sigma(A_2^*(G_\omega), CV_2(G_\omega))$. Let us define $B_\omega : CV_p(G_\omega) \to CV_p(G)$ by

$$\forall f \in A_p(G), \forall S \in CV_p(G_\omega), \quad \langle B_\omega(S), f \rangle = \langle fS, \omega \rangle.$$ 

This mapping has the following properties:

(i) $\|B_\omega\|_{CV_p(G_\omega) \to CV_p(G)} \leq 1$.

(ii) $B_\omega$ restricted to finitely supported measures is identity.

(iii) $B_\omega$ commutes with multiplication by elements of $B_p(G)$.

(iv) If $\Lambda \subset G$ and $\bar{\Lambda}$ is the closure of $\Lambda$ in $G$, $B_\omega$ maps $CV_p(\Lambda)$ into $CV_p(\bar{\Lambda})$.

(v) $B_\omega$ is one to one on $X_p(G)$ and sends $X_p(G)$ into $Y_p(G)$.

**Proof.**

(i) By definition $\omega \in \mathcal{F}_0^0(G_\omega) \subset \mathcal{F}_p^0(G_\omega)$. By [Ey] theorem 1, $A_p(G)$ is a subspace of $B_p(G_\omega)$ hence $\langle fS, \omega \rangle$ is well defined and

$$|\langle fS, \omega \rangle| \leq \|fS\|_{CV_p(G_\omega)} \leq \|S\| \|f\|_{A_p(G)}.$$ 

(ii) As $P_F(x) \to 1(F \in \mathcal{F})$ for every $x \in G$,

$$\langle f\delta_x, \omega \rangle = f(x) = \langle \delta_x, f \rangle$$

for every $f \in A_p(G)$ hence $B_\omega(\delta_x) = \delta_x$.

(iii) By [Ey] theorem 1, $B_p(G)$ is a subspace of $B_p(G_\omega)$ hence (iii) holds by the definition of $B_\omega$.

(iv) is obvious from the definitions.

(v) Let $S \subset CV_p(G_\omega)$, $S \neq 0$. Hence there exists $x_0 \in G$ such that $\langle S, 1_{[x_0]} \rangle \neq 0$. If moreover $S \subset X_p(G)$, $M_p^G(S_x) = \langle S, 1_{[x]} \rangle$ for every $x \in G$. By Lemma 1.8 for every $\varepsilon > 0$ and $x \in G$ there exists $\varphi \in \mathcal{F}_p(G)$ such that $\|\varphi_xS - \langle S, 1_{[x]} \rangle \delta_x\|_{CV_p(G_\omega)} \leq \varepsilon$. By (i), (ii), (iii) $\|\varphi_xB_\omega(S) - \langle S, 1_{[x]} \rangle \delta_x\|_{CV_p(G)} \leq \varepsilon$ which implies by lemma 1.8 again that $B_\omega(S) \in Y_p(G)$ and that $\varphi_{x_0}B_\omega(S)$ is not zero for a suitable $\varphi$.

The following lemma is proved in [Lohl] chap. 2, theorem 1.1, proposition 3.2.0. Actually a more general result is proved there and we recall a short proof for this particular case.
LEMMA 3.2. — Let $G$ be a lca group, $1 \leq p \leq 2$. Let $\mu$ be a finitely supported measure on $G$. Then $\|\mu\|_{CV_p(G)} = \|\mu\|_{CV_p(G_d)}$.

Proof. — The inequality $\|\mu\|_{CV_p(G_d)} \leq \|\mu\|_{CV_p(G)}$ is proved by a computation similar to the proof of lemma 2.1: Let $k, k'$ be finitely supported functions in the unit sphere of $L^p(G_d)$ and $L^p(G_d)$ respectively. Let $W$ be an open neighborhood of $\{0\}$ in $G$ such that the $x_i + W - W$ are pairwise disjoint for $x_i$ lying in the union of the supports of $k, k', \mu$. Hence

(i) $\langle \mu, k * k' \rangle = \langle \mu, (k * k')(\varphi) \rangle$

(ii) $(k * k')(\varphi) = \left( |W|^{-1/p} \sum_{k(x) \neq 0} k(x_1)(1\!w) x_1 \right) \ast \left( |W|^{-1/p} \sum_{k'(x) \neq 0} k'(x_1)(1\!\!w) x_1 \right)$

(iii) $1 = \left\| |W|^{-1/p} \sum_{k(x) \neq 0} k(x_1)(1\!w) x_1 \right\|_{L^p(G)}$

hence $(k * k')(\varphi)$ belongs to the unit ball of $\ell^p(G)$.

