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<http://www.numdam.org/item?id=AIF_1990__40_4_951_0>
REMARKS ON THE LICHNEROWICZ-POISSON COHOMOLOGY

by Izu VAISMAN

The Lichnerowicz-Poisson (LP) cohomology of a Poisson manifold was defined in [L], and it provides a good framework to express deformation and quantization obstructions [L], [VK], [H], [V2]. The LP cohomology spaces are, generally, very large, and their structure is known only in some particular cases [VK], [X]. The homological algebraic place of these spaces was clarified in [H]. In the present note, we make a number of further remarks on the LP cohomology, most of them related with a certain natural spectral sequence which shows that, in the case of a regular Poisson manifold, the LP cohomology is connected with the cohomology of the sheaves of germs of foliated (i.e., projectable) forms of the symplectic foliation of the manifold (e.g., [V1]).

1. General remarks.

Let $M^m$ be a Poisson manifold with the Poisson bivector $\Pi$, and put $\mathcal{V}^0(M) \overset{\text{def}}{=} C^\infty(M), \mathcal{V}(M) = \mathcal{V}^1(M) = \text{the space of } C^\infty \text{ vector fields of } M, \mathcal{V}^k(M) = \text{the space of } k\text{-vector fields (i.e., antisymmetric } k\text{-contravariant tensor fields of } M), \mathcal{V}^*(M) = \text{the space of Pfaff forms of } M, \text{ and, finally } \mathcal{L}(M) = \bigoplus_{k=0}^m \mathcal{V}^k(M) = \text{the contravariant Grassmann algebra of } M. \text{ The bivector } \Pi \text{ has an associated morphism } \#: T^*M \to TM, \text{ defined by } \beta(\alpha^*) = \Pi(\alpha, \beta), \forall \alpha, \beta \in T^*M, \text{ and it yields the Poisson bracket of functions } \{f, g\} = \Pi(df, dg), \text{ as well as Hamiltonian vector fields } X_f, \forall f \in \mathcal{V}^0(M), \text{ given by } X_f g = \{f, g\}. \text{ These fields define a generalized foliation with symplectic leaves called the symplectic foliation of } (M, \Pi) \text{ (i.e., } \{X_f\} \text{ generate the tangent spaces of }

Key-words: Poisson manifolds - LP cohomology.
A.M.S. Classification: 58A12 - 58F05.
the leaves). It is important to remember that the Poisson bracket induces a bracket of Pfaff forms which is the unique natural extension of the formula \( \{ df, dg \} = d\{ f, g \} \), and is given by

\[
\{ \alpha, \beta \} = L_\alpha \beta - L_\beta \alpha - d(\Pi(\alpha, \beta)).
\]

The basic Poisson condition \([\Pi, \Pi] = 0\), where \([ , , ]\) denotes the Schouten-Nijenhuis bracket, ensures that \((\mathcal{V}^0(M), \{ , \})\) and \((\mathcal{V}^*(M), \{ , \})\) are Lie algebras. The same condition also shows that the operator \(\sigma Q = -[\Pi, Q]\) is a coboundary on \(\mathcal{L}(M)\) (i.e., \(\sigma^2 = 0\)), and the cohomology of the cochain complex \((\mathcal{L}, \sigma)\) is, by definition, the LP cohomology of \((M, \Pi)\). Its spaces will be denoted by \(H^k_{LP}(M, \Pi)\). It is also important to remind that, for \(Q = \mathcal{V}^k(M)\), one has \([BV]\)

\[
(1.1) \quad \{ \alpha, \beta \} = \sum_{i=1}^{k} \alpha_i^k (\mathcal{Q}(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_k))
+ \sum_{i<j=0}^{k} (-1)^{i+j} Q(\{\alpha_i, \alpha_j\}, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_k),
\]

where \(\alpha_i \in \mathcal{V}^*(M)\), and \(\hat{\cdot}\) denotes the absence of an argument.

