Annales de l'institut Fourier

Plamen Stefanov

Stability of the inverse problem in potential scattering at fixed energy

Annales de l'institut Fourier, tome 40, nº 4 (1990), p. 867-884

<http://www.numdam.org/item?id=AIF_1990__40_4_867_0>

Ann. Inst. Fourier, Grenoble
40, 4 (1990), 867-884.

STABILITY OF THE INVERSE PROBLEM IN POTENTIAL SCATTERING AT FIXED ENERGY

by Plamen STEFANOV

1. Introduction.

Let $q(x)$ be a potential supported in the ball $B_\rho = \{x; |x| \leqslant \rho\}$ and consider the related quantum-mechanical scattering amplitude $A_q(\omega,\theta,k)$. The inverse problem under consideration is to recover the potential $q(x)$ if $A_q(\omega,\theta,k)$ is known at a fixed energy $k > 0$. The uniqueness of this inverse problem has been established by R. G. Novikov [No] (see also [SU4], [R]). Novikov's proof (as well as the other approaches) reduces the problem to the uniqueness of the inversion $\Lambda_{q-k^2} \to q$, where Λ_{q-k^2} is the Dirichlet-to-Neumann map on $S_R = \partial B_R$ (see sect. 3) for some fixed $R > \rho$. The latter problem has been investigated in a series of papers by Sylvester, Uhlmann, Nachman [SU1]-[SU4], [NSU], [Na1] (see also [No]).

The aim of the present work is to derive a stability estimate related to the map $A_q(\omega,\theta,k) \to q(k\text{-fixed})$. More precisely, we find a special norm of A_q with respect to which that map is continuous. In order to define it, given a function $A(\omega,\theta)$, $\omega \in S^2$, $\theta \in S^2$ let us expand $A(\omega,\theta)$ in the spherical harmonics Y_n^m:

$$(1.1)\quad A(\omega,\theta) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} a_{n_1 m_1 n_2 m_2} Y_{n_1}^{m_1}(\omega) Y_{n_2}^{m_2}(\theta)$$

This work is partially supported by the Bulgarian Committee of Sciences, Grant 52/1990.

Key-words : Inverse scattering problem - Fixed energy stability.
A.M.S. Classification : 35P25.

(see section 2 for more details). Then we set

$$\|A\|_{R,s_1,s_2} = \left\{ \sum \left(\frac{2n_1+1}{ekR}\right)^{2n_1+2s_1} \left(\frac{2n_2+1}{ekR}\right)^{2n_2+2s_2} |a_{n_1m_1n_2m_2}|^2 \right\}^{1/2}.$$

Obviously, $\|\cdot\|_{R,s_1,s_2}$ is a norm which depends also on k (assumed to be fixed). If A_q is a scattering amplitude associated with some q with $\operatorname{supp} q \subset B_\rho$, then $\|A_q\|_{R,s_1,s_2} < \infty$ for any $R > \rho$. On the other hand, even for smooth A, $\|A\|_{R,s_1,s_2}$ may be infinite. This norm admits also the following equivalent interpretation. Let Δ_{LB} be the Laplace-Beltrami operator on S^2. Denote $B = (1-\Delta_{LB})^{1/2}$. Since for the eigenvalues of B we have $(n(n+1)+1)^{1/2} = n + 1/2 + O(n^{-1})$, $\|A\|_{R,s_1,s_2}$ is equivalent to the following norm

$$\left\| \left(\frac{2B_\omega}{ekR}\right)^{B_\omega+s_1-1/2} \left(\frac{2B_\theta}{ekR}\right)^{B_\theta+s_2-1/2} A \right\|_{L^2(S^2)\times L^2(S^2)}.$$

Note that the expressions in the brackets are well-defined by means of the spectral theorem.

Our main result is the following.

THEOREM. 1.1. — *Let* $\operatorname{supp} q_0 \subset B_\rho$, $q = \bar{q}$ *and fix* $k > 0$. *Then for any* $R > \rho$ *we have :*

a) *If* $q_0 \in H^s(\mathbb{R}^3)$, $s > 3/2$, *then there exists a neighborhood of* q_0 *of the kind* $\mathcal{O} = \{q \in H^s(\mathbb{R}^3)\,;\, \operatorname{supp} q \subset B_R, \|q-q_0\|_{H^s} < E\}$, *such that if* $q_1 \in \mathcal{O}$, $q_2 \in \mathcal{O}$, *then*

$$\|q_1 - q_2\|_{L^\infty} \leqslant C\phi(\|A_{q_1}(\cdot,\cdot,k) - A_{q_2}(\cdot,\cdot,k)\|_{R,3/2,-1/2}),$$

where $\phi(t) = (-\ln t)^{-\delta}$, $0 < \delta < 1$ *for* $t > 0$ *sufficiently small.*

b) *If* $q_0 \in L^\infty(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$, $s > 0$, *then there exists a neighborhood of* q_0 *of the kind* $\{q \in L^\infty(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)\,;\, \operatorname{supp} q \subset B_R, \|q-q_0\|_{L^\infty} + \|q-q_0\|_{H^s} < E\}$, *such that for all* q_1, q_2 *from that neighborhood we have*

$$\|q_1 - q_2\|_{L^2} \leqslant C\phi(\|A_{q_1}(\cdot,\cdot,k) - A_{q_2}(\cdot,\cdot,k)\|_{R,3/2,-1/2}).$$

Our approach is inspired by Nachman's constructive method [Na2] of reconstructing $q(x)$ from the scattering amplitude A_q at a fixed $k > 0$, and by the paper of Alessandrini [A]. A brief description of Nachman's method is the following. Given $A_q(\omega,\theta,k)$ at a fixed k we first recover the Green's function $G_q(x,y,k)$ for $|x| > \rho$, $|y| > \rho$ (see Proposition 2.2). To this end we expand A_q and G_q in special functions and use the fact that the leading term of the asymptotic of $G_q(x,y,k)$ as $|x| \to \infty$, $|y| \to \infty$ is $[e^{ik|x|}\, e^{ik|y|}/(4\pi|x||y|)]\, A_q\left(\frac{x}{|x|}, -\frac{y}{|y|}, k\right)$ (see Proposition 2.1). Next, since $G_q(x,y,k)$ is known for $|x| = R$, $|y| = R$, $R > \rho$, we reconstruct Λ_{q-k^2} by means of explicit formulae (see (3.1)). And finally, we can use the results of [Na1] to perform the inversion $\Lambda_{q-k^2} \to q$. In order to prove Theorem 1.1, we first get a stability estimate related to the map $A_q \to \Lambda_{q-k^2}$. Next, following essentially [A] we get a stability estimate corresponding to the map $\Lambda_{q-k^2} \to q$.

