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THE ASYMPTOTICS OF THE RAY-SINGER ANALYTIC TORSION OF THE SYMMETRIC POWERS OF A POSITIVE VECTOR BUNDLE

by J.-M. BISMUT and E. VASSEROT

Let X be a compact complex manifold, equipped with a smooth Hermitian metric. Let $(E, \|\cdot\|_E)$, $(\xi, \|\cdot\|_\xi)$ be holomorphic Hermitian vector bundles on X . Assume that $(E, \|\cdot\|_E)$ is positive, i.e. if L^E is the curvature of the holomorphic Hermitian connection on $(E, \|\cdot\|_E)$, for any $U \in TX \setminus \{0\}$, $e \in E \setminus \{0\}$, then $\langle L^E(U, \bar{U})e, \bar{e} \rangle > 0$. By [K] Theorem III 6.19, E is an ample vector bundle on X .

For $p \in \mathbb{N}$, let $S^p(E)$ be the p^{th} symmetric tensor power of E . Then by a result of Le Potier [LP], [K] Theorem III.6.25, for p large enough, and $q > 0$, $H^q(S^p(E) \otimes \xi) = 0$. Let τ_p be the Ray-Singer analytic torsion of the Dolbeault complex $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi)$ [RS]. The purpose of this paper is to establish an asymptotic formula for $\text{Log}(\tau_p)$ as $p \rightarrow +\infty$. This extends an earlier result by ourselves [BV] Theorem 8, in the case where E is a positive line bundle.

The general strategy is the same as in [BV]. Namely if $\bar{\square}_p^{X,q}$ denotes the Hodge Laplacian acting on $\Omega^{(0,q)}(S^p(E) \otimes \xi)$, we first establish in Theorem 1 an asymptotic formula for $\text{Tr} \left[\exp \left(-\frac{t}{p} \bar{\square}_p^{X,q} \right) \right]$ as $p \rightarrow +\infty$. In Theorem 8 we prove that if λ_p^q is the lowest eigenvalue of $\bar{\square}_p^{X,q}$, if $q > 0$, as $p \rightarrow +\infty$, λ_p^q grows at least like p . The combination of these two results leads us in Theorem 11 to an asymptotic formula for $\text{Log}(\tau_p)$ very much like in [BV].

To establish these intermediary results, we use a trick due to Getzler [Ge] in a different context. In [Ge], Getzler extended a result of Bismut [B2] on the asymptotics of certain heat equation operators, which

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is valid for line bundles, to vector bundles, associated with representations of the structure group with weight $p\lambda$ as $p \rightarrow +\infty$. Here if μ is the dual of the universal line bundle on $\mathbb{P}(E^*)$, we consider $S^p(E)$ as the direct image of $\mu^{\otimes p}$ by the map $\pi: \mathbb{P}(E^*) \rightarrow X$. We then use Getzler's trick to lift our initial problem to a corresponding problem on the line bundle $\mu^{\otimes p}$ on $\mathbb{P}(E^*)$, to which the techniques of [BV] can be applied.

Our paper is organized as follows. In §1, we introduce our main assumptions and notation. In §2, we calculate the asymptotics of $\text{Tr} \left[\exp \left(-\frac{t}{p} \bar{\square}_p^{X,q} \right) \right]$ as $p \rightarrow \infty$. Finally in §3, we establish our main result on the asymptotics of $\text{Log}(\tau_p)$ as $p \rightarrow +\infty$.

As was pointed out by the referee, the results contained in this paper can be extended to other irreducible representations of E , which are associated with the weights pa (where a is a given weight) when p tends to $+\infty$. The corresponding vector bundles can be expressed as direct images of the p^{th} power of a certain line bundle over the corresponding flag manifold. Arguments of Demailly [De] Lemma 3.7, can then be used to prove the positivity of this Hermitian line bundle when $(E, \|\cdot\|_E)$ is positive, and the trick of Getzler [Ge] together with the techniques used in our paper still apply. This extension of our main result is left to the reader.

1. Assumptions and notation.

Let X be a compact complex manifold of complex dimension ℓ . Let TX be the complex holomorphic tangent space.

Let E be a holomorphic vector bundle on X , of complex dimension k . Let E^* be the dual of E . For $p \in \mathbb{N}$, $S^p(E)$ denotes the p^{th} symmetric tensor power of E .

Let $\mathbb{P}(E^*)$ denote the projectivization of E^* , and let π be the projection $\mathbb{P}(E^*) \rightarrow X$. Let μ be the dual of the universal line bundle on $\mathbb{P}(E^*)$.

Let $\|\cdot\|_E$ be a smooth Hermitian metric on E . Let $\|\cdot\|_{S^p(E)}$, $\|\cdot\|_{E^*}$, $\|\cdot\|_\mu$ be the Hermitian metrics on $S^p(E)$, E^* , μ induced by the metric $\|\cdot\|_E$.