The converse inequality $\|\mu\|_{CV_p(G)} \leq \|\mu\|_{CV_p(G_d)}$ comes from theorem 3.1 (i) and (ii).

We can now prove a consequence of theorem 2.8 and 2.14; for $p = 2$ it was proved in [L-P1] theorem 3 and partly in [L-P3] theorem 2.2, by two different methods.

THEOREM 3.3. — Let $G$ be a discrete abelian group and $\Lambda \subset G$. We assume that there exists a lca group $H$ such that $G \to H$ (as it was defined in part 1) and the closure $\Lambda$ of $\Lambda$ in $H$ is compact and scattered. Then $CV_p(\Lambda)$ is the norm closure in $CV_p(G)$ of finitely supported measures on $\Lambda$; it has the Radon-Nikodym and the Schur property.

We give a first proof which is similar to [L-P1] proposition 2, theorem 3, but simpler, owing to corollary 2.4.

Proof. — By assumption $G$ is a closed subgroup of $H_d$ hence by [H1] theorem 3, $CV_p(G)$ is a closed subspace of $CV_p(H_d)$ and $CV_p(\Lambda)$ is a closed subspace of $CV_p((\Lambda)_d) \subset CV_p(H_d)$. By theorem 3.1 (iv) and theorem 2.8, $B_\Lambda: CV_p((\Lambda)_d) \to \ell^1(\Lambda)_{CV_p(H)}$. By lemma 3.2 there exists an isometry which we denote by $A: \ell^1(\Lambda)_{CV_p(H)} \to \ell^1(\Lambda)_{CV_p(H_d)}$ which is identity when restricted to finitely supported measures.
By corollary 2.4 \( CV_p((\Lambda)_d) \) lies in \( X_p(H) \), hence with the notations of the proof of theorem 3.1 (v) for every \( S \in CV_p((\Lambda)_d) \) and \( x \in G \)

\[
\langle A \circ B_\sigma(S), 1_{[x]} \rangle - \langle S, 1_{[x]} \rangle \leq \|A\| \|\phi_x B_\sigma(S) - S, 1_{[x]} \|_{CV_p(H)} \leq \varepsilon
\]

which implies that \( A \circ B_\sigma \) is identity on \( CV_p((\Lambda)_d) \). This proves that \( CV_p((\Lambda)_d) = \ell^1((\Lambda))^{cvp(H)} \); as \( \|B_\sigma\| \leq 1 \) this proves also that \( B_\sigma \) is an isometry: \( CV_p((\Lambda)_d) \to CV_p(\Lambda) \). Hence theorem 2.8 and 2.14 imply that \( CV_p((\Lambda)_d) \) and its subspace \( CV_p(\Lambda) \) have RNP and the Schur property.

Alternatively theorem 3.3 has another proof which is similar to [L-P3] theorem 2.2: We keep the previous notations. By lemma 3.2 the spaces \( \ell^1((\Lambda)_d) \) and \( \ell^1((\Lambda))^{cvp(H)} \) are isometric, hence by theorem 2.8 and 2.14 the first one has RNP and the Schur property. It remains to prove that this space is the same as \( CV_p((\Lambda)_d) \) which is a consequence of the following lemma, a generalization of [L-P3] theorem 2.1:

**Lemma 3.4.** — Let \( G \) be a discrete abelian group, \( \Lambda \subset G \), \( 1 \leq p \leq 2 \). Then \( \ell^1((\Lambda))^{cvp(G)} \) has RNP iff it coincides with \( CV_p(\Lambda) \).

**Proof.** — Let \( S \in CV_p(\Lambda) \). It defines a bounded multiplier: \( A_2(G) \to CV_p(\Lambda), f \mapsto fS \). As functions with finite support are dense in \( A_2(G) \) the range of this multiplier lies in \( \ell^1((\Lambda))^{cvp(G)} \). If this space has RNP there exists a bounded strongly measurable function \( F: \hat{G} \to \ell^1((\Lambda))^{cvp(G)} \) such that

\[
\forall f \in A_2(G), \quad fS = \int_{\hat{G}} \hat{f}(\gamma)F(\gamma) d\gamma.
\]

In particular for every \( \gamma' \in \hat{G} \)

\[
\int_{\hat{G}} \hat{f}(\gamma)S(\gamma' - \gamma) d\gamma = fS(\gamma') = \int_{\hat{G}} \hat{f}(\gamma)\hat{F}(\gamma)(\gamma') d\gamma
\]

hence for almost all \( \gamma \in \hat{G}, \quad \hat{F}(\gamma)(\gamma') = (\gamma S)(\gamma') \) and \( F(\gamma) = \gamma S \). In particular \( S \in \ell^1((\Lambda))^{cvp(G)} \).