Now, the definitions given above have some easy consequences such as

a) \([X], [VK]. \quad H^0_{LP}(M, \Pi) = \{ f \in C^\infty(M) / \forall g \in C^\infty(M), \quad X_gf = 0 \}. \) (Since \(\sigma^0 = 0\).)

b) \([X], [VK]. \quad H^1_{LP}(M, \Pi) = \mathcal{V}_{\pi}(M)/\mathcal{V}_{\pi}(M), \) where

\[
\mathcal{V}_{\pi}(M) = \{ X \in \mathcal{V}(M)/L_x \Pi = 0 \}, \quad \mathcal{V}_{\pi}(M) = \{ X_f/ f \in \mathcal{V}^0(M) \}.
\]

(Since \(\sigma X = -L_x \Pi \) [L].)

c) \([L], \quad \sigma \Pi = 0, \) and \(\Pi\) defines a fundamental class \([\Pi] \in H^2_{LP}(M, \Pi)\).

d) The LP cohomology satisfies the Mayer-Vietoris exact sequence property i.e., if \(U, V\) are open subsets of \(M\), there is an exact sequence of the form

\[
(1.3) \quad \ldots \to H^k_{LP}(U \cup V, \Pi) \to H^k_{LP}(U, \Pi) \oplus H^k_{LP}(V, \Pi) \to H^k_{LP}(U \cap V, \Pi) \to H^{k+1}_{LP}(U \cup V, \Pi) \to \ldots
\]

The definition of the arrows and the proof of the exactness are the same as for the de Rham cohomology (e.g., [BT]).
Natural homomorphisms $\varrho: H^k(M,\mathbb{R}) \to H^k_L(M,\Pi)$, which are isomorphisms in the symplectic case, exist. Namely, $\varrho$ is defined by the extension of $\#$ to $k$-forms $\lambda$ by

$$\lambda^\#(\alpha_1, \ldots, \alpha_k) = (-1)^k \lambda(\alpha_1^\#, \ldots, \alpha_k^\#),$$

since (1.2) shows that $\sigma(\lambda^\#) = (-1)^k (d\lambda)^\#$.

Because of $e)$, it is natural to ask for a covariant interpretation of the whole LP cohomology via a Riemannian metric, and such an interpretation can be obtained by using Koszul’s generating operators of the Schouten-Nijenhuis bracket. If we change signs such as to agree with $[L]$, Koszul’s formula for $[A,B]$ where $A \in \mathcal{X}^{-1}(M)$, $B \in \mathcal{X}^{-1}(M)$ is $[K]$

$$[A,B] = D_A^B - (D_A)^B - (-1)^{|A|} A \wedge (D_B^A),$$

where $V$ is a torsionless linear connection on $M$, and $D_V$ is defined by the coordinatewise formula

$$\varphi^h_\ast \cdots \varphi^h_i = \nabla_k A^{kh} \cdots \varphi^h_i.$$

If $V$ is the Riemannian connection of a metric $g$, (1.6) means $D_V = -\#_g \delta_g \#_g^{-1}$, where $\#_g: T^*M \to TM$ is the well known musical isomorphism, and $\delta_g$ is the codifferential of $(M,g)$. Now, if we denote $\pi = \#_g^{-1} \Pi$, $B = \#_g \lambda$, and take $A = \Pi$ in (1.5), we obtain $\sigma(\#_g \lambda) = \#_g \delta_\pi$, where, if $e (i)$ denotes the exterior (interior) multiplication by a form, one has

$$\delta_\pi = \delta_g e(\pi) - e(\pi) \delta_g - e(\delta_g \pi).$$

Hence, $H^k_L(M,\Pi)$ are isomorphic to the cohomology spaces of the Grassmann complex $\Lambda M$ endowed with the coboundary $\delta_\pi$.

Of course, $\pi$ must satisfy the condition $\delta_\pi \pi = 0$, which is equivalent to $[\pi, \Pi] = 0$ i.e., we must have

$$\delta_g (\pi \wedge \pi) = 2 \pi \wedge (\delta_g \pi),$$

and this is a new characterization of a Poisson structure which may have some usefulness. For instance, it shows that the parallel 2-forms of a Riemannian manifold (if any) and the harmonic 2-forms of a compact Riemannian symmetric space (where the exterior product of two harmonic forms is again a harmonic form) define Poisson structures. Formulas (1.7), (1.8) may also be used if we are looking for compatible
Poisson structures on a given symplectic manifold $M$ with symplectic form $\omega$ i.e., Poisson bivectors $\Pi$ such that $[\omega^{-1}, \Pi] = 0$ (e.g., [G]). After the choice of a metric $g$ on $M$, this problem amounts to solving the equations

$$\delta_{\#^{-1}(\omega^{-1})} = 0, \quad \delta_g (\pi \wedge \pi) = 2\pi \wedge \delta_g \pi,$$