Let ∇^E, ∇^{E^*} be the holomorphic Hermitian connections on $(E, \|\cdot\|_E), (E^*, \|\cdot\|_{E^*})$, and let L^E, L^{E^*} be the corresponding curvatures. Let ∇^μ be the holomorphic Hermitian connection on $(\mu, \|\cdot\|_\mu)$ and let r be its curvature.

We first calculate r . Let $T^V\mathbb{P}(E^*)$ be the relative tangent bundle to the fibres of $\pi : \mathbb{P}(E^*) \rightarrow X$. The connection ∇^{E^*} induces a horizontal subbundle $T^H\mathbb{P}(E^*)$ of $T\mathbb{P}(E^*)$.

Let r^V be the restriction of r to $T^V\mathbb{P}(E^*)$. r^V is explicitly known by a formula given in [GrH] p. 30, and defines the Fubini-Study metric along the fibres of $\mathbb{P}(E^*)$. r^V extends into a (1,1) form on $\mathbb{P}(E^*)$ such that if $U \in T^H\mathbb{P}(E^*), i_U r^V = 0$.

Set

$$(1) \quad r^H = -\pi^* \frac{\langle L^{E^*} y, \bar{y} \rangle}{|y|^2}; \quad y \in E^* \setminus \{0\}.$$

Then r^H is a (1,1) form on $\mathbb{P}(E^*)$. Also by [K] p. 90,

$$(2) \quad r = r^V + r^H$$

r^H is then the restriction of r to $T^H\mathbb{P}(E^*) \times T^H\mathbb{P}(E^*)$.

For $p \geq 1$, the connection ∇^E induces on $S^p(E)$ the holomorphic Hermitian connection $\nabla^{S^p(E)}$ on $(S^p(E), \|\cdot\|_{S^p(E)})$.

Let $(\xi, \|\cdot\|_\xi)$ be a holomorphic Hermitian vector bundle on X . Let ∇^ξ be the corresponding holomorphic Hermitian connection.

Let $\|\cdot\|_{TX}$ be a Hermitian metric on X .

We then equip $\Lambda(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$ with the tensor product of the metrics induced by $\|\cdot\|_{TX}$ on $\Lambda(T^{*(0,1)}X)$, of the metric $\|\cdot\|_{S^p(E)}$ and of the metric $\|\cdot\|_\xi$.

For $0 \leq q \leq \ell$, let $\Omega^{(0,q)}(S^p(E) \otimes \xi)$ be the set of C^∞ sections of $\Lambda^q(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$ over X . Set $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi) = \bigoplus_{q=0}^{\ell} \Omega^{(0,q)}(S^p(E) \otimes \xi)$.

Let dx be the volume form on X associated with the metric $\|\cdot\|_{TX}$. We equip $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi)$ with the L_2 Hermitian product

$$(3) \quad \alpha, \alpha' \in \Omega^{(0,\cdot)}(S^p(E) \otimes \xi) \rightarrow \langle \alpha, \alpha' \rangle = \int_X \langle \alpha, \alpha' \rangle(x) \frac{dx}{(2\pi)^{\dim X}}$$

Let $\bar{\partial}_p^X$ be the Dolbeault operator acting on $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi)$, and let $\bar{\partial}_p^{X^*}$ be the formal adjoint of $\bar{\partial}_p^X$ with respect to the Hermitian product (3). Set

$$(4) \quad \bar{\square}_q^X = (\bar{\partial}_p^X + \bar{\partial}_p^{X^*})^2.$$

For $0 \leq q \leq \ell$, let $\bar{\square}_p^{X,q}$ be the restriction of $\bar{\square}_p^X$ to $\Omega^{(0,q)}(S^p(E) \otimes \xi)$.

2. The asymptotics of the trace of certain heat kernels as $p \rightarrow +\infty$.

If $U \in T_R X$, let U^H be its horizontal lift in $T_R^H \mathbb{P}(E)$, so that $U^H \in T_R^H \mathbb{P}(E)$, $\pi_* U^H = U$.

Let w_1, \dots, w_ℓ be an orthonormal base of TX , let w^1, \dots, w^ℓ be the corresponding dual base of T^*X . If $z \in \mathbb{P}(E^*)$, r_z acts as a derivation $r_{d,z}^H$ of $\Lambda_{\pi z}^{(0,1)}(T^{*(0,1)}X)$ by the formula

$$(5) \quad r_{d,z}^H = - \sum r_z(w_i^H, \bar{w}_j^H) \bar{w}^j \wedge i_{\bar{w}_i}.$$

We identify r_z^H with the self-adjoint matrix $r_z^H \in \text{End}_{\pi x}(\overline{TX})$ such that if $U, V \in T_{\pi z} X$,

$$r_z^H(U^H, \bar{V}^H) = \langle U^H, r_z^H \bar{V}^H \rangle.$$

For $0 \leq q \leq \ell$, let $r_d^{H,q}$ be the restriction of r_d^H to $\Lambda^q(T^{*(0,1)}X)$. As $t \rightarrow 0$, we have the asymptotic expansion

$$(6) \quad \int_{\mathbb{P}(E^*)} \frac{\text{Tr}[e^{tr_d^{H,q}}]}{\det(1 - e^{-tr^H})} \exp\left(\frac{-r}{2i\pi}\right) = \sum_{j=-\ell}^m a_j^q t^j + o(t^m).$$