Conversely if \( \ell^1((\Lambda))^{cvp(G)} = CV_p(\Lambda) \) the same equality is true for every countable subset \( \Lambda' \subset \Lambda \); hence \( CV_p(\Lambda') \) is a separable dual and

has RNP. This implies that every separable subspace of $CV_p(\Lambda)$ (which is a subspace of a $CV_p(\Lambda')$ where $\Lambda'$ is countable) has RNP, hence $CV_p(\Lambda)$ has RNP.

**Definition 3.5.** Let $G$ be a discrete group, $\Lambda \subset G$, $1 \leq p \leq 2$. If $\ell^1(\Lambda)_{\| CV_p(G)} = CV_p(\Lambda)$ we call $\Lambda$ a $p$-Rosenthal set.

Obviously every $\Lambda$ is a $1$-Rosenthal set and a $2$-Rosenthal set is usually called a Rosenthal set. Theorem 3.3 gives examples of sets $\Lambda$ which are $p$-Rosenthal for every $1 \leq p \leq 2$. We do not know whether « $\Lambda$ is $p$-Rosenthal » implies « $\Lambda$ is $q$-Rosenthal » for $1 < q < p$, but we have the following result:

**Lemma 3.6.** Let $G$ be a countable discrete abelian group and $\Lambda \subset G$. Let $1 < q < p \leq 2$. Let $\Lambda$ be a $p$-Rosenthal set.

a) Every bounded sequence in $A_p(\Lambda)$ has a weak Cauchy subsequence.

b) If $\ell^1(\Lambda)_{\| CV_p(G)}$ is weakly complete $\Lambda$ is $q$-Rosenthal.

**Proof.** a) By assumption $CV_p(\Lambda)$ is a separable dual. Hence its predual $A_p(\Lambda)$ has no $\ell^1$-sequence. Rosenthal's theorem [R] implies the claim.

b) Let $(P_{F_n})_{n \geq 1}$ be an approximate identity in $A_2(G)$. By (a) the sequence $(R(P_{F_n}))_{n \geq 1}$ of restrictions to $\Lambda$ has a weak-Cauchy subsequence in $A_p(\Lambda)$. As identity: $CV_q(\Lambda) \rightarrow CV_p(\Lambda)$ is continuous, so is: $A_p(\Lambda) \rightarrow A_q(\Lambda)$. Hence $(R(P_{F_n}))_{n \geq 1}$ has a weak Cauchy subsequence in $A_q(\Lambda)$. For every $S \in CV_q(\Lambda)$, $n \geq 1$, $P_{F_n} S = R(P_{F_n}) S \in \ell^1(\Lambda)_{\| CV_q(G)}$ and $P_{F_n} S \rightarrow S$, $\sigma(CV_q(\Lambda), A_q(\Lambda))$. It also has a weak Cauchy subsequence in $\ell^1(\Lambda)_{\| CV_q(G)}$ hence it converges weakly to $S$ and $S$ lies in $\ell^1(\Lambda)_{\| CV_q(G)}$. \[\Box\]

If $\Lambda \subset G$ is a Sidon set identity is continuous (by definition):

$$\ell^1(\Lambda) \rightarrow CV_p(\Lambda) \rightarrow CV_2(\Lambda) \rightarrow \ell^1(\Lambda).$$

If $\Lambda_2 \subset G_1$ and $\Lambda_2 \subset G_2$ are two Sidon sets we have

$$\ell^1(\Lambda_1 \times \Lambda_2) \rightarrow CV_p(\Lambda_1 \times \Lambda_2) \rightarrow CV_2(\Lambda_1 \times \Lambda_2) = \ell^1 \otimes \ell^1.$$

Is $\Lambda_1 \times \Lambda_2$ a $p$-Rosenthal set for $1 < p < 2$? (This is true if $\Lambda_1$ and $\Lambda_2$ satisfy the assumptions of theorem 3.3 because $\Lambda_1 \times \Lambda_2$ also satisfy them.) We can also define $p$-Riesz sets as follows:
DEFINITION 3.7. — Let $G$ be a discrete abelian group and $\Lambda \subset G$. $1 < p \leq 2$. $\Lambda$ is a $p$-Riesz set if every $f \in B_p(G)$ which is supported on $\Lambda$ lies in $A_p(G)$.

A 2-Riesz set is usually called a Riesz set. We do not define 1-Riesz sets because $A_1(G) = C_0(G)$, $B_1(G) = \ell^\infty(G)$ hence no infinite set is 1-Riesz. In order to generalize results on Riesz sets for $p$-Riesz sets ($1 < p < 2$) it is necessary to know whether $A_p(G)$ is weakly complete or not when $G$ is discrete, which the author does not know.