where also, if we ask $g$ to be almost Hermitian $\omega$-compatible, then $\#^{-1}(\omega^{-1}) = \omega$. For instance, (1.9) shows that, if $M$ is a compact Hermitian symmetric space, and $\omega$ is its Kähler form, then any harmonic form of $M$ defines an $\omega$-compatible Poisson structure. On the other hand, we shall notice that, in case $M$ is compact and oriented, $\delta_\pi$ has the formal adjoint

$$d_\pi = i(\pi)d - di(\pi) - i(\delta_g \pi),$$

and we may expect to be able to apply the abstract Hodge decomposition theorem of [LT]. (From the expression of the Schouten-Nijenhuis bracket [L], it follows easily that the complex

$$\ldots \to \mathcal{C}^k(M) \xrightarrow{\sigma} \mathcal{C}^{k+1}(M) \to \ldots$$

is elliptic along the leaves of the symplectic foliation of $(M, \Pi)$.)

Finally, we make a remark which will be important for the next sections of this paper. Namely, that there is a Serre-Hochschild spectral sequence associated with the LP cohomology. Let $\mathcal{C}^k(M) = \ker \# = \text{the space of conormal 1-forms}$ of the symplectic foliation of $(M, \Pi)$. Since the bracket (1.1) satisfies $\{\alpha, \beta\} = [\alpha, \beta]$, $\mathcal{C}^k(M)$ is an abelian ideal of $(\mathcal{C}^*(M), \{ , \})$, and we may define the filtration degree of $Q \in \mathcal{C}^k(M)$ to be $h$ if $Q(\alpha_1, \ldots, \alpha_k) = 0$ as soon as $\geq k - h + 1$ of the arguments are conormal. This yields a differential filtration of the LP complex $\mathcal{L}(M)$, where $S^k(M) = \text{the space of k-vector fields}$ of filtration degree $h$ is equal to the locally finite span of $\{ f_{x_1} \wedge \ldots \wedge x_{j_1} \wedge y_1 \wedge \ldots \wedge y_{k-h-1}, f_{i} \in \mathcal{C}^0(M), y_j \in \mathcal{C}^{-1}(M) \}$. Now, the spectral sequence which we have in mind, and which we shall denote by $E^p_q(M, \Pi)$, is the one associated with this filtration i.e., the Serre-Hochschild sequence of the pair of Lie algebras $(\mathcal{C}^*(M), \mathcal{C}^*(M), \{ , \})$. This sequence converges to $H^*_\mathcal{L}(M, \Pi)$, and one has (e.g., [F])

$$E^2_{pq}(M, \Pi) = H^p(V^*(M)/\mathcal{C}^q(M); H^q(\mathcal{C}^*(M); C^\infty(M))).$$
2. The regular case.

In the remaining part of this paper we assume that $\Pi$ is of the constant rank $2n$, and $m = 2n + s$. This is the regularity condition, and then the symplectic foliation of $(M, \Pi)$, hereafter to be denoted by $\mathcal{F}$, is regular. Hence, we can and shall define a transversal distribution $\mathcal{F}'$, and $TM = \mathcal{F}' \oplus T\mathcal{F}$, $T^*M = \mathcal{F}'^* \oplus T^*\mathcal{F}$ induce a bigrading of the covariant and contravariant tensors of $M$. A tensor whose transversal degree is $p$ and whose leafwise degree is $q$ is said to be of the type $(p,q)$. We shall denote by $\mathcal{V}^{p,q}(M)$ and $\Lambda^{p,q}(M)$ the spaces of $k$-vector fields and $k$-forms ($k = p + q$) of the type $(p,q)$ of $M$, respectively. For instance, it is easy to understand that ker $\#$ (i.e., $\mathcal{V}^*_0(M)$) is just $\mathcal{F}'^* = \text{the space of the 1-forms of type (1,0), and that type } \Pi = (0,2).

(E.g., see [VI] for details on the bigrading of differential forms.)