Also for $p \in \mathbb{N}$, $0 \leq q \leq \ell$, $t > 0$, let $\text{Tr}[\exp(-t \bar{\square}_p^{X,q})]$ be the trace of the operator $\exp(-t \bar{\square}_p^{X,q})$. For any $m \in \mathbb{N}$, as $t \rightarrow 0$

$$(7) \quad p^{-(\dim X + \dim E - 1)} \text{Tr}\left[\exp\left(\frac{-t}{p} \bar{\square}_p^{X,q}\right)\right] = \sum_{j=-\ell}^m a_{p,j}^q t^j + o(t^m).$$

THEOREM 1. — *For any $t > 0$, $0 \leq q \leq \ell$, the following identity holds*

$$(8) \quad \lim_{p \rightarrow +\infty} p^{-(\dim X + \dim E - 1)} \text{Tr}\left[\exp\left(\frac{-t}{p} \bar{\square}_p^{X,q}\right)\right] = rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\text{Tr}[e^{r_d^{H,q}}]}{\det(1 - e^{-tr^H})} \exp\left(\frac{-r}{2i\pi}\right)$$

and the convergence in (8) is uniform as t varies in compact sets of R_+^* . For any $j \geq -\dim X$, $0 \leq q \leq \dim X$, as $p \rightarrow +\infty$

$$(9) \quad a_{p,j}^q = \text{rk}(\xi) a_j^q + O\left(\frac{1}{\sqrt{p}}\right).$$

In (7), for any $m \in \mathbb{N}$, $o(t^m)$ is uniform with respect to $p \in \mathbb{N}$.

Proof. — By [GrH] p. 165 and [K] Theorem III.4.10, for any $x \in X$, $p \in \mathbb{N}$

$$(10) \quad \begin{aligned} H^q(\mathbb{P}_x(E^*), \mu_{\mathbb{P}_x(E^*)}^{\otimes p}) &= S^p(E)_x \quad \text{if } q = 0 \\ H^q(\mathbb{P}_x(E^*), \mu_{\mathbb{P}_x(E^*)}^{\otimes p}) &= 0 \quad \text{if } q > 0. \end{aligned}$$

To prove (8), we will use (10) together with a procedure used by Getzler [Ge] in a similar situation, to transform the initial problem into a corresponding problem on $\mathbb{P}(E^*)$ associated with $\mu^{\otimes p}$, to which we can apply results of Bismut [B2] and Bismut-Vasserot [BV].

Let $U(E^*)$ be the bundle of orthonormal frames in E^* . We identify $U(E^*)$ with the set of linear isometries from \mathbb{C}^k into E^* . Clearly

$$\mathbb{P}(E^*) = U(E^*) \times_{U(k)} \mathbb{P}(\mathbb{C}^k).$$

The connection ∇^{E^*} on $U(E^*)$ induces a connection on the fibration $\pi: \mathbb{P}(E^*) \rightarrow X$. The associated horizontal subbundle of $T\mathbb{P}(E^*)$ is exactly the vector bundle $T^H\mathbb{P}(E^*)$ considered in § 1.

We then have the identification of C^∞ vector bundles

$$(11) \quad \begin{aligned} T\mathbb{P}(E^*) &\cong T^H\mathbb{P}(E^*) \oplus T^V\mathbb{P}(E^*) \\ T^H\mathbb{P}(E^*) &\cong \pi^*TX. \end{aligned}$$

From (11), we deduce the identification of C^∞ vector bundles

$$(12) \quad \Lambda(T^{*(0,1)}\mathbb{P}(E^*)) \cong \pi^*(\Lambda(T^{*(0,1)}X)) \widehat{\otimes} \Lambda(T^{V*(0,1)}\mathbb{P}(E^*)).$$

We equip $T^V\mathbb{P}(E^*)$ with the Fubini-Study metric $\|\cdot\|_{T^V\mathbb{P}(E^*)}$. Let $\|\cdot\|_{T\mathbb{P}(E^*)}$ be the metric on $T\mathbb{P}(E^*) \cong T^H\mathbb{P}(E^*) \oplus T^V\mathbb{P}(E^*)$ which is the orthogonal sum of $\pi^*\|\cdot\|_{TX}$ and $\|\cdot\|_{T^V\mathbb{P}(E^*)}$. For $p \in \mathbb{N}$, set

$$(13) \quad \xi_p = \mu^{\otimes p} \otimes \pi^*\xi.$$

Let $\bar{\partial}_p^V$ be the $\bar{\partial}$ operator along the fibres of $\mathbb{P}(E^*)$ acting on smooth sections of $\pi^*(\Lambda(T^{*(0,1)}X)) \widehat{\otimes} \Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$, and let $\bar{\partial}_p^{V*}$ be its formal adjoint with respect to the considered metrics.