Acknowledgments. – The author would like to thank Prof. A. Nachman for explaining him the method in [Na2] and for pointing him out Ref. [L] which is the essential part of the proof of Lemma 3.2.

2. Properties of the Green's function.

Throughout this section we assume $k > 0$ fixed and $q \in L^\infty(\mathbb{R}^3)$, $\operatorname{supp} q \subset B_\rho$, $q = \bar{q}$. Recall that the Green's functions $G_\pm(x,y,k)$ can be defined as the kernels of $(-\Delta + q - k^2 \mp i0)^{-1}$ in $\mathscr{L}(L^2_\delta, L^2_{-\delta})$, $\delta > 1/2$ (see e.g. [Na1]) or, equivalently, as the solution of the problem

$$(-\Delta_x + q(x) - k^2)G_\pm(x,y,k) = \delta(x-y),$$

satisfying the outgoing (incoming) Sommerfeld condition at infinity. The first definition implies $G_+(x,y,k) = \overline{G_-(y,x,k)}$, while the second one yields $G_+(x,y,k) = \overline{G_-(x,y,k)}$. Therefore, $G_\pm(x,y,k) = G_\pm(y,x,k)$. Since in what follows we shall deal only with the outgoing Green's function G_+ we shall drop the sign « + ». To emphasize the dependence on q we shall use the notation G_q instead of G_+. It is known, that $G_q(x,y,k)$ is a function that satisfies the following inequality [S]

$$c|x-y|^{-1} \leqslant G_q(x,y,k) \leqslant c|x-y|^{-1}, \qquad x \in \mathbb{R}^3,\ y \in \mathbb{R}^3,$$

which, in particular, implies $G_q \in L^2_{\mathrm{loc}}(\mathbb{R}^3_x \times \mathbb{R}^3_y)$. Let

$$G_0(x,y,k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

be the outgoing Green's function corresponding to $q = 0$. Since $(\Delta_x + k^2)(G_q - G_0) = (\Delta_y + k^2)(G_q - G_0) = 0$ for $|x| > \rho$, $|y| > \rho$, it follows that $G_q - G_0$ is smooth for such x, y.

The following asymptotic is known and we include its proof only for the sake of completeness.

PROPOSITION 2.1. — *We have*

$$(G_q - G_0)(x,y,k) = \frac{1}{4\pi}\frac{e^{ik|x|}}{|x|}\frac{e^{ik|y|}}{|y|}A_q\left(\frac{x}{|x|}, -\frac{y}{|y|}, k\right) + \frac{1}{|x|\,|y|}B(x,y,k),$$

where $B(x,y,k) \to 0$, *as* $|x| \to \infty$, $|y| \to \infty$.

Proof. — It is known that

$$(2.1)\quad G_0(x,y,k) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}$$

$$= \frac{\exp(ik(|x| - y\cdot x/|x|))}{4\pi|x|} + O(|x|^{-2}),$$

as $|x| \to \infty$, and the convergence is uniform in $y \in B_\rho$. Combining (2.1) with the formula

$$(2.2)\quad (G_q - G_0)(x,y,k) = -\int G_0(x,z,k)q(z)G_q(z,y,k)\,dz,$$

we get

$$(2.3)\quad G_q(x,y,k) = \frac{e^{ik|x|}}{4\pi|x|}v(y, -\frac{x}{|x|}, k) + O(|x|^{-2}),\ \text{as } |x| \to \infty.$$

Here $v(x,\omega,k)$ is the generalized eigenfunction, which is the solution of the Lippmann-Schwinger equation

$$v(x,\omega,k) = \exp(ikx\cdot\omega) - \int G_0(x,y,k)q(y)v(y,\omega,k)dy.$$

The convergence in (2.3) is uniform in $y \in B_\rho$. Substituting (2.1) and (2.3) into the formula

$$(2.4)\quad (G_q - G_0)(x,y,k) = -\int G_q(x,z,k)q(z)G_0(z,y,k)dz,$$

we get

$$(G_q - G_0)(x,y,k) = -\frac{e^{ik|x|}}{4\pi|x|}\frac{e^{ik|y|}}{4\pi|y|}$$

$$\times\left(\int \exp(-ikz\cdot y/|y|)q(z)v(z, -\frac{x}{|x|}, k)dz + B(x,y,k)\right),$$

where B has the needed properties. Applying the formula

$$A_q(\omega,\theta,k) = -\frac{1}{4\pi}\int e^{-ik\omega\cdot x}q(x)v(x,\theta,k)dx$$

and using the equality $A_q(\omega,\theta,k) = A_q(-\theta,-\omega,k)$, we get the required result.

Next we are going to expand $G_q(x,y,k)$ in special functions. Recall that the spherical harmonics are defined as follows :

$$Y_n^m(\omega) = \left[\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}\right]^{1/2} P_{n,|m|}(\cos\alpha)e^{im\beta},$$

$n = 0, 1, \ldots, m = -n, \ldots, n$, where $P_{n,m}(t) = \dfrac{(1-t^2)^{m/2}}{2^n n!}\dfrac{d^n}{dt^n}(t^2-1)^n$, and $\omega = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$. It is well-known that $\{Y_n^m;\ n=0,1,\ldots,m=-n,\ldots,n\}$ form an orthonormal base in $L^2(S^2)$ and if Δ_{LB} is the Laplace-Beltrami operator on S^2, then

$$-\Delta_{LB}Y_n^m = n(n+1)Y_n^m. \tag{2.5}$$

Let $h_n^{(1)}(r) = \left(\dfrac{\pi}{2r}\right)^{1/2} H_{n+1/2}^{(1)}(r)$ be the spherical Hankel function with asymptotic

$$h_n^{(1)}(r) = (-i)^{n+1}\frac{e^{ir}}{r} + O(r^{-2}),\quad \text{as } r \to \infty. \tag{2.6}$$

Now, let us expand the scattering amplitude $A_q(\omega,\theta,k)$ in spherical harmonics

$$A_q(\omega,\theta,k) = \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2} a_{n_1m_1n_2m_2} Y_{n_1}^{m_1}(\omega)\, Y_{n_2}^{m_2}(\theta)$$

where

$$a_{n_1m_1n_2m_2} = \int_{S^2}\int_{S^2} A_q(\omega,\theta,k)\,\overline{Y_{n_1}^{m_1}(\omega)}\;\overline{Y_{n_2}^{m_2}(\theta)}\,d\omega\, d\theta. \tag{2.7}$$

The main result in this section is the following.