Now, if $Q \in \mathcal{V}^{-k}(M)$ is of type $(p,q)(p+q = k)$, and if we use bihomogeneous arguments $\alpha_i$ in (1.2), we see that $\sigma = \sigma' + \sigma''$ where type $\sigma' = (-1,2)$, type $\sigma'' = (0,1)$, and, for arguments $\alpha$ of type $(1,0)$ and $\beta$ of type $(0,1)$, one has

\begin{align*}
(2.1) \quad (\sigma' Q)(\alpha_0, \ldots, \alpha_{p-2}, \beta_0, \ldots, \beta_{q+1}) &= \sum_{i<j=0}^{q+1} (-1)^{i+j} Q(\beta_i, \beta_j), \\
\alpha_0, \ldots, \alpha_{p-2}, \beta_0, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_{q+1}),
\end{align*}

\begin{align*}
(2.2) \quad (\sigma'' Q)(\alpha_0, \ldots, \alpha_{p-1}, \beta_0, \ldots, \beta_q) &= \sum_{i=0}^{q} (-1)^{p+i} \beta_i^\# (Q(\alpha_0, \ldots, \alpha_{p-1}, \\
\beta_0, \ldots, \beta_i, \ldots, \beta_q) + \sum_{i=0}^{p-1} \sum_{j=0}^{q} (-1)^{p+i+j} Q(\alpha_i, \beta_j), \alpha_0, \ldots, \alpha_{p-1}, \\
\alpha_0, \ldots, \beta_j, \ldots, \beta_q) + \sum_{i<j=0}^{q} (-1)^{p+i+j} Q(\alpha_0, \ldots, \alpha_{p-1}, \\
\beta_i, \ldots, \beta_j, \ldots, \beta_q).
\end{align*}

Remember that type $\alpha = (1,0)$ means $\alpha \in \mathcal{V}^*_0(M)$, and that the latter is an ideal of $\mathcal{V}^*(M)$. On the other hand, we denoted by $\{ , , \}$, $\{ , \}$ the type $(1,0)$ and $(0,1)$ components of $\{ , , \}$. Particularly, if type $X = (1,0)$, we get easily

\begin{align*}
(2.3) \quad \{\beta_1, \beta_2\}'(X) &= (L_X \pi)(\beta_1, \beta_2).
\end{align*}
In this section we use the type decomposition of $\sigma$ in order to indicate a recurrent computational process of the LP cohomology which, in fact, is similar to the one used in [VK] for the case where $\mathcal{S}$ is a fibration. Take $Q \in \mathcal{V}^{k}(M)$, and decompose it as

\[(2.4) \quad Q = Q^{k,0} + Q^{k-1,1} + \ldots + Q^{0,k},\]

where the indices denote the type of the components. Then, $\sigma Q = 0$ means

\[(2.5) \quad \sigma''Q^{i,k-i} + \sigma'Q^{i+1,k-i-1} = 0 \quad (i = 0, \ldots, k).\]

For $i = k$, (2.5) gives $\sigma''Q^{k,0} = 0$, and, on the other hand, $(Q + \bar{Q})^{k,0} = Q^{k,0}$, $\forall \bar{Q} \in \mathcal{V}^{k-1}(M)$. Therefore, there exist homomorphisms

\[(2.6) \quad p_{k,0} : H_{\text{LP}}^{k}(M,\Pi) \to \mathcal{V}^{k,0}_{0}(M),\]

where $\mathcal{V}^{k,0}_{0}(M)$ is the space of $\sigma''$-closed $k$-vectors of type $(k,0)$, and, furthermore, (2.5) shows that $\text{im} \ p_{k,0}$ consists of $k$-vectors $Q^{k,0} \in \mathcal{V}^{k,0}_{0}(M)$ which satisfy the following sequence of existence conditions of $k$-vectors $Q^{k-1,1}, \ldots, Q^{0,k}$ such that

\begin{align*}
(c_1) & \quad \sigma'Q^{k,0} = \text{exact} \quad \overset{\text{def}}{=} - \sigma''Q^{k-1,1}, \\
(c_2) & \quad \sigma'Q^{k-1,1} = \text{exact} \quad \overset{\text{def}}{=} - \sigma''Q^{k-2,2}, \\
& \quad \vdots \\
(c_k) & \quad \sigma'Q^{1,k-1} = \text{exact} \quad \overset{\text{def}}{=} - \sigma''Q^{0,k}.
\end{align*}

In this case we shall say that $\sigma'Q^{k,0}$ satisfies $k$ times the $\sigma''$-exactness condition, and we shall denote by $\mathcal{V}^{k,0}_{0(\Pi)}(M)$ the space of such $Q^{k,0}$. If we also denote $\ker p_{k,0} = \mathcal{V}^{k}_{0}\mathcal{L}(M,\Pi) = \mathcal{V}^{k,0}(M)$ the space of $k$-dimensional LP cohomology classes whose cocycles are (2.4) with $Q^{k,0} = 0$, we obtain the result of the first recurrence step

\[(2.7) \quad H_{\text{LP}}^{k}(M,\Pi) \approx \mathcal{V}^{k}_{0}\mathcal{L}(M,\Pi) \oplus \mathcal{V}^{k,0}_{0(\Pi)}(M).\]