Let $\nabla^{T^V\mathbb{P}(E^*)}$, ∇^{ξ_p} be the holomorphic Hermitian connections on $T^V\mathbb{P}(E^*)$, ξ_p respectively. These connections induce a natural connection on $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$, which we note $\nabla^{\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p}$.

DEFINITION 2. — *If α is a smooth section of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$, if $U \in T_R X$, set*

$$(14) \quad \tilde{\nabla}_{p,U} \alpha = \nabla_{UH}^{\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p} \alpha.$$

We extend $\tilde{\nabla}_p$ to a differential operator acting on smooth sections of $\pi^*(\Lambda(T_R^*X)) \widehat{\otimes} \Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$, with the convention that if ω is a smooth section of $\Lambda(T_R^*X)$, and if α is a smooth section of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$, then

$$(15) \quad \tilde{\nabla}_p(\omega\alpha) = \pi^*(d\omega)\alpha + (-1)^{\text{deg } \omega} \omega \wedge \tilde{\nabla}_p \alpha.$$

Let $\tilde{\nabla}'_p, \tilde{\nabla}''_p$ be the holomorphic and antiholomorphic parts of $\tilde{\nabla}_p$, so that

$$(16) \quad \tilde{\nabla}_p = \tilde{\nabla}'_p + \tilde{\nabla}''_p.$$

For $0 \leq q' \leq k + \ell - 1$, let $\Omega^{(0,q)}(\xi_p)$ be the set of smooth sections of $\Lambda(T^{*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$. Set $\Omega^{(0,\bullet)}(\xi_p) = \bigoplus_0^{k+\ell-1} \Omega^{(0,q)}(\xi_p)$. Let $\bar{\partial}_p^{P(E^*)}$ be the classical Dolbeault operator acting on $\Omega^{(0,\bullet)}(\xi_p)$.

Using the identification (12), it is clear that $\tilde{\nabla}''_p, \bar{\partial}_p^V, \bar{\partial}_p^{V*}$ act on $\Omega^{(0,\bullet)}(\xi_p)$.

If A, B are operators acting on the \mathbb{Z} -graded vector space $\Omega^{(0,\bullet)}(\xi_p)$, $[A,B]$ denotes the supercommutator of A and B in the sense of [Q].

PROPOSITION 3. — *The following identities of operators acting on $\Omega^{(0,\bullet)}(\xi_p)$ hold*

$$(17) \quad \begin{aligned} (\tilde{\nabla}''_p)^2 &= 0 \\ [\tilde{\nabla}''_p, \bar{\partial}_p^V] &= [\tilde{\nabla}''_p, \bar{\partial}_p^{V*}] = 0 \\ \bar{\partial}_p^{P(E^*)} &= \tilde{\nabla}''_p + \bar{\partial}_p^V. \end{aligned}$$

Proof. — Assume first that $\xi = \mathbb{C}$. Let F be the vector space of smooth sections of $\Lambda(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \mu^{\otimes p}$ over $\mathbb{P}(E^*)$. As in [B1], Section 1f), we view F as an infinite dimensional vector bundle over X . If $x \in X$, the fibre F_x is simply the vector space of smooth sections $\Lambda(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \mu^{\otimes p}$ on the fibre $\mathbb{P}(E^*)_x$.

Clearly F is a $U(k)$ -equivariant vector bundle on X , in the sense it comes from a representation space for $U(k)$. $\tilde{\nabla}_p$ is then a connection on the vector bundle F , which is inherited from the original connection ∇^E on $U(E)$. Since $(\nabla^{E''})^2 = 0$, then $(\tilde{\nabla}_p'')^2 = 0$. It is now trivial to prove in full generality the equation $(\tilde{\nabla}_p'')^2 = 0$.

Since $U(k)$ acts on $\mathbb{P}(\mathbb{C}^{k^*})$ by holomorphic isometries which lift unitarily to the dual of the universal bundle on $\mathbb{P}(\mathbb{C}^{k^*})$, we get

$$(18) \quad [\tilde{\nabla}_p, \bar{\partial}_p^V] = 0; \quad [\tilde{\nabla}_p, \bar{\partial}_p^{V^*}] = 0.$$

In particular, the second equation in (17) holds.

If $B \in \text{End}(E^*)$ is skew-adjoint, let B_y be the holomorphic Killing vector field on $\mathbb{P}(E^*)$ induced by the corresponding vector field on E^* . The vector field B_y lies in $T^V\mathbb{P}(E^*)$. Then $L^{E^*}y$ is a (1,1) form on X taking values in vector fields in $T^V\mathbb{P}(E^*)$. $L^{E^*}y$ lifts to a (1,1) form on $\mathbb{P}(E^*)$.