PROPOSITION. 2.2. – *For* $|x| > \rho$, $|y| > \rho$ *we have the expansion*

$$(2.8)\quad G_q(x,y,k) - G_0(x,y,k) = \sum_{n_1=0}^{\infty} \sum_{m_1=-n_1}^{n_1} \sum_{n_2=0}^{\infty} \sum_{m_2=-n_2}^{n_2} \gamma_{n_1m_1n_2m_2} \times h_{n_1}^{(1)}(k|x|)\, Y_{n_1}^{m_1}\left(\frac{x}{|x|}\right) h_{n_2}^{(1)}(k|y|)\, Y_{n_2}^{m_2}\left(\frac{y}{|y|}\right),$$

where $\gamma_{n_1m_1n_2m_2} = -\dfrac{k^2}{4\pi}(-1)^{n_2}\, i^{n_1+n_2}\, a_{n_1m_1n_2m_2}$.

The series converges uniformy and absolutely for $|x| \geqslant R$, $|y| \geqslant R$ *for any* $R > \rho$ *and can be differentiated termwise.*

Before giving the proof, we note that this proposition enables us to find the values of $G_q(x,y,k)$ for $|x| > \rho$, $|y| > \rho$, if A_q is known for all ω, θ and $k > 0$ fixed (see also [Na2]).

Proof. – We use the formula [EMOT]

$$(2.9)\quad G_0(x,y,k) = \sum_{n,m} \mathscr{E}_{nm}(y,k)\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right), \qquad |x| > |y|,$$

where $\mathscr{E}_{nm} = ikj_n(k|y|)\, Y_n^m\left(\dfrac{y}{|y|}\right)$ and $j_n(r)$ is the spherical Bessel function. From (2.2) and (2.4) we get $G_q - G_0 = I_1 + I_2$, where

$$I_1 = -\int G_0(x,z,k)q(z)G_0(z,y,k)\, dz\,,$$

$$I_2 = \int\!\!\int G_0(x,z_1,k)q(z_1)G_q(z_1,z_2,k)q(z_2)G_0(z_2,y,k)\, dz_1\, dz_2\,.$$

Taking into account (2.9), we get for $|x| > \rho$, $|y| > \rho$

$$I_1 = \sum \alpha_{n_1m_1n_2m_2} h_{n_1}^{(1)}(k|x|)\, Y_{n_1}^{m_1}\left(\frac{x}{|x|}\right) h_{n_2}^{(1)}(k|y|)\, Y_{n_2}^{m_2}\left(\frac{y}{|y|}\right),$$

where $\alpha_{n_1m_1n_2m_2} = -\int \mathscr{E}_{n_1m_1}(z,k)q(z)\mathscr{E}_{n_2m_2}(z,k)\, dz$. Here and below we use the sign $\sum$ to denote the sum in (2.8). The coefficients $\alpha_{n_1m_1n_2m_2}$ admit the following estimate

$$(2.10)\quad |\alpha_{n_1m_1n_2m_2}| \leqslant \|q\|_{L^\infty} \|\mathscr{E}_{n_1m_1}\|_{L^2(B_\rho)} \|\mathscr{E}_{n_2m_2}\|_{L^2(B_\rho)}\,.$$

Similarly, for $|x| > \rho$, $|y| > \rho$

$$I_2 = \sum \beta_{n_1 m_1 n_2 m_2} h_{n_1}^{(1)}(k|x|)\, Y_{n_1}^{m_1}\left(\frac{x}{|x|}\right) h_{n_2}^{(1)}(k|y|)\, Y_{n_2}^{m_2}\left(\frac{y}{|y|}\right),$$

where

$$\beta_{n_1 m_1 n_2 m_2} = \int\int \mathscr{E}_{n_1 m_1}(z_1, k) q(z_1) G_q(z_1, z_2, k) q(z_2) \mathscr{E}_{n_2 m_2}(z_2, k)\, dz_1\, dz_2 .$$

Since $q(z_1)G_q(z_1, z_2, k)q(z_2) \in L^2(\mathbb{R}^6)$, we get

$$|\beta_{n_1 m_1 n_2 m_2}| \leqslant C\, \|\mathscr{E}_{n_1 m_1}\|_{L^2(B_\rho)}\, \|\mathscr{E}_{n_2 m_2}\|_{L^2(B_\rho)} . \tag{2.11}$$

Therefore, $G_q - G_0$ admits the required expansion with some coefficients $\gamma_{n_1 m_1 n_2 m_2}$, and relations (2.10), (2.11) imply the same estimate for $\gamma_{n_1 m_1 n_2 m_2}$:

$$|\gamma_{n_1 m_1 n_2 m_2}| \leqslant C\, \|\mathscr{E}_{n_1 m_1}\|_{L^2(B_\rho)}\, \|\mathscr{E}_{n_2 m_2}\|_{L^2(B_\rho)} . \tag{2.12}$$

Let us estimate $\|\mathscr{E}_{nm}\|$. Using Lemma A1, we get

$$\begin{aligned} \|\mathscr{E}_{nm}\|_{L^2(B_\rho)}^2 &\leqslant k^2 \int_{B_\rho} |j_n(k|y|)|^2 \left| Y_n^m\left(\frac{y}{|y|}\right)\right|^2 dy \\ &= k^2 \int_0^\rho \int_{S^2} |j_n(kr)|^2 |Y_n^m(\theta)|^2 r^2\, d\theta\, dr = k^2 \int_0^\rho |j_n(kr)|^2 r^2\, dr \\ &\leqslant \frac{C}{n^2} \int_0^\rho \left(\frac{ekr}{2n+1}\right)^{2n} r^2\, dr = C' \frac{(2n+1)^3}{n^2(2n+3)} \left(\frac{ek\rho}{2n+1}\right)^{2n+3} . \end{aligned}$$

Therefore, for $n \geqslant 1$ we get

$$\|\mathscr{E}_{nm}\|_{L^2(B_\rho)} \leqslant C\left(\frac{ek\rho}{2n+1}\right)^{n+3/2} . \tag{2.13}$$

A straightforward calculation shows that (2.12) remains true for $n = 0$. Combining (2.12) and (2.13), we arrive at the estimate

$$|\gamma_{n_1 m_1 n_2 m_2}| \leqslant C\left(\frac{ek\rho}{2n_1+1}\right)^{n_1+3/2} \left(\frac{ek\rho}{2n_2+1}\right)^{n_2+3/2} . \tag{2.14}$$