Now, in the next step we have to compute $\mathcal{V}^{k}_{0}\mathcal{L}(M,\Pi)$, and for this purpose we take the subcomplex $\mathcal{V}^{k}_{0}\mathcal{L}(M)$ of $\mathcal{L}(M)$ consisting of multivectors $Q$ with a vanishing $(.,0)$ component, and denote by $H^{k}(\mathcal{L}(M))$ its cohomology spaces. Then $\mathcal{V}^{k}_{0}\mathcal{L}(M,\Pi)$ is the image of $H^{k}(\mathcal{L}(M))$ with respect to the inclusion $\mathcal{V}^{k}_{0}\mathcal{L}(M) \subseteq \mathcal{L}(M)$. It is clear that the complex $\mathcal{L}(M)^{0}\mathcal{L}(M)$ has coboundary zero, therefore, $H^{k}(\mathcal{L}(M)^{0}\mathcal{L}(M)) = (\mathcal{L}^{0}\mathcal{L})^{k} = \mathcal{V}^{k,0}(M)$. This gives us the exact sequence
\[ \mathcal{V}^{-k-1,0}(M) \xrightarrow{\sigma} H^k(\mathcal{L}(M)) \xrightarrow{\iota} H^k(\mathcal{L}(M)), \text{ and we get} \]

\[ (2.8) \quad 0H^1_{LP}(M,\Pi) \approx H^k(\mathcal{L}(M))/\sigma(\mathcal{V}^{-k-1,0}(M)). \]

Hence, the second step will have to consist of an analysis of \( H^k(\mathcal{L}(M)) \), which can be made in the same way as in step 1, and resulting in a formula similar to (2.7), and so on.

For \( k = 1 \), we get easily

\[ (2.9) \quad 0H^1_{LP}(M,\Pi) = \{X \in \mathcal{V}^{-0,1}(M)/\sigma''X = 0\}/\sigma''(\mathcal{V}^{-0}(M)). \]

For \( k = 2 \), we have first

\[ (2.10) \quad H^2(\mathcal{L}(M)) = \left\{ \frac{Q^{1,1} + Q^{0,2}/\sigma''Q^{1,1} = 0, \sigma''Q^{0,2} + \sigma'Q^{1,1} = 0}{\sigma'' \mathcal{V}^{0,1}} \right\}, \]

and the analysis which gave (2.7) now yields

\[ (2.11) \quad H^2(\mathcal{L}(M)) \approx "H^2(\mathcal{L}^{0,*}(M)) \oplus \mathcal{V}^{-1,1}_{0(1)}(M), \]

where \( \mathcal{L}^{0,*}(M) = \bigoplus_k \mathcal{V}^{-0,k}(M) \), and "\( H \) is its cohomology with respect to \( \sigma'' \), and

\[ (2.12) \quad \mathcal{V}^{-1,1}_{0(1)}(M) = \{Q^{1,1}/\sigma''Q^{1,1} = 0 \text{ and } \sigma'Q^{1,1} = \sigma''\text{-exact}\}. \]

(We shall see in Section 3 that, if the foliation \( \mathcal{F} \) is either transversally Riemannian or transversally symplectic, then

"\( H^i(\mathcal{L}^{0,*}(M)) \approx H^i(M,\Phi^0(\mathcal{F})), \)

where \( \Phi^0(\mathcal{F}) \) is the sheaf of germs of functions which are constant along the leaves of \( \mathcal{F} \).) Summing up the results we get

\[ (2.13) \quad H^2_{LP}(M,\Pi) \approx ("H^2(\mathcal{L}^{0,*}(M)) \oplus ((\mathcal{V}^{-1,1}_{0(1)}(M))/\sigma(\mathcal{V}^{-1,0}(M))) \oplus \mathcal{V}^{-2,0}_{0(2)}(M)), \]

Etc.