Assume first that $p = 0$, and $\xi = \mathbb{C}$. Let d^V be the de Rham operator along the fibres of $\mathbb{P}(E^*)$, and let d be the de Rham operator on $\mathbb{P}(E^*)$. Similarly $\tilde{\nabla}_0$ can be made to act on the de Rham complex of $\mathbb{P}(E^*)$. Then by [B1] eq. (1.30) and [BGS1] eq. (1.26), we find that

$$(19) \quad d = \tilde{\nabla}_0 + d^V + i_{L^{E^*}y}.$$

Since L^{E^*} is of type (1,1), we deduce from (19) that we have the identity of operators acting on $\Omega^{(0,\cdot)}(\xi_0)$

$$(20) \quad \bar{\partial}_0^{P(E^*)} = \tilde{\nabla}_0'' + \bar{\partial}^V.$$

Extending (20) to $\Omega^{(0,\cdot)}(\xi_p)$ is easy and is left to the reader. □

Remark 4. — The fibration $\pi : \mathbb{P}(E^*) \rightarrow X$ is locally Kähler in the sense of Bismut-Gillet-Soulé [BGS1], [BGS2]. Part of the identities in (17) follows from [BGS1], Theorem 2.6.

Let $\tilde{\nabla}_p^{''*}, \bar{\partial}_p^{V*}, \bar{\partial}_p^{P(E^*)}$, be the formal adjoints of $\tilde{\nabla}_p'', \bar{\partial}_p^V, \bar{\partial}_p^{P(E^*)}$, with respect to the obvious Hermitian product on $\Omega^{(0,\cdot)}(\xi_p)$ associated to the various metrics.

Observe that $\bar{\partial}_p^{V*}$ restricts on each fibre of $\pi : \mathbb{P}(E^*) \rightarrow E^*$ to the fibrewise adjoint of $\bar{\partial}_p^V$.

THEOREM 5. — *The following identities of operators acting on $\Omega^{(0,\cdot)}(\xi_p)$ hold*

$$(21) \quad (\bar{\partial}_p^{P(E^*)} + \bar{\partial}_p^{P(E^*)*})^2 = (\tilde{\nabla}_p'' + \tilde{\nabla}_p^{''*})^2 + (\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2$$

$$[(\tilde{\nabla}_p'' + \tilde{\nabla}_p^{''*})^2, (\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2] = 0.$$

Proof. — (21) follows from Proposition 3. □

For $0 \leq q \leq \dim X, 0 \leq q' \leq \dim E - 1$, let $\Omega^{(0,q,q')}(\xi_p)$ be the set of smooth section of $\pi^*(\Lambda^q(T^{*(0,1)}X)) \widehat{\otimes} \Lambda^{q'}(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$. By (12), we know that $\Omega^{(0,q,q')}(\xi_p)$ is a vector subspace of $\Omega^{(0,q+q')}(\xi_p)$. More precisely, for any $q, 0 \leq q \leq \dim X + \dim E - 1$

$$(22) \quad \Omega^{(0,q)}(\xi_p) = \bigoplus_{q'+q''=q} \Omega^{(0,q',q'')}(\xi_p).$$

Set

$$(23) \quad \square_p^{P(E^*)} = (\bar{\partial}_p^{P(E^*)} + \bar{\partial}_p^{P(E^*)*})^2.$$

By Theorem 5, we find that

$$(24) \quad \square_p^{P(E^*)} = \tilde{\nabla}_p'' \tilde{\nabla}_p^{''*} + \tilde{\nabla}_p^{''*} \tilde{\nabla}_p'' + \bar{\partial}_p^V \bar{\partial}_p^{V*} + \bar{\partial}_p^{V*} \bar{\partial}_p^V.$$

From (24), it is clear that the operator $\square_p^{P(E^*)}$ acts on each $\Omega^{(0,q,q')}(\xi_p)$. Let $\square_p^{P(E^*),q,q'}$ be the restriction of $\square_p^{P(E^*)}$ to $\Omega^{(0,q,q')}(\xi_p)$.

We now have the following result directly inspired by Getzler [Ge].

THEOREM 6. — *For any $p \in N, 0 \leq q \leq \dim X, t > 0$, the following identity holds*

$$(25) \quad \text{Tr} [\exp(-t \square_p^{X,q})] = \sum_{q'=0}^{\dim E-1} (-1)^{q'} \text{Tr} [\exp(-t \square_p^{P(E^*),q,q'})].$$

Proof. — Let F_p be the vector space of smooth sections of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$. As in the proof of Proposition 3, we regard F_p as an infinite dimensional vector bundle over X . If $x \in X$,

the fibre $F_{p,x}$ is the set of smooth sections of $\Lambda(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)_x$. F_p is a $U(k)$ -equivariant Hermitian vector bundle on X , and $\tilde{\nabla}_p$ is the corresponding holomorphic Hermitian connection. Also $\Omega^{(0,\cdot)}(F_p)$ is canonically isomorphic to $\Omega^{(0,\cdot)}(\xi_p)$.

The operator $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$ is $U(k)$ -equivariant. Therefore the spectrum of $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$ acting on a fibre $F_{p,x}$ does not depend on $x \in X$. In the sequel $\lambda \geq 0$ varies in the spectrum of $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$.