In order to prove that the series (2.8) is uniformly and absolutely convergent, let us estimate each term. Recall the estimate

$$|Y_n^m(\omega)| \leqslant \left(\frac{2n+1}{4\pi}\right)^{1/2}$$

(see e.g. Lemma 13, [M]). Combining it with Lemma A2 and (2.14), we see that the right hand side of (2.8) can be estimated termwise by the series

$$C\sum_{n_1,n_2=0}^{\infty}\left(\frac{ek\rho}{2n_1+1}\right)^{n_1+3/2}\left(\frac{ek\rho}{2n_2+1}\right)^{n_2+3/2} n_1(2n_1+1)^{1/2}n_2(2n_2+1)^{1/2} \times\left(2+\frac{2n_1}{ekR}\right)^{n_1}\left(2+\frac{2n_2}{ekR}\right)^{n_2},$$

provided $|x|\geqslant R$, $|y|\geqslant R$. The series above does not depend on x, y and if $R>\rho$, then it converges. This proves the last assertion of the proposition.

Now, in order to complete the proof, take the limit $|x|\to\infty$, $|y|\to\infty$ in (2.8). Then, using asymptotic (2.6), Proposition 2.1, and the equality $Y_n^m(-\theta)=(-1)^n Y_n^m(\theta)$, we get the desired relation between $a_{n_1m_1n_2m_2}$ and $\gamma_{n_1m_1n_2m_2}$.

3. Stability of the inversion $A_q\to\Lambda_{q-k^2}$.

Let $A(\omega,\theta)$ be a smooth function on $S^2\times S^2$ (not necessarily a scattering amplitude). Let us expand A in spherical harmonics as in (1.1) and let $a_{n_1m_1n_2m_2}$ be the corresponding coefficients (see (2.7)). We then define the following map ($k>0$ is fixed).

$$\mathcal{N}:\ A(\omega,\theta)\ \to\ a_{n_1m_1n_2m_2}\ \to\ -\frac{k^2}{4\pi}\sum(-1)^{n_2}i^{n_1+n_2}a_{n_1m_1n_2m_2} \times h_{n_1}^{(1)}(k|x|)\,Y_{n_1}^{m_1}\left(\frac{x}{|x|}\right)h_{n_2}^{(1)}(k|y|)\,Y_{n_2}^{m_2}\left(\frac{y}{|y|}\right)=:F(x,y).$$

Now, suppose that A_q is a scattering amplitude at a fixed k corresponding to some q with $\operatorname{supp} q\subset B_\rho$. Then $a_{n_1m_1n_2m_2}$ satisfy the estimate (2.14). Therefore, in that case $F=\mathcal{N}A_q$ is a well-defined function for $|x|>\rho$, $|y|>\rho$, which coincides with $(G_q-G_0)(\cdot,\cdot,k)$. Next, we shall prove that in fact the map $\mathcal{N}$ is continuous in an appropriate topology. Recall the norm $\|\cdot\|_{R,s_1,s_2}$ defined in the Introduction. Estimate (2.14) (which is satisfied also by $a_{n_1m_1n_2m_2}$) shows that $\|A_q(\cdot,\cdot,k)\|_{R,s_1,s_2}<\infty$ for any $R>\rho$, s_1, s_2, $k>0$, provided that A_q is a scattering amplitude corresponding to a potential supported in a ball with a radius ρ. Below we denote $S_R=\partial B_R=\{x\,;\,|x|=R\}$. We have the following.

PROPOSITION. 3.1. — *For any $R > 0$ and for all s_1, s_2 we have*

$$\|\mathcal{N} A(\cdot,\cdot)\|_{H^{s_1}(S_R)\times H^{s_2}(S_R)} \leqslant C_{R,s_1,s_2} \|A\|_{R,s_1,s_2}.$$

Proof. — Recall that $B = (1-\Delta_{LB})^{1/2}$, thus by (2.5) $BY_n^m = (n^2+n+1)^{1/2} Y_n^m$. Therefore

$$B_\omega^{s_1} B_\theta^{s_2} \mathcal{N} A(R\omega, R\theta) = -\frac{k^2}{4\pi}\sum(-1)^{n_2} i^{n_1+n_2} a_{n_1m_1n_2m_2}(n_1^2+n_1+1)^{s_1/2} \times h_{n_1}^{(1)}(kR)\, Y_{n_1}^{m_1}(\omega)(n_2^2+n_2+1)^{s_2/2}\, h_{n_2}^{(1)}(kR)\, Y_{n_2}^{m_2}(\theta).$$

Hence, applying lemma A2, we get

$$\int_{S^2}\int_{S^2} |B_\omega^{s_1} B_\theta^{s_2} \mathcal{N} A(R\omega, R\theta)|^2\, d\omega\, d\theta$$

$$= \frac{k^2}{4\pi}\sum |a_{n_1m_1n_2m_2}|^2 (n_1^2+n_1+1)^{s_1} |h_{n_1}^{(1)}(kR)|^2 (n_2^2+n_2+1)^{s_2} |h_{n_2}^{(1)}(kR)|^2$$

$$\leqslant C\sum |a_{n_1m_1n_2m_2}|^2 (n_1^2+n_1+1)^{s_1}\left(2+\frac{2n_1}{ekR}\right)^{2n_1}(n_2^2+n_2+1)^{s_2}\left(2+\frac{2n_2}{ekR}\right)^{2n_2}$$

$$\leqslant C\sum |a_{n_1m_1n_2m_2}|^2 \left(\frac{2n_1+1}{ekR}\right)^{2n_1+2s_1}\left(\frac{2n_2+1}{ekR}\right)^{2n_2+2s_2},$$

which completes the proof.

Let us recall the Dirichlet to Neumann map Λ_{q-k^2} (see [SU1]-[SU4], [NSU]). Let $q \in L^\infty$, $\operatorname{supp} q \subset B_R$ and assume that k^2 is not a Dirichlet eigenvalue of $-\Delta + q$ in B_R. Let u solve the problem

$$\begin{cases} -\Delta u + qu = k^2 u, \\ u|_{S_R} = f. \end{cases}$$

Then we set $\Lambda_{q-k^2} f := \partial_N u|_{S_R}$, ∂_N being the derivative with respect to the outer normal to S_R. According to [Na1], $\Lambda_{q-k^2} : H^{3/2}(S_R) \to H^{1/2}(S_R)$ is a bounded operator. Below we define the operator norm in $\mathcal{L}(H^{s_1}(S_R), H^{s_2}(S_R))$ by $\|\cdot\|_{s_1,s_2}$. Note that it is different from the norm $\|\cdot\|_{R,s_1,s_2}$ given in Introduction.