### 3. The spectral sequence.

In this section we continue to refer to a regular Poisson manifold \((M,\Pi)\), and use the notation introduced in Section 2, while we are focussing on the spectral sequence \( E^p_{pq}(M,\Pi) \) defined at the end of Section 1. We have:
**Proposition 3.1.** — *The first terms of the LP Serre-Hochschild spectral sequence of a regular Poisson manifold \((M, \Pi)\) are given by*

\[
E_0^{pq}(M, \Pi) = E_1^{pq}(M, \Pi) = \mathcal{V}^{q-p}(M), \quad E_2^{pq}(M, \Pi) = H^p(\mathcal{V}^{q,*}, \sigma^\sigma).
\]

The reader can prove this by noticing that the \(h\)-filtering subcomplex of \(\mathcal{L}(M)\) as defined in Section 1 is equal to \(S_h(M) = \bigoplus_{i \geq h} \mathcal{V}^{p,i}(M)\), and then following the usual definition of \(E_0^{pq}\). Here, we just prefer to observe that \(\mathcal{L}(M) = \bigoplus \mathcal{W}^{-i,j}(M), \sigma = \Sigma d_{hk}\), where \(\mathcal{W}^{-i,j}(M) = \mathcal{V}^{j,*}(M)\), and the terms of \(\sigma\) are \(d_{0i} = 0, \ d_{10} = \sigma'' \), \(d_{2,-1} = \sigma'\), is a double semipositive cochain complex in the sense of [VI], p. 76-77, and then (3.1) follows from this reference.

Now, let \(G\) be a metric of the vector bundle \(\mathcal{S}^\ast\) of Section 2, and let \(\# \stackrel{\text{def}}{=} \#_G \oplus \# : \mathcal{S}^\ast \oplus T^\ast \mathcal{S} \to \mathcal{S}^\ast \oplus T \mathcal{S}\) be the corresponding musical isomorphism also extended to \(\Lambda^k(M) \to \mathcal{V}^k(M)\). Then, if \(\lambda\) is a differential form of type \((p,q)\), \(\lambda^\#\) is a multivector of the same type, and we have

\[
(\#^{-1} \sigma'' \lambda^\#)(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q) = (-1)^{q+1}(\sigma'' \lambda^\#)(\#_G^{-1}X_0, \ldots, \#_G^{-1}X_{p-1}, \#^{-1}Y_0, \ldots, \#^{-1}Y_q).
\]

In this relation, and in the sequel, we agree that type \(X = (1,0)\) and type \(Y = (0,1)\). Furthermore, in order to compute \(\sigma'' \lambda^\#\) by (2.2) we establish first

\[
\{\#^{-1} Y_i, \#^{-1} \tilde{Y}_j\}'' = \{\#^{-1} Y_i, \#^{-1} \tilde{Y}_j\}'' = [Y_i, Y_j]
\]

(remember that \(\{\alpha, \beta\}'' = [\alpha'', \beta'']\) [BV]), and using (1.1))

\[
\{\#_G^{-1} X_i, \#^{-1} \tilde{Y}_j\}(X) = - (L_{\tilde{Y}_j} G^*)(X, X) - G^*([Y_i, X], X),
\]

where \(G^*\) is the dual metric of \(G\) on \(\mathcal{S}'\). If these formulas are used, and the result is compared with the formula of the \(\mathcal{S}\)-leafwise exterior differential \(d_f\) [VI], p. 184, one gets

\[
(\#^{-1} \sigma'' \lambda^\#)(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q) = - (d_f \lambda)(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q)
\]

\[
+ \sum_{i=0}^{p-1} \sum_{j=0}^{q} (-1)^{p+i+j} \lambda_i([L_{\tilde{Y}_j} G^*](X_i, \cdot)'' G^*,
\]

\[
X_0, \ldots, \tilde{X}_i, \ldots, X_{p-1}, Y_0, \ldots, \tilde{Y}_j, \ldots, Y_q).
\]
Remark. — The same result holds if $G$ is a symplectic structure on $\mathcal{S}^\ast$.

This computation leads to

**Proposition 3.2.** — If the symplectic foliation $\mathcal{S}$ of the regular Poisson manifold $(M, \Pi)$ is either transversally Riemannian or transversally symplectic, one has

$$E^n_p(M, \Pi) = E^n_p(\mathcal{S}) = H^p(M, \Phi^q(\mathcal{S}))$$

where $E^n_p(\mathcal{S})$ is the spectral sequence of the foliation $\mathcal{S}$ (e.g., [KT]), and $\Phi^q(\mathcal{S})$ is the sheaf of germs of $\mathcal{S}$-foliated $q$-forms of $M$ (e.g., [VI]). Particularly, (3.4) holds if $\mathcal{S}$ is a fibration.