The vector bundle F_p over X then splits into a direct orthonormal sum of finite dimensional vector spaces F_p^λ which are eigenspaces of $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$ associated with the eigenvalues λ , i.e.

$$(26) \quad F_p = \bigoplus_{\lambda \geq 0} F_p^\lambda.$$

For $0 \leq q' \leq \dim E - 1$, let $F_p^{q'}$ be the set of smooth sections of $\Lambda^{q'}(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$. Clearly $F_p = \bigoplus_{q'=0}^{\dim E - 1} F_p^{q'}$. Also the operator $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$ preserves each $F_p^{q'}$. To the splitting (26) of F_p corresponds the splitting

$$(27) \quad F_p^{q'} = \bigoplus_{\lambda \geq 0} F_p^{q',\lambda}$$

of each $F_p^{q'}$. Using (10) and Hodge theory, we know that

$$(28) \quad \begin{aligned} F_p^{q',(0)} &= S^p(E) \otimes \xi & \text{if } q' = 0 \\ &= 0 & \text{if } q' > 0. \end{aligned}$$

Moreover since $U(k)$ acts irreducibly on $S^p(E)$, the metric on $S^p(E)$ induced from the L_2 metric on the fibers of $\mathbb{P}(E^*)$ coincides (up to an irrelevant constant) with the metric $\| \cdot \|_{S^p(E)}$.

Let $\tilde{\square}_p^{q',\lambda}$ be the restriction of the operator $(\tilde{\nabla}_p'' + \tilde{\nabla}_p''^*)^2$ to the set of smooth sections of $\Lambda^q(T^*(0,1)X) \widehat{\otimes} F_p^{q',\lambda}$ over X . From Theorem 5, we get

$$(29) \quad \begin{aligned} \sum_{q'=0}^{\dim E - 1} (-1)^{q'} \text{Tr} [\exp(-t \tilde{\square}_p^{(E^*),q,q'})] \\ = \sum_{\lambda \geq 0} \exp(-t\lambda) \sum_{q'=0}^{\dim E - 1} (-1)^{q'} \text{Tr} [\exp(-t \tilde{\square}_p^{q,q',\lambda})]. \end{aligned}$$

From (28) and from the considerations which follow, we find that

$$(30) \quad \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr} [\exp(-t \tilde{\square}_p^{q, q', 0})] = \operatorname{Tr} [\exp(-t \bar{\square}_p^{X, q})].$$

On the other hand, for $\lambda > 0$, we have a $U(k)$ -equivariant exact sequence of vector bundles on X

$$(31) \quad 0 \rightarrow F_p^{0, \lambda} \xrightarrow{\partial_p^V} F_p^{1, \lambda} \rightarrow \dots \xrightarrow{\partial_p^V} F_p^{\dim E-1, \lambda} \rightarrow 0.$$

From (31), we easily deduce that for $\lambda > 0$

$$(32) \quad \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr} [\exp(-t \tilde{\square}_p^{q, q', \lambda})] = 0.$$

Using (30), (32), we get (25). □

Remark 7. — As $t \rightarrow 0$, the left-hand side of (25) has a singularity $t^{-\dim X}$. A priori, the right-hand side has a singularity $t^{-(\dim X + \dim E-1)}$. Therefore a cancellation process occurs in the right-hand side of (25) as $t \rightarrow 0$.

Proof of Theorem 1. — Let r_d be the analogue of r_d^H on $\mathbb{P}(E^*)$. Namely if $w'_1, \dots, w'_{\ell+k-1}$ is an orthonormal base of $T\mathbb{P}(E^*)$, if $w'^1, \dots, w'^{\ell+k-1}$ is the corresponding base of $T^*\mathbb{P}(E^*)$, set

$$(33) \quad r_d = - \sum r(w'_i, \bar{w}'_j) \bar{w}'^{ij} \wedge i_{\bar{w}'_i}.$$

Then r_d acts as a derivation of

$$\Lambda(T^{*(0,1)\mathbb{P}}(E^*)) = \pi^*(\Lambda(T^{*(0,1)X})) \widehat{\otimes} \Lambda(T^{V^*(0,1)\mathbb{P}}(E^*)).$$

We identify r with the self-adjoint matrix $\mathring{r} \in \operatorname{End}(\overline{T\mathbb{P}(E)})$ such that $U, V \in T\mathbb{P}(E)$

$$(34) \quad r(U, V) = \langle U, \mathring{r} \bar{V} \rangle.$$

By (1), (2), it is clear that r_d preserves $\pi^*(\Lambda^q(T^{*(0,1)X})) \otimes \Lambda^{q'}(T^{V^*(0,1)\mathbb{P}}(E^*))$. Let $r_d^{q, q'}$ be the corresponding restriction of r_d .

Let dz be the volume form on $\mathbb{P}(E^*)$ with respect to the metric $\| \cdot \|_{T\mathbb{P}(E^*)}$.