Since we deal with a fixed $k > 0$ and with q in a neighborhood of a fixed q_0, the following question arises. Given $k > 0$, $q \in L^\infty$ with $\operatorname{supp} q \subset B_\rho$, can we choose $R > \rho$ such that k^2 is not a Dirichlet eigenvalue of $-\Delta + q$ in B_R? The answer is positive. Indeed, let $\lambda_1(R) \leqslant \lambda_2(R) \leqslant \ldots$ be the Dirichlet eigenvalues of $-\Delta + q$ in B_R, $R > \rho$, counted according to their multiplicities. It is well-known that $\lambda_k(R)$ are non-increasing functions of R (see e.g. [RS], § XIII.15). However, in fact they are *strictly* decreasing, according to [L]. This fact leads us immediately to the following.

LEMMA 3.2. — *The set of all $R > \rho$, such that k^2 is a Dirichlet eigenvalue of $-\Delta + q$ in B_R, is discrete.*

Proof. — Suppose the contrary and let $R_1 < R_2 < \ldots$ be an infinite sequence bounded above by some R_0 such that $k^2 = \lambda_{n_m}(R_m)$ for some n_m. Since $\lambda_n(R)$ is a strictly decreasing function of R and a non-decreasing function of n, we get $n_1 < n_2 < \ldots n_m < \ldots \to \infty$. Thus $k^2 = \lambda_{n_m}(R_m) > \lambda_{n_m}(R_0) \to \infty$, as $m \to \infty$, which leads to a contradiction. The lemma is proved.

The following proposition is the main result of this section.

PROPOSITION 3.3. — *Let $q_0 \in L^\infty$, $\operatorname{supp} q_0 \subset B_\rho$. Assume that $k^2 > 0$ is not a Dirichlet eigenvalue of $-\Delta$ and $-\Delta + q_0$ in B_R, $R > \rho$. Then there exists a neighborhood of q_0 of the kind $\mathcal{O}' = \{q \in L^\infty\,;\, \operatorname{supp} q \subset B_R, \|q-q\|_{L^\infty} < E'\}$ and a constant $C > 0$, such that for all q_1, q_2 belonging to that neighborhood we have*

$$\|\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2}\|_{3/2,1/2} \leqslant C\|A_{q_1}(\cdot,\cdot,k) - A_{q_2}(\cdot,\cdot,k)\|_{R,3/2,-1/2}.$$

Proof. — Let $\mathcal{G}_{q,k} : H^{1/2}(S_R) \to H^{3/2}(S_R)$ be the operator

$$\mathcal{G}_{q,k} f(x) = \int_{S_R} G_q(x,y,k) f(y)\, dS_y.$$

$\mathcal{G}_{q,k}$ is a bounded invertible operator [Na1]. Furthermore, as proved in [Na1],

$$\Lambda_{q-k^2} - \Lambda_{-k^2} = \mathcal{G}_{q,k}^{-1} - \mathcal{G}_{0,k}^{-1}. \tag{3.1}$$

Let us denote the Dirichlet Laplacian in B_R by Δ_D. We can show that the map $L^\infty(B_R) \ni q \to (-\Delta_D + q - k^2)^{-1}$ is a continuous one if $q \in \mathcal{O}' = \{q \in L^\infty\,;\, \operatorname{supp} q \subset B_R, \|q - q_0\|_{L^\infty} < E'\}$ and E' is sufficiently small. Thus k^2 is not a Dirichlet eigenvalue of $-\Delta + q$ in B_R and Λ_{q-k^2} is well-defined for $q \in \mathcal{O}'$. Moreover $\|\mathcal{G}_{q,k}^{-1}\|_{3/2,1/2} \leqslant C$,

$\|\Lambda_{q-k^2}\|_{3/2,1/2} \leqslant C$ for $q \in \mathcal{O}'$. Therefore, if q_1 and q_2 lie in $\mathcal{O}'$ we have according to (3.1)

$$\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2} = \mathcal{G}_{q_1,k}^{-1}(\mathcal{G}_{q_2,k} - \mathcal{G}_{q_1,k})\mathcal{G}_{q_2,k}^{-1}.$$

Hence

$$\begin{aligned}\|\Lambda_{q_1-k^2}-\Lambda_{q_2-k^2}\|_{3/2,1/2} &\leqslant \|\mathcal{G}_{q_1,k}^{-1}\|_{3/2,1/2}\,\|\mathcal{G}_{q_1,k}-\mathcal{G}_{q_2,k}\|_{1/2,3/2}\,\|\mathcal{G}_{q_2,k}^{-1}\|_{3/2,1/2}\\ &\leqslant C\|\mathcal{G}_{q_1,k} - \mathcal{G}_{q_2,k}\|_{1/2,3/2}.\end{aligned}$$

In order to estimate the last term we consider

$$\begin{aligned}[B^{3/2}(\mathcal{G}_{q_1,k}-\mathcal{G}_{q_2,k})f]\,(R\omega) &= B_\omega^{3/2}\int_{S^2}(G_{q_1}-G_{q_2})(R\omega, R\theta, k)f(R\theta)\,R^2\,d\theta\\ &= \int_{S^2}[B_\omega^{3/2}B_\theta^{-1/2}(G_{q_1}-G_{q_2})(R\omega, R\theta, k)]\,B_\theta^{1/2}f(R\theta)\,R^2\,d\theta.\end{aligned}$$

Here $G_{q_i}(x, y, k)$ is the Green's function corresponding to q_i, $i = 1, 2$. Thus we get

$$\begin{aligned}\|\mathcal{G}_{q_1,k} - \mathcal{G}_{q_2,k}\|_{1/2,3/2} &\leqslant C\|(G_{q_1} - G_{q_2})(R\cdot, R\cdot, k)\|_{H^{3/2}(S^2)\otimes H^{-1/2}(S^2)}\\ &\leqslant C'\|(A_q - A_{q_2})(\cdot, \cdot, k)\|_{R,3/2,-1/2}.\end{aligned}$$

In the last step we applied Proposition 3.1.