Indeed, under the hypotheses, $L_y G = 0$ in (3.3), and in view of (3.1) we get an isomorphism $E^n_p(M, \Pi) = H^p(\oplus \Lambda^q(\mathcal{S}), d)$. But then (3.4) is known [VI], p. 216, 222, 77. (Remember that an $\mathcal{S}$-foliated $q$-form is a $q$-form which, locally, is the pull-back of a form of a local transversal manifold of the foliation $\mathcal{S}$.)

Now, let us define an interesting special class of Poisson manifolds. A vector field $V$ of $M$ is $\mathcal{S}$-foliated if it sends leaves to leaves or, equivalently, $\forall Y \in T\mathcal{S}$, $[V, Y] \in T\mathcal{S}$. For instance, this happens if $V$ is an infinitesimal automorphism of $\Pi$ i.e., $L_y \Pi = 0$, a condition which is easily seen to be equivalent to each of the following two conditions, where $f, g \in C^\infty(M)$,

$$V(f, g) = [V, X_{(f)}](g) - [V, X_{(g)}](f),$$

$$[V, X_{(f)}] = X_{V(f)}.$$

A regular Poisson structure $\Pi$ of $M$ will be called transversally constant if $\mathcal{S}$ has a transversal distribution $\mathcal{S}'$ such that every local foliate vector field $V \in \mathcal{S}'$ is a local infinitesimal automorphism of $\Pi$. For instance, if $M = S \times N$, and $\Pi$ is defined by a symplectic structure of $S$, the distribution $\mathcal{S}' = TN$ has this property. Particularly, the existence of the local canonical coordinates of $\Pi$ in the sense of [L] p. 256-257, shows that every regular Poisson manifold is locally transversally constant. Another example is the Dirac bracket defined as follows. Let $(M, \omega)$ be a symplectic manifold endowed with a foliation $\mathcal{F}$ such that $\omega$ induces symplectic structures of its leaves. These induced structures yield a Poisson bivector $\Pi$ such that $\mathcal{S}(\Pi) = \mathcal{F}$, and $\{ \quad, \}$ is the Dirac bracket of $(M, \omega, \mathcal{F})$. It follows that every $\mathcal{F}$-foliate vector
field $V$ which is $\omega$-orthogonal to $\mathcal{F}$ is an infinitesimal automorphism of $\Pi$. Indeed, for such $V$, (3.5) is equivalent to $(L_{\xi}\omega)(X_\ell, X_\rho) = 0$, and this is an easy consequence of $d\omega = 0$. Using this definition, we have

**Proposition 3.3.** If $\Pi$ is transversally constant, $\sigma' = 0$, and

$$H^k_{LP}(M, \Pi) = \bigoplus_{q=0}^q E^{k-q,q}_{2}(M, \Pi).$$

**Proof.** Of course, the proposition refers to $\sigma'$ of (2.1) taken with respect to the distribution $\mathcal{F}'$ involved in the definition of a transversally constant Poisson structure. Let us use the notation of (2.1), and evaluate there $[\beta_1, \beta_2]_\rho(X_p) (p \in M, X_p \in \mathcal{F}_\rho)$. This may be done by extending $X_p$ to a local foliate $(1,0)$-vector field $X$, and using (2.3). Since $\Pi$ is transversally constant, $L_{\xi}\Pi = 0$ and we get $\sigma' = 0$. Then, (3.7) follows from (3.1). Q.e.d.

We shall finish by giving various corollaries of Propositions 3.1, 3.2, 3.3.

**Corollary 3.1.** If $(M, \Pi)$ is a transversally constant Poisson manifold whose symplectic foliation is either transversally Riemannian or transversally symplectic, one has

$$H^k_{LP}(M, \Pi) = \bigoplus_{q=0}^q E^{k-q,q}_{2}(\mathcal{F}) = \bigoplus_{q=0}^q H^q(M, \Phi^{k-q}(\mathcal{F})).$$

**Corollary 3.2.** Let $\Pi$ be a Dirac bracket of a symplectic manifold $(M, \omega)$ endowed with a leafwise symplectic foliation $\mathcal{F}$, and its $\omega$-orthogonal distribution $\mathcal{F}'$. Assume that the bihomogeneous components of $\omega$ with respect to the decomposition $TM = \mathcal{F}' \oplus T\mathcal{F}$ are closed. Then, again, formula (3.8) holds good.