(This is true if $G$ is compact by [L-P4] theorem 4.)

If there exists $f \in B_2(G)$ which is supported on $\Lambda$ and such that $f \notin C_0(G)$ $\Lambda$ is not a $p$-Riesz set for any $1 < p \leq 2$ because $f$ is not in $A_p(G)$. This is the case if $\Lambda$ contains the spectrum of a Riesz product.

4. Transfer theorems.

We have already proved one transfer theorem, namely theorem 3.1. We now prove a «converse» one, by defining mappings $A_{\tilde{m}} : CV_p(G) \to CV_p(G_d)$. Actually all these mappings will coincide on $\ell^1(G)_{||CV_p(G)}$ and their common restriction is the mapping $A$ which we already used in the proof of theorem 3.3. Mappings $A$ and $B_n$ were already used implicitly in [Lohl], [Loh2]. For $p = 2$ $A_{\tilde{m}}$ was defined in [W2], p. 104 and [W1], p. 292, on $UC_2(G)$ and it was defined in full generality in [L-P2]. The proof below is different.

THEOREM 4.1. — Let $G$ be a lcagroup, $1 \leq p \leq 2$. Let $(P_\gamma)_{\gamma \in I}$ be an approximate identity in $A_2(G_d)$. Let $\tilde{m}$ be a topological mean on $CV_p(G)$. The linear mapping $A_{\tilde{m}} : CV_p(G) \to CV_p(G_d)$ is defined by

$$\forall S \in CV_p(G), \forall f \in A_p(G_d), \langle A_{\tilde{m}}(S), f \rangle = \lim_{\gamma} \langle (\tilde{m} * P_\gamma)S, f \rangle$$

$A_{\tilde{m}}$ has the following properties:

(i) $\|A_{\tilde{m}}\|_{CV_p(G) \to CV_p(G_d)} \leq 1$.

(ii) $A_{\tilde{m}}$ restricted to finitely supported measures on $G$ is identity.

(iii) $A_{\tilde{m}}$ commutes with multiplication by functions of $B_p(G)$.

(iv) If $E \subset G$ is a closed subset

$$A_{\tilde{m}} : CV_p(E) \to CV_p(E_d).$$

(v) $A_{\tilde{m}}$ maps $Y_p(G)$ into $X_p(G)$. 
Proof. — We first explain the definition of $A_m$. $(\hat{\mu} \ast P_x)S$ is defined as in proposition 2.3, lemma 2.6 and part 1 by

\[(vi) \quad \forall f \in A_p(G), \quad \langle (\hat{\mu} \ast P_x)S, f \rangle = \langle fS, \hat{\mu} \ast P_x \rangle = \sum_{P_x(x_i) \neq 0} P_x(x_i) \langle S, \hat{\mu}_{x_i} \rangle \langle \delta_{x_i}, f \rangle.
\]

It is a finitely supported measure on $G$. By lemmas 2.2 and 3.2

\[(vii) \quad \|S\|_{CV_p(G)} \geq \| (\hat{\mu} \ast P_x)S \|_{CV_p(G)} = \| (\hat{\mu} \ast P_x)S \|_{CV_p(G_d)}.
\]

Let $f \in A_p(G_d)$ with a finite support. Then

\[(viii) \quad \langle (\hat{\mu} \ast P_x)S, f \rangle = \sum_{f(x_i) \neq 0} P_x(x_i) f(x_i) \langle S, \hat{\mu}_{x_i} \rangle \geq \sum_{f(x_i) \neq 0} f(x_i) \langle S, \hat{\mu}_{x_i} \rangle.
\]

Hence $(\hat{\mu} \ast P_x)S = \sum_{P_x(x_i) \neq 0} P_x(x_i) \langle S, \hat{\mu}_{x_i} \rangle \delta_{x_i} (F \in \mathcal{F})$ is a bounded net in $CV_p(G_d)$ which converges for $\sigma(CV_p(G_d), A_p(G_d))$ to a limit which we denote by $A_m(S)$. $A_m$ is clearly a linear mapping.

(i) is implied by (vii) and (viii); (ii) is implied by (viii) because $\langle \mu, \hat{\mu}_{x_i} \rangle = \mu(x_i)$ if $\mu$ is finitely supported.