4. Stability of the inversion $\Lambda_{q-k^2} \rightarrow q$.

In this section we prove that q depends continuously on Λ_{q-k^2} in an appropriate sense. We note that the theorem below has been obtained by Alessandrini [A] for potentials q of the kind $q = \Delta\gamma^{1/2}/\gamma^{1/2}$, $\gamma > \text{const.} > 0$. A slight modification of the proof in [A] yields the desired estimate in a neighborhood of those q, for which k^2 is not a Dirichlet eigenvalue of $-\Delta + q$ in B_R.

THEOREM 4.1. – *Let $q_0 \in H^s(\mathbb{R}^3)$, $s > 3/2$, supp $q_0 \subset B_\varrho$ and suppose that $k^2 > 0$ is not a Dirichlet eigenvalue of $-\Delta + q_0$ in B_R for some $R > \rho$. Then there exists a neighborhood of q_0 of the kind $\mathcal{O} = \{q \in H^s(\mathbb{R}^3)$; supp $q \subset B_R$, $\|q - q_0\|_{H^s} < E\}$, such that if $q_1 \in \mathcal{O}$, $q_2 \in \mathcal{O}$, then*

$$\|q_1 - q_2\|_{L^\infty} \leqslant C\,\phi\,(\|\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2}\|_{3/2,1/2}),$$

where ϕ is the same function as in Theorem 1.1.

If we assume the weaker condition $q_0 \in L^\infty(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$, *with some* $s > 0$, *then there exists a neighborhood of* q_0 *of the kind* $\{q \in L^\infty(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)\,;\ \text{supp}\ q \subset B_R,\ \|q - q_0\|_{L^\infty} + \|q - q_0\|_{H^s} < E\}$, *such that for all* q_1, q_2 *from that neighborhood we have*

$$\|q_1 - q_2\|_{L^2} \leqslant C\,\phi\,(\|\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2}\|_{3/2,1/2}),$$

Proof. – We follow essentially [A], Lemma 2. Given $q \in \mathrm{L}^\infty$, we set $\tilde{q}(x) = q(x) - k^2$ for $(x) \leqslant \rho$, $\tilde{q}(x) = 0$ for $|x| \geqslant \rho$. According to [NSU], Lemma 2.2, there exists $\varepsilon_0 > 0$, such that if $\|\tilde{q}\|_{L^\infty} < \varepsilon_0|\zeta|$, supp $q \subset B_\rho$, then there is a unique solution to the equation $(-\Delta + \tilde{q})\,u = 0$ in $\mathbb{R}^3$ of the form $u(x, \zeta) = e^{\zeta\cdot x}(1 + \psi(x, \zeta))$, where $\zeta\cdot\zeta = 0$, $\zeta \in \mathbb{C}^3$ and

$$\|\psi(\cdot\,, \zeta)\|_{L^2(B_{2R})} \leqslant \frac{C}{|\zeta|}\|\tilde{q}\|_{L^2}. \tag{4.1}$$

Let $\mathcal{O}'$ be as in Proposition 3.3. As mentioned above if $E' > 0$ is sufficiently small Λ_{q-k^2} is well-defined for arbitrary $q \in \mathcal{O}'$. Let $q_1 \in \mathcal{O}'$, $q_2 \in \mathcal{O}'$ and denote by $u_i(x, \zeta)$ the solution of the kind described above related to q_i, $i = 1, 2$.

Set $c_0 = 2^{1/2}\varepsilon_0^{-1} \sup\{\|\tilde{q}\|_{L^\infty}, q \in \mathcal{O}'\}$ and assume $|p| > c_0$. Put

$$\zeta_1 = -\frac{i}{2}p - (ir\eta + \xi), \qquad \zeta_2 = -\frac{i}{2}p + (ir\eta + \xi),$$

where $r > 0$, $p\cdot\eta = p\cdot\xi = \eta\cdot\xi = 0$, $|\eta| = 1$, $|\xi|^2 = r^2 + |p|^2/4$. Note that $|\zeta_j|^2 = 2|\mathrm{Re}\,\zeta_j|^2 = 2(r^2 + |p|^2/4)$, $\xi_j\cdot\xi_j = 0$, $j = 1, 2$. The inequality $|p| > c_0$ guarantees that $u_j(x, \xi_j)$ are well-defined. According to [A], we have

$$\int_{B_R} (q_1 - q_2)(x)\, u_1(x, \xi_1)\, u_2(x, \zeta_2)\, dx = \int_{S_R} u_1(x, \zeta_1)\,(\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2})\, u_2(x, \zeta_2)\, dS_x.$$

Note that $u_j(x, \zeta_j)$, $j = 1, 2$ are harmonic functions in a neighborhood of S_R, hence the right hand side of the equality above is well-defined. We get

$$\int_{B_R} (q_1 - q_2)(x)\, e^{-ip\cdot x} dx = \int_{S_R} u_1(x, \zeta_1)\,(\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2})\, u_2(x, \zeta_2)\, dS_x$$
$$- \int_{B_R} (q_1 - q_2)(x)\, e^{-ip\cdot x}[\psi_1(x, \zeta_1) + \psi_2(x, \zeta_2) + \psi_1(x, \zeta_1)\psi_2(x, \zeta_2)]\, dx.$$

Thus, setting $q := q_1 - q_2$ and using (4.1), we obtain

$$|\hat{q}(p)| \leqslant C \|u_1(\cdot, \zeta_1)\|_{H^{-1/2}(S_R)} \| u_2(\cdot, \zeta_2)\|_{H^{3/2}(S_R)} \times \|\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2}\|_{3/2,1/2} + C(|\zeta_1|^{-1} + |\zeta_2|^{-1} + |\zeta_1|^{-1}|\zeta_2|^{-1}).$$

Here and in what follows we denote by C various constants depending only on $\mathcal{O}'$, which may vary from line to line. Next, since $(-\Delta_x + \tilde{q}_2) u_2 = 0$, we get

$$\|u_2(\cdot, \zeta_2)\|_{H^{3/2}(S_R)} \leqslant C\|u_2(\cdot, \zeta_2)\|_{H^2(B_R)} \leqslant C\|u_2(\cdot, \zeta_2)\|_{L^2(B_{2R})} \leqslant C \exp(2R|\operatorname{Re}\zeta_2|)(1 + |\zeta_2|^{-1}).$$

Similarly,

$$\|u_1(\cdot, \zeta_1)\|_{H^{-1/2}(S_R)} \leqslant C \|u_1(\cdot, \zeta_1)\|_{H^{3/2}(S_R)} \leqslant C \exp(2R|\operatorname{Re}\zeta_1|)(1 + |\zeta_1|^{-1}).$$

Taking into account that $\frac{1}{8}(|p| + r) \leqslant |\operatorname{Re}\zeta_j| = |\xi| \leqslant |p| + r$, we get for $|p| > c_0$

$$|\hat{q}(p)| \leqslant C(e^{4R(|p|+r)}\lambda + r^{-1}). \tag{4.2}$$

Here and in what follows we denote for simplicity

$$\lambda := \|\Lambda_{q_1-k^2} - \Lambda_{q_2-k^2}\|_{3/2,1/2}.$$

Recall that (4.2) holds for any $q_1 \in \mathcal{O}'$, $q_2 \in \mathcal{O}'$ and the constant C depends only on $\mathcal{O}'$, but not on q_1, q_2, $r \in \mathbb{R}$, $|p| > c_0$. In order to obtain a same kind of estimate for $|p| \leqslant c_0$, we apply the following lemma (see [A], Lemma 3).