Clearly as $t \rightarrow 0$, we have the asymptotic expansion

$$(35) \quad (2\pi)^{-(\dim X + \dim E - 1)} \int_{\mathbb{P}(E^*)} \det(\dot{r}) \frac{\text{Tr}[e^{t_d^{q,q'}}]}{\det(1 - e^{-tr})} dz = \sum_{j=-\ell-k+1}^m b_j^{q,q'} t^j + o(t^m).$$

For any $p \in \mathbb{N}$, $0 \leq q \leq \dim X$, $0 \leq q' \leq \dim E - 1$, as $t \rightarrow 0$, we have the asymptotic expansion

$$(36) \quad p^{-(\dim X + \dim E - 1)} \text{Tr} \left[\exp \left(-\frac{t}{p} \square_p^{\mathbb{P}(E^*), q, q'} \right) \right] = \sum_{j=-\ell-k+1}^m b_{p,j}^{q,q'} t^j + o(t^m).$$

By a straightforward adaptation of [B2]. Theorem 1.5, and [BV] Theorem 2, we know that for any $t > 0$

$$(37) \quad \lim_{p \rightarrow +\infty} p^{-(\dim X + \dim E - 1)} \text{Tr} \left[\exp \left(-\frac{t}{p} \square_p^{\mathbb{P}(E^*), q, q'} \right) \right] = (2\pi)^{-(\dim X + \dim E - 1)} rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\det(\dot{r}) \text{Tr}[e^{t_d^{q,q'}}]}{\det(1 - e^{-tr})} dz$$

and the convergence is uniform as t varies in compact subsets of R_+^* . Also as $p \rightarrow +\infty$

$$(38) \quad b_{p,j}^{q,q'} = rk(\xi) b_j^{q,q'} + O\left(\frac{1}{\sqrt{p}}\right).$$

Moreover in (36), $o(t^m)$ is uniform with respect to $p \in \mathbb{N}$.

By (2) \dot{r} map $T^V \mathbb{P}(E^*)$ into itself. Let \dot{r}^V be the restriction of \dot{r} to $T^V \mathbb{P}(E^*)$. We then find that

$$(39) \quad \sum_0^{\dim E - 1} (-1)^{q'} \text{Tr}[e^{tr_d^{q,q'}}] = \text{Tr}[e^{tr_d^{H,q}}] \det(1 - e^{-tr^V})$$

$$\det(1 - e^{-tr}) = \det(1 - e^{-tr^H}) \det(1 - e^{-tr^V}).$$

By (25), (37), (39), we get

$$(40) \quad \lim_{p \rightarrow +\infty} p^{-(\dim X + \dim E - 1)} \text{Tr} \left[\exp \left(-\frac{t}{p} \square_p^{X, q, q'} \right) \right] = rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\text{Tr}[e^{tr_d^{H,q}}]}{\det(1 - e^{-tr^H})} \det\left(\frac{\dot{r}}{2\pi}\right) dz.$$

Clearly

$$(41) \quad \det \left(\frac{\dot{r}}{2\pi} \right) dz = \left[\exp \left(\frac{-r}{2i\pi} \right) \right]^{\max}.$$

Using (40), (41), we get (8). From the previous considerations, we also obtain the full proof of Theorem 1. □

3. The asymptotics of the Ray-Singer analytic torsion as $p \rightarrow \infty$.

From now on, we assume that the holomorphic Hermitian vector bundle $(E, \| \cdot \|_E)$ is positive, i.e. that if $U \in TX \setminus \{0\}$, $e \in E \setminus \{0\}$

$$(42) \quad \langle L^E(U, \bar{U})e, \bar{e} \rangle > 0.$$

From (2), we find that if $y \in E^* \setminus \{0\}$ represent $z \in \mathbb{P}(E^*)$, then

$$(43) \quad r_z = r^V + \pi^* \frac{\langle L^E \bar{y}, y \rangle}{|y|^2}.$$

Classically [GrH] p. 30, the restriction of the line $(\mu, \| \cdot \|_\mu)$ to the fibres $\mathbb{P}(E^*)$ is positive, i.e. if $U \in T^V \mathbb{P}(E^*) \setminus \{0\}$, $r^V(U, \bar{U}) > 0$. From (43), we deduce that the Hermitian line bundle $(\mu, \| \cdot \|_\mu)$ is positive on $\mathbb{P}(E^*)$. This is of course a well-known result [K] Theorem III 6.19.