(iii) Let $F \in B_p(G)$ and $\varphi \in \mathcal{S}_p(G)$. For every $x \in G$, $\varphi \ast F \in A_p(G)$. As $x$ is a point of synthesis for $A_p(G)$ lemma 1.5 (b) implies $\langle \varphi \ast F, \hat{\mu}_x \rangle = F(x) \langle S, \hat{\mu}_x \rangle$. As $\langle \varphi \ast F, \hat{\mu}_x \rangle = \langle F, \hat{\mu}_x \rangle$ (viii) implies (iii).

(iv) By lemma 1.5 (b) $\langle S, \hat{\mu}_x \rangle = 0$ if $x$ lies outside the support of $S$. Hence (viii) implies (iv).

(v) If we write $M_p(S_x)$ instead of $\langle S, 1_{[x]} \rangle$ the proof of (v) is similar to the proof of (v) in theorem 3.1.

Let us notice however that $A_m$ is not one to one on $Y_p(G)$: e.g. if $\mu \in M(G)$ is a diffuse measure $A_m(\mu) = 0$. This will be precised in theorem 4.2 below.

Theorem 4.2 provides an Eberlein p-decomposition for elements of $Y_p(G)$.

**Theorem 4.2.** — Let $G$ be a lca group, $1 \leq p \leq 2$. Let $\hat{\mu}$ be any topological mean on $CV_p(G)$, let $A_m$ and $B$ be as in theorems 3.1, 4.1.

a) $A_m \circ B_m$ is identity on $X_p(G)$; $B_m$ is an isometry on $X_p(G)$, $A_m$ is an isometry on $B_m(X_p(G))$. 
b) For every $S \in Y_p(G)$, $S = B_\omega \circ A_m(S) + S'_\omega$ where

$$B_\omega \circ A_m(S) \in Y_p(G)$$

and does not depend on $\dot{m}$, and $A_m(S'_\omega) = 0$.

c) If $\dot{m}$ is a topological mean on $CV_2(G)$, $X_p(G)$ and $Y_p(G)$ can be replaced by $X_2(G) \cap CV_p(G_d)$ and $Y_2(G) \cap CV_p(G)$ in the assertions above.

For $p = 2$ this result was partly proved in [W2] corollary 2, and proved in [L-P2] theorem 7.

Proof. - a) Let $S \in X_p(G)$. By the proof of theorem 3.1 (v), for every $\varepsilon > 0$ and $x \in G$ there exists $\varphi \in \mathcal{S}_p(G)$ such that

$$\|\varphi \cdot B_\omega(S) - \langle S, 1_{|x|}\rangle \delta_x\|_{CV_p(G)} \leq \varepsilon$$

hence by theorem 4.1

$$\|\varphi \cdot A_m \circ B_\omega(S) - \langle S, 1_{|x|}\rangle \delta_x\|_{CV_p(G_d)} \leq \varepsilon$$

hence

$$\forall x \in G, \quad \langle A_m \circ B_\omega(S), 1_{|x|}\rangle = \langle S, 1_{|x|}\rangle.$$ 

As $\|B_\omega\|$, $\|A_m\| \leq 1$ the rest of the claim is now obvious.

b) Let $S \in Y_p(G)$. By theorems 4.1 (v) and 3.1 (v) $A_m(S) \in X_p(G)$ and $B_\omega \circ A_m(S) \in Y_p(G)$. By (a) $(A_m \circ B_\omega) \circ A_m(S) = A_m(S)$ hence $S - B_\omega \circ A_m(S) \in \ker A_m$. On the other hand all $A_m$ coincide on $Y_p(G)$ for topological means $\dot{m}$ on $CV_p(G)$.

c) By (a) $A_m \circ B_\omega$ is identity on $X_2(G)$ hence on $X_2(G) \cap CV_p(G_d)$. The rest of the proof is similar to the proof of (a), (b).

Theorem 4.2 (c) implies [Loh1] chap. 2, corollaire de la proposition III. 2.0, p. 56, where $\ell^1(G) \cap CV_2(G)$ is shown to be isometric to $\ell^2(G) \cap CV_2(G_d) \cap CV_2(G_d)$. We do not know whether $X_2(G) \cap CV_p(G_d)$ is strictly larger than $X_p(G)$ or not (and the same question for $Y_2(G) \cap CV_p(G)$ and $Y_p(G)$). However let $1 \leq q < p$ and let $S \in CV_p(G_d)$. Lemma 1.8 and the interpolation inequality

$$\left( \frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2} \right)$$

$$\|S\|_{CV_p(G_d)} \leq \|S\|_{CV_q(G_d)}^{\theta} \|S\|_{CV_2(G_d)}^{1-\theta}$$
MEANS ON $CV_p(G)$.