LEMMA 4.2. – *There exists $\alpha \in (0, 1)$, $c > 0$, such that for any holomorphic function $F(z)$ over $\mathbb{C}^3$, we have*

$$\max_{|z|\leqslant 1} |F(z)| \leqslant C\left(\max_{\substack{1\leqslant|z|\leqslant 2\\ \operatorname{Im}z=0}} |F(z)|\right)^{\alpha} (\max_{|z|\leqslant 4} |F(z)|)^{1-\alpha}.$$

Since $\hat{q}$ is a holomorphic function over $\mathbb{C}^3$, applying Lemma 4.2 together with (4.2), we get

$$\max_{|p|<c_0} |\hat{q}(p)| \leqslant C(\max_{c_0\leqslant|p|\leqslant 2c_0} |\hat{q}(p)|)^{\alpha} \leqslant C(e^{4R(r+2c_0)}\lambda + r^{-1})^{\alpha}.$$

Here we have used the fact $|\hat{q}(p)|$ remains bounded if $q \in \mathcal{O}'$, $p \in \mathbb{C}^3$, $|p| \leqslant 4c_0$. Since $0 < \alpha < 1$, it is easy to show that the following inequality $(a+b)^\alpha \leqslant a^\alpha + b^\alpha$ holds for $a > 0$, $b > 0$, thus

$$\max_{|p|<c_0} |\hat{q}(p)| < C(e^{4R(r+2c_0)}\lambda^\alpha + r^{-\alpha}). \tag{4.3}$$

Combining (4.2) and (4.3) together with the fact that $\|\Lambda_{q-k^2}\|_{3/2,1/2}$ remains bounded when q runs over $\mathcal{O}'$, we get

$$\begin{aligned} |\hat{q}(p)| &\leqslant C\{\exp[4R(\max(|p|,\, 2c_0) + r)]\lambda^\alpha + r^{-\alpha}\} \\ &\leqslant C(e^{4R(r+|p|)}\lambda^\alpha + r^{-\alpha}) \end{aligned} \tag{4.4}$$

for any $p \in \mathbb{R}^3$ with a constant C depending only on $\mathcal{O}'$. Now assume $q \in H^s$, $s = 3/2 + 2\eta$, $\eta \geqslant 0$. Setting $\langle p\rangle = (1 + |p|^2)^{1/2}$, we have $\hat{q}\langle p\rangle^\eta = [\hat{q}\langle p\rangle^s]\langle p\rangle^{-3/2-\eta} \in L^1$ and moreover $\|\hat{q}\langle p\rangle^\eta\|_{L^1} \leqslant C\|q\|_{H^s}$. Since $\|f\|_{L^\infty} \leqslant C'\|f\|_{H^S}$, we have $\mathcal{O} := \{q \in H^s(\mathbb{R}^3);\ \operatorname{supp} q \subset B_R,\ \|q - q_0\|_{H^S} < E = E'/C'\} \subset \mathcal{O}$. Therefore for any $\mu > 0$ and $q \in \mathcal{O}$, we get

$$\int_{|p|>\mu} |\hat{q}(p)|\,dp \leqslant \mu^{-\eta} \int \langle p\rangle^\eta |\hat{q}(p)|\,dp < C\mu^{-\eta}. \tag{4.5}$$

From (4.4) and (4.5) we obtain for $r > 0$, $\mu > 0$

$$\|q\|_{L^\infty} \leqslant C\|\hat{q}\|_{L^1} \leqslant C(e^{c(r+\mu)}\lambda^\alpha + \mu^3 r^{-\alpha} + \mu^{-\eta}), \tag{4.6}$$

where $q = q_1 - q_2$, $q_1 \in \mathcal{O}$, $q_2 \in \mathcal{O}$, $c = 4R + 1$, $C = C(\mathcal{O})$. Putting $\mu = r^{\alpha/6}$, we get

$$\|q\|_{L^\infty} \leqslant C(\exp[c(r + r^{\alpha/6})]\lambda^\alpha + r^{-\alpha/2} + r^{-\alpha\eta/6}).$$

Set $r = -\dfrac{\alpha}{4c}\ln\lambda$. For λ sufficiently small we have $r > 1$. Then $r^{\alpha/6} < r$ and

$$\|q\|_{L^\infty} \leqslant C(\lambda^{\alpha/2} + (-\ln\lambda)^{-\alpha/2} + (-\ln\lambda)^{-\alpha\eta/6}).$$

Applying the inequality $\lambda < (-\ln\lambda)^{-1}$, $(0<\lambda<1)$, we finally get for $\lambda > 0$ sufficiently small

$$\|q\|_{L^\infty} \leqslant C(-\ln\lambda)^{-\delta}, \qquad 0 < \delta < 1.$$

This completes the proof of the first part of Theorem 4.1. Now, to prove (b), suppose that $q \in L^\infty \cap H^s$, $s > 0$. Then similarly to (4.5), we can easily prove that

$$\int_{|p|>\mu} |\hat{q}(p)|^2\, dp \leqslant C\mu^{-\eta}, \qquad \eta = 2s > 0,$$

for all $q \in \mathcal{O}$, $\mathcal{O}$ being a neighborhood of the kind given in (b). From (4.3) and the above estimate we see that $\|\hat{q}\|_{L^2}$ (and therefore $\|q\|_{L^2}$) admits an estimate similar to (4.6). Now, repeating the arguments following (4.6), we complete the proof of Theorem 4.1.