THEOREM 8. — *There exists $C > 0$, $c > 0$, $c' > 0$ such that for any $p \in \mathbb{N}$, $1 \leq q \leq \ell$, $t \geq 1$, then*

$$(44) \quad p^{-(\dim X + \dim E - 1)} \text{Tr} \left[\exp \left(- \frac{t}{p} \square_p^{X, q, q} \right) \right] \leq C \exp \left(- \left(c - \frac{c'}{p} \right) t \right).$$

Proof. — By [BV] Theorems 1 and 2, there exist $C > 0$, $c > 0$, $c' > 0$ such that for $p \in \mathbb{N}$, $0 \leq q \leq \ell$, $0 \leq q' < k - 1$, $q + q' \geq 1$, $t \geq 1$

$$(45) \quad \text{Tr} \left[\exp \left(- \frac{t}{p} \square_p^{P(E^*), q, q'} \right) \right] \leq C \exp \left(- \left(c - \frac{c'}{p} \right) t \right).$$

Using (25) and (45), (44) follows. □

Remark 9. — Let λ_p^q be the lowest eigenvalue of $\bar{\square}_p^{X,q}$. From (44), we deduce that if $q \geq 1$

$$(46) \quad \lambda_p^q \geq cp - c'.$$

(46) is also an easy consequence of Theorem 5, of the considerations in the proof of Theorem 6 and of [BV], Theorem 1. [BV] Theorem 1 is itself a consequence of the Bochner-Kodaira-Nakano formula of Demailly [De] for the operator $\bar{\square}_p^{p(E^*)}$. Strangely enough, (46) does not seem to be a straightforward consequence of a similar formula for $\bar{\square}_p^X$.

By Theorem 8 or by (46), there exists $p_0 \in \mathbb{N}$ such that if $p \geq p_0$, $1 \leq q \leq \ell$, the operator $\bar{\square}_p^{p(E^*),q}$ is invertible.

DEFINITION 10. — For $p \geq p_0$, $s \in \mathbb{C}$, $\text{Re}(s) \geq \ell$, set

$$(47) \quad \zeta_p(s) = \frac{-1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left(\sum_1^\ell (-1)^q q \text{Tr} [\exp(-t \bar{\square}_p^{X,q})] \right) dt.$$

By a well-known result of Seeley [Se], $\zeta_p(s)$ extends into a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$. By definition $\exp(-\zeta_p'(0))$ is the Ray-Singer analytic torsion [RS] of the Hermitian vector bundle $S^p(E) \otimes \xi$.

We now state the main result of this paper.

THEOREM 11. — As $p \rightarrow +\infty$

$$(48) \quad \zeta_p'(0) = rk(\xi) \frac{1}{2} \int_{\mathbb{P}(E^*)} \text{Log} \left[\det \left(\frac{p^{\circ H}}{2\pi} \right) \right] \exp \left(\frac{-pr}{2i\pi} \right) + o(p^{(\dim X + \dim E - 1)}).$$

In particular as $p \rightarrow +\infty$

$$(49) \quad \zeta_p'(0) = O(p^{\dim X + \dim E - 1} \text{Log } p).$$

Proof. — In view of Theorems 1 and 8, which are the obvious extensions of [BV] Theorem 2, the proof of Theorem 11 proceeds formally as the proof of [BV] Theorems 4 and 8. Details are left to the reader. □

BIBLIOGRAPHY

- [B1] J. M. BISMUT, The index Theorem for families of Dirac operators : two heat equation proofs, *Invent. Math.*, 83 (1986), 91-151.
- [B2] J. M. BISMUT, Demailly's asymptotic Morse inequalities : a heat equation proof, *J. Funct. Anal.*, 72 (1987), 263-278.
- [BGS1] J. M. BISMUT, H. GILLET, C. SOULÉ, Analytic torsion and holomorphic determinant bundles. II, *Comm. Math. Phys.*, 115 (1988), 79-126.
- [BGS2] J. M. BISMUT, H. GILLET, C. SOULÉ, Analytic torsion and holomorphic determinant bundles. III, *Comm. Math. Phys.*, 115 (1988), 301-351.
- [BV] J. M. BISMUT, E. VASSEROT. The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle, *Comm. Math. Phys.*, 125 (1989), 355-367.
- [De] J. P. DEMAILLY, Vanishing theorems for tensor powers of a positive vector bundle. In *Geometry and Analysis*, T. Sunada, ed., pp. 86-106, *Lecture Notes in Math.* Berlin-Heidelberg-New York, Springer-Verlag, 1988.
- [Ge] E. GETZLER, Inégalités asymptotiques de Demailly pour les fibrés vectoriels, *C.R. Acad. Sci., Série I. Math.*, 304 (1987), 475-478.
- [GrH] P. GRIFFITHS, J. HARRIS, *Principles of algebraic geometry*, New York, Wiley, 1978.
- [K] S. KOBAYASHI, *Differential geometry of complex vector bundles*, Iwanami Shoten and Princeton University Press, 1987.
- [LP] J. LE POTIER, Théorèmes d'annulation en cohomologie, *C.R. Acad. Sci. Paris, Série A*, 276 (1976), 535-537.
- [Q] D. QUILLEN, Superconnections and the Chern character, *Topology*, 24 (1985), 89-95.
- [RS] D. B. RAY, I. M. SINGER, Analytic torsion for complex manifolds, *Ann. of Math.*, 98 (1973), 154-177.
- [Se] R. T. SEELEY, Complex powers of an elliptic operator, *Proc. Symp. Pure and Appl. Math.*, Vol. 10, 288-307, Providence, Am. Math. Soc., (1967).

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