Then $S \in X_p(G)$ and $S \in X_p(G)$. In the same way $CV_q(G) \cap Y_p(G) \subset Y_p(G)$. The following result is a generalization of [Gl] theorem 4, where $p = 2$.

**THEOREM 4.3.** Let $G$ be a lca group, let $E \subset G$ be closed and scattered. Let $1 \leq p \leq 2$. Then $CV_p(E)$ and $CV_p(E_d)$ are isometric.

**Proof.** By corollary 2.4, $CV_p(E)$ is a closed subspace of $X_p(G)$. By theorem 4.2 (a), $B_0$ is an isometry: $CV_p(E_d) \rightarrow CV_p(E)$ and $A_m$ is an isometry: $CV_p(E) \rightarrow CV_p(E_d)$ if $CV_p(E) \subset B_0(X_p(G))$ hence if $A_m$ is one to one on $CV_p(E)$. Let now $S \in CV_p(E), S \neq 0$. Hence the support $E'$ of $S$ is a closed non empty subset of $E$. As $E$ is scattered let $x$ be an isolated point of $E'$. Let $V$ be a neighborhood of $x$ in $G$ such that $V \cap E' = \{x\}$. By assumption there exists $\varphi_V \in A_p(G)$ which is supported on $V$ and such that $\langle S, \varphi_V \rangle$ is not zero. The support of $\varphi_VS$ is $\{x\}$ hence $\varphi_VS = \langle S, \varphi_V \rangle \delta_x$ and $A_m(\varphi_VS)$ is not zero. By theorem 4.1 (iii) $A_m(\varphi_VS) = \varphi_VA_m(S)$ which proves the claim.

Alternatively we could have used Glowacki's result (whose proof is the same as above, for $p = 2$) and theorem 4.2 (c).

Theorem 3.3 is an obvious consequence of theorems 4.3, 2.8, 2.14. But we prefered to give a direct simpler proof.

Our next aim is to precise the Eberlein decomposition of $S \in CV_p(G)$ when $S$ is $p$-weak almost periodic. We first establish a general lemma:

**LEMMA 4.4.** Let $G$ be a lca group and $1 \leq p \leq 2$. $CV_p(G)$ is isometric to the space of multipliers: $A_p(G) \rightarrow CV_p(G)$ and to the space of multipliers: $A_2(G) \rightarrow CV_p(G)$ provided with operator norm.

**Proof.** (i) For every $f \in A_p(G), g \in A_2(G), S \in CV_p(G)$

$$\langle fS, g \rangle = \langle gS, f \rangle = \langle S, gf \rangle$$

hence

$$\|S\|_{A_p-CV_2} = \|S\|_{A_2-CV_p} \leq \|S\|_{A_p-CV_p} \leq \|S\|_{CV_p}.$$

(ii) Conversely let $S$ be a multiplier : $A_2(G) \rightarrow CV_p(G)$. Let $(\varphi_x)_{x \in A}$ be an approximate identity with compact support in $S_2(G)$. Hence $\|S(\varphi_x)\|_{CV_p(G)} \leq \|S\|_{A_2-CV_p}$. For every $f \in A_p(G)$ with a compact support $K$ there exists $g_K \in A_2(G)$ such that $g_K = 1$ on $K$. Hence as $\|\varphi_xg_K - g_K\|_{A_2(G)} \rightarrow 0$

$$\langle S(\varphi_x, f) = \langle S(\varphi_x, g_Kf) = \langle S(g_K), f \rangle \rightarrow \langle S(g_K), f \rangle.$$
It implies that \((S(\varphi_s))_{s \in A}\) converges for \(\sigma(CV_p(G), A_p(G))\); let \(s \in CV_p(G)\), \(S_{CV_p(G)} \leq \|S\|_{A_2-CV_p} \) be the limit. In particular for \(f\) as above
\[
\langle S(g_s), f \rangle = \langle g_{ks}, f \rangle.
\]
We now verify that \(hs = S(h)\) in \(CV_p(G)\) when \(h \in A_2(G)\). It is sufficient to prove it when \(h\) has a compact support \(K\). Then for every \(f \in A_p(G)\), as \(g_{sh} = h\)
\[
\langle hs - S(h), f \rangle = \langle g_{ks} - S(g_{ks}), hf \rangle = 0.
\]
It implies the above claim hence \(\|S\|_{A_2-CV_p} \leq \|S\|_{CV_p(G)}\).

The assertion of the lemma is now obvious. \(\square\)

Let us recall the definition of \(p\text{-WAP}(G)\), the weak \(p\)-almost periodic elements of \(CV_p(G)\):

**Definition 4.5** [Gra]. — Let \(G\) be a lca group, \(1 \leq p \leq 2\). \(p\text{-WAP}(G)\) is the subspace of \(CV_p(G)\) of elements \(S\) which define weakly compact multipliers: \(A_p(G) \to CV_p(G)\).