Now we are ready to prove the main result of the paper. Let $q_0 \in H^s(\mathbb{R}^3)$, $s > 3/2$ ($q_0 \in L^\infty \cap H^s$, $s>0$), $\operatorname{supp} q_0 \subset B_\rho$ and $k > 0$ is fixed. Fix $R > \rho$. By Lemma 3.2, there exists $R_0 > R$ such that k^2 is not a Dirichlet eigenvalue of $-\Delta + q_0$, $-\Delta$ in B_{R_0}. Since ϕ is an increasing function and for sufficiently small $t > 0$ we have $\phi(Ct) \leqslant C'\phi(t)$, we can apply Proposition 3.3 and Proposition 4.1 to get the estimates of Theorem 1.1 with R_0 instead of R. Now observe that $\|A\|_{R_0,s_1,s_2} \leqslant \|A\|_{R,s_1,s_2}$ for $R \leqslant R_0$ and $s_1 + s_2 \geqslant 0$. On the other hand, $\|A_q\|_{R,s_1,s_2} < \infty$ because $R > \rho$. Therefore the estimates of Theorem 1.1 hold for all $R > \rho$.

Appendix.

Here we derive some uniform estimates for the spherical Bessel function $j_n(r)$, and for the spherical Hankel function $h_n^{(1)}(r)$. In particular, we are interested in the behavior of these functions as $n \to \infty$ and r belongs to some bounded interval $r \in [0, a]$ for j_n and $r > a$ for $h_n^{(1)}$. It is clear that the well-known asymptotics

$$h_n^{(1)} \sim -\,2^{1/2}e^{-1/2}r^{-1}\left(\frac{2n+1}{er}\right)^n, \quad j_n(r) \sim 2^{-1/2}e^{-1/2}r^{-1}\left(\frac{er}{2n+1}\right)^{n+1},$$

as $n \to \infty$, $r -$ fixed, lead to such kind of estimates. However, we wish to obtain estimates that are uniform in r.

LEMMA A1. — *For $n = 1, 2, \ldots$, and for $r > 0$ we have*

$$|j_n(r)| \leqslant \frac{e}{n\sqrt{2}}\left(\frac{er}{2n+1}\right)^n.$$

Proof. – We start with the formula

$$j_n(r) = \frac{r^n}{2^{n+1}n!}\int_0^\pi \cos(r\cos\alpha)\sin^{2n+1}\alpha\, d\alpha.$$

Therefore,

$$|j_n(r)| \leqslant \frac{r^n}{2^{n+1}n!}\int_0^\pi \sin^{2n+1}\alpha\, d\alpha.$$

An elementary calculation shows that

$$I_{2n+1} := \int_0^\pi \sin^{2n+1}\alpha\, d\alpha = -\int_0^\pi \sin^{2n}\alpha\, d\cos\alpha$$

$$= 2n\int_0^\pi (1-\sin^2\alpha)\sin^{2n-1}\alpha\, d\alpha = 2nI_{2n-1} - 2nI_{2n+1}.$$

Therefore, $(2n+1)I_{2n+1} = 2nI_{2n-1}$, which, together with the fact that $I_1 = 2$, leads to

$$I_{2n+1} = 2\frac{2n(2n-2)\dots 2}{(2n+1)(2n-1)\dots 3} = 2^{2n+1}\frac{(n!)^2}{(2n+1)!}.$$

From the Stirling formula $n! = \sqrt{2\pi}\, n^{n+1/2}\exp(-n+\theta(n)/12n)$, $n > 0$, $0 < \theta(n) < 1$, we get

$$\text{(A1)}\quad \sqrt{2\pi}\, n^{n+1/2}e^{-n} < n! < 2\sqrt{2\pi}\, n^{n+1/2}e^{-n},\quad n = 1, 2, \dots$$

Hence

$$|j_n(r)| \leqslant \frac{r^n}{2^{n+1}n!}I_{2n+1} = \frac{(2r)^n n!}{(2n+1)!} < \frac{(2r)^n 2\sqrt{2\pi}\, n^{n+1/2}e^{-n}}{\sqrt{2\pi}(2n+1)^{2n+3/2}e^{-2n-1}}$$

$$< \frac{e^{n+1}r^n}{\sqrt{n}(2n+1)^{n+1/2}} < \frac{e}{n\sqrt{2}}\left(\frac{er}{2n+1}\right)^n.$$

The proof is complete.

LEMMA A2. – *For any $r > 0$ we have*

$$|h_n^{(1)}(r)| \leqslant \frac{2\sqrt{2}}{r}\left(2+\frac{2n}{er}\right)^n.$$

Proof. – Let us use the following representation of $h_n^{(1)}$

$$h_n^{(1)}(r) = i^{-n-1}r^{-1}e^{ir}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}(-2ir)^{-k}.$$

Since for $n=0$ the desired estimate holds, we can assume $n \geqslant 1$. We have

$$\frac{(n+k)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}\frac{(n+k)!}{n!},$$

where, by virtue of (A1)

$$\frac{(n+k)!}{n!} \leqslant 2\frac{(n+k)^{n+k+1/2}e^{-n-k}}{n^{n+1/2}e^{-n}} \leqslant 2^{n+3/2}\left(\frac{2n}{e}\right)^k \quad \text{for} \quad k \leqslant n.$$

Therefore,

$$|h_n^{(1)}(r)| \leqslant 2^{n+3/2}r^{-1}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{n}{er}\right)^k = 2^{n+3/2}r^{-1}\left(1+\frac{n}{er}\right)^n,$$

which completes the proof.

BIBLIOGRAPHY

[A] S. Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal., 27 (1988), 153-172.

[EMOT] A. Erdélyi, W. Magnus, F. Oberhettinger and F. C. Tricomi, Higher Transcendental Functions, Vol. 2, McGraw Hill, New York, 1953.

[L] R. Leis, Zur Monotonie der Eigenwerte selbstadjungierter elliptischer Differentialgleichungen, Math. Z., 96 (1967), 26-32.

[M] C. Müller, Grundprobleme der Mathematischen Theorie Elektromagnetischer Schwingungen, Springer, 1957.

[Na1] A. Nachman, Reconstruction from boundary measurements, Ann. Math., 128 (1988), 531-576.

[Na2] A. Nachman, Exact reconstruction procedures for some inverse scattering and inverse boundary problems, Meeting on inverse problems, Montpellier, 1989.

[NSU] A. Nachman, J. Sylvester and G. Uhlmann, An n-dimensional Borg-Levison theorem, Comm. Math. Physics, 115 (1988), 595-605.

[NO] R. G. Novikov, Multidimensional inverse spectral problems for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Funkt. Analizi i Ego Prilozheniya, 22 (4) (1988), 11-12; Translation in Funct. Anal. and its Appl., 22 (4) (1988), 263-272.