Let \(S \in CV_p(G)\). By remark 1.2 it is easy to see that \(S \in p\text{-WAP}(G)\) iff \(\{fS\}_{f \in \mathcal{Y}_p(G)}\) is relatively compact for \(\sigma(CV_p(G), A_p(G))\) hence iff \(\{fS\}_{f \in \mathcal{Y}_p(G)}\) is relatively weakly compact in \(C(\mathcal{Y}_p(G))\), which means by [BJM] chapter 3, definition 8.1, that \(S\) is a weak almost periodic function on the semi-group \(\mathcal{Y}_p(G)\).

In the same way \(S\) is a compact multiplier: \(A_p(G) \to CV_p(G)\) iff \(S\) is an almost periodic function on the semi-group \(\mathcal{Y}_p(G)\) [BJM] 3, definition 9.1.

By [Gra], proposition 9, \(p\text{-WAP}(G)\) is a closed subspace of \(Y_p(G)\).

By [Gra] proposition 7, \(M(G)\) is a subspace of \(p\text{-WAP}(G)\).

Assertion (c) \(\iff\) (d) in the next theorem is Eberlein’s decomposition of WAP function on \(\hat{G}\) [Eb2] when \(p = 2\). (b) \(\iff\) (d) is a particular case of [BJM] chapter 3, corollary 16.14.

**Theorem 4.6.** — Let \(G\) be a lca group, \(G \to H\), \(1 \leq p \leq 2\). Let \(S \in CV_p(G)\). The following assertions are equivalent:

a) \(S \in p\text{-WAP}(G)\).

b) \(\mathcal{Y}_p(G)S\) is relatively weakly compact in \(CV_p(G)\).

c) \(\mathcal{Y}_p(H)S\) is relatively weakly compact in \(CV_p(G)\).
d) \( S = B_\omega \circ A_\hat{m}(S) + S' \) where \( \hat{m} \) is a topological mean on \( CV_p(G) \).
\( B_\omega, A_\hat{m} \) are defined as in theorems 3.1, 4.1, \( B_\omega \circ A_\hat{m}(S) \) belongs to \( \ell^1(G)^{\| \cdot \|_{CV_p(G)}} \) and does not depend on \( \omega \) nor on \( \hat{m} \), \( S' \in p\text{-WAP}(G) \) and \( A_\hat{m}(S') = 0 \).

**Proof.** — (a) \( \Rightarrow \) (b) is obvious.

(a) \( \Leftarrow \) (b) is easy by remark 1.2 as we already told above.

(b) \( \Rightarrow \) (c) : When we studied \( K_S \) in part 1 we saw that \( \mathcal{S}_p(H)S \) lies in \( K_S \).

If (b) holds \( K_S \) is the norm closure of \( \mathcal{S}_p(G)S \) and \( K_S \) is weakly compact in \( CV_p(G) \).

(c) \( \Rightarrow \) (b) : By lemma 1.14, \( \mathcal{S}_p(H)S \) is dense in \( K_S \) for \( \sigma(CV_p(G), A_p(G)) \).

If (c) holds \( K_S \) is the norm closure of \( \mathcal{S}_p(H)S \) and \( K_S \) is weakly compact.

(b) \( \Rightarrow \) (d) : the assumption implies that \( S \in Y_p(G) \) hence theorem 4.2 (b) holds. We claim that \( A_\hat{m}(S) \) lies in \( \ell^1(G)^{\| \cdot \|_{CV_p(G)}} \) : by definition and lemma 2.2 \{\( (\hat{m} * P_F)S \mid F \subset G, F \) finite\} lies in \( K_S \) and in \( \ell^1(G)^{CV_p(G)} \) (see the proof of theorem 4.1 (a)). By assumption it is relatively weakly compact in \( CV_p(G) \) hence in \( \ell^1(G)^{\| \cdot \|_{CV_p(G)}} \) hence in \( \ell^1(G)^{CV_p(G)} \) by lemma 3.2. The definition of \( A_\hat{m} \) (see the proof of theorem 4.1) now proves the claim. As \( B_\omega \) is identity on \( \ell^1(G) \)
\[ B_\omega \circ A_\hat{m}(S) \in \ell^1(G)^{\| \cdot \|_{CV_p(G)}} , \]
it does not depend on \( \omega \), nor on \( \hat{m} \) by theorem 4.2 (b), it lies obviously in \( p\text{-WAP}(G) \).

\( d \Rightarrow a \) is obvious. \( \square \)

Motivated by lemma 4.4 and a result of Lohoué on compact multipliers : \( A_p(G) \to CV_p(G) \) [Loh1] chap. 2, theorem III.1, p. 50, we also consider elements of \( CV_p(G) \) which are weakly compact multipliers : \( A_2(G) \to CV_p(G) \). We do not know if they are weakly compact multipliers : \( A_p(G) \to CV_p(G) \), but they have analogous properties. In particular they lie in \( Y_p(G) \) : let \( W \) be a decreasing basis of neighborhoods of \( \{0\} \) in \( G \). If \( S \in CV_p(G) \) and if \( \mathcal{S}_2(G)S \) is relatively weakly compact in \( CV_p(G) \) (\( \varphi _WS \) weak cluster point which must be a scalar multiple of \( \delta_0 \) and which belongs to the norm closure of \( \mathcal{S}_2(G)S \). Lemma 1.8 finishes the proof.
Theorem 4.7. — Theorem 4.6 holds true if we replace \( \mathcal{F}_p(G) \) by \( \mathcal{F}_2(G) \) and \( p\)-WAP\((G) \) by the set of weakly compact multipliers: \( A_2(G) \rightarrow CV_p(G) \).

Proof. — By lemma 4.4 such a multiplier is given by an element \( S \in CV_p(G) \). The proof then follows the same lines as the proof of theorem 4.6. It is even simpler: for example lemma 1.14 is obvious for \( p = 2 \), it implies that \( \mathcal{F}_2(H)S \) and \( \mathcal{F}_2(G)S \) have the same closure for \( \sigma(CV_p(G), A_p(G)) \). If \( \hat{m} \) is a topological mean on \( CV_2(G) \) and if \( \mathcal{F}_2(G)S \) is relatively weakly compact in \( CV_p(G) \) \((\hat{m} * P_p)S \) lies in the norm closure of \( \mathcal{F}_2(G)S \) hence \( A_{\hat{m}}(S) \in \bar{\mathcal{F}_2(G)} \) by the same proof as in theorem 4.6. As \( S \in Y_p(G) A_{\hat{m}}(S) \) does not depend on \( \hat{m} \) when \( \hat{m} \) is a topological mean on \( CV_p(G) \).

Theorem 4.7 implies the following improvement of [Lohl] chap. 2, theorem III.1:

Theorem 4.8. — Let \( G \) be a lca group, \( G \rightarrow H, 1 \leq p \leq 2 \), let \( S \in CV_p(G) \).

The following assertions are equivalent:

(a) \( S \in \overline{\mathcal{F}_2(G)} \).  
(b) \( S \) is a compact multiplier: \( A_p(G) \rightarrow CV_2(G) \).
(c) \( \mathcal{F}_p(G)S \) is relatively compact in \( CV_p(G) \).
(d) \( \mathcal{F}_2(G)S \) is relatively compact in \( CV_p(G) \).
(e) \( \mathcal{F}_2(H)S \) is relatively compact in \( CV_p(G) \).
(f) \( \mathcal{F}_2(G)S \) is relatively weakly compact in \( CV_p(G) \) and relatively compact in \( CV_2(G) \).

Proof. — (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) \( \Rightarrow \) (f) are obvious.

(e) \( \Leftrightarrow \) (d) by the proof of theorem 4.7.

(f) \( \Rightarrow \) (a): By theorem 4.7

\[
S = B_0 \circ A_{\hat{m}}(S) + S' \quad \text{and} \quad B_0 \circ A_{\hat{m}}(S) \in \bar{\mathcal{F}_2(G)}
\]

We only have to prove that \( S' = 0 \) in \( CV_p(G) \) or that \( S' = 0 \) in \( CV_2(G) \). We know that \( A_{\hat{m}}(S') = 0 \) and that \( \mathcal{F}_2(G)S' \) is relatively compact in \( CV_2(G) \) because \( \mathcal{F}_2(G)S' \) is \( \sigma(CV_2(G), A_2(G)) \) dense in the \( \sigma(CV_2(G), A_2(G)) \) closure of \( \mathcal{F}_2(G)S' \). Hence \( S' \) is an almost periodic function on \( \bar{G} \) in the usual sense and \( \langle \chi S',m \rangle = 0 \) for every character \( \chi \) on \( \bar{G} \) and every mean \( m \) on \( L^\infty(\bar{G}) \). Hence \( S' = 0 \) by classical results.
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Françoise Lust-Piquard,
Université de Paris-Sud
Mathématiques
Bâtiment 425
91405 Orsay Cedex.