**Proof.** Being a Dirac bracket, $\Pi$ is transversally constant. On the other hand, if $\omega = \omega_{(2,0)} + \omega_{(0,2)}$; the hypothesis $d\omega_{(2,0)} = 0$ implies $(L_{\xi}\omega_{(2,0)})(X_1, X_2) = 0$ for $(Y \in T\mathcal{F}, X_1, X_2 \in \mathcal{F})$, and we see that $\omega_{(2,0)}$ defines a transversal symplectic structure of $\mathcal{F}$. Q.e.d.

**Corollary 3.3 [X].** Let $\Pi$ be the Poisson structure defined on $M = S \times N$ by a fixed symplectic structure of $S$, and assume that $S$ has finite Betti numbers. Then one has

$$H^k_{LP}(M, \Pi) = \bigoplus_{q=0}^k [H^q(S, \mathbb{R}) \otimes \Lambda^{k-q}(N)].$$
This result follows from (3.8) and from

**Proposition 3.4.** — Let $\mathcal{F}$ be the foliation of $M = F \times N$ by the leaves $F \times \{x\} (x \in N)$, and assume that $F$ has finite Betti numbers. Then

$$H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N).$$

**Proof.** — For $q = 0$ the result was proven in [E] by a spectral sequence argument. Generally, we have the following straightforward argument. By the foliated de Rham theorem [VI], p. 216, we have

$$H^q(M, \Phi^p(\mathcal{F})) = \ker \left[ d_f : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M) \right] / \text{im} \left[ d_f : \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M) \right].$$

In our case, $\Lambda^{p,q}(M)$ is isomorphic to the space $\Lambda^q(F, \Lambda^p(N))$ of $\Lambda^p(N)$-valued $q$-forms on $F$ by the mapping which sends $\lambda \in \Lambda^{p,q}(M)$ to $\tilde{\lambda} \in \Lambda^p(F, \Lambda^p(N))$ defined by

$$\tilde{\lambda}(y)_{x}(Y_1, \ldots, Y_q) = (-1)^p \lambda_{(x,y)}(X_1, \ldots, X_p, Y_1, \ldots, Y_q),$$

where the last equality follows from the hypothesis on $F$. Q.e.d.

**Remark.** — If $M = S \times N$ of Corollary 3.3 is given a Poisson structure $\Pi$ which has the symplectic foliation $S \times \{x\} (x \in N)$, but where each leaf has a different symplectic structure (e.g., the structure studied in [X]), $\Pi$ is no more transversally constant, but we may use Propositions 3.2. and 3.4, and get

$$E^p_2(M, \Pi) = H^p(S, \mathbb{R}) \otimes \Lambda^q(N).$$

**Corollary 3.4.** — Let $(M, \Pi)$ be an arbitrary regular Poisson manifold. Then every $x \in M$ has a connected open neighbourhood $Y$ such that

$$H^k_{\text{Lie}}(U, \Pi|_U) = \Gamma(\Phi^k(\mathcal{S}|_U)),$$

i.e., the space of the $\mathcal{S}$-foliated $k$-forms over $U$. 
Indeed, we may take $U = S \times N$ where $S$ is contractible, and such that the product coordinates are canonical for $\Pi$ in the sense of [L], p. 257. Then Corollary 3.3 holds on $U$, and we get (3.13). We shall say that such a neighbourhood $U$ is LP-simple.

**Corollary 3.5 (The LP Poincaré Lemma [L]).** Let $(M, \Pi)$ be a regular Poisson manifold, and $x \in M$. Then, there exists an open neighbourhood $U$ of $x$ in $M$ such that, if $Q \in \mathcal{V}^{k}(U)$ and $\sigma Q = 0$, one has $Q = A + \sigma B$ for some $B \in \mathcal{V}^{k-1}(U)$ and a $k$-vector field $A$ over $U$ which is projectable to a $k$-vector field of a local transversal submanifold of $\mathcal{S}$ in $U$.

**Proof.** Take $U$ LP-simple, and with $\Pi$-canonical coordinates. The latter define a bigrading, and we may write $Q = \sum_{p=0}^{k} (\lambda^{p,k-p})^\#$, where $\lambda$ are differential forms, and $\#$ is like in (3.2). The use of the canonical coordinates makes $\Pi/\nu$ transversally constant and transversally Riemannian hence, by Proposition 3.3 and formula (3.3), $\sigma = \sigma^n$, and $\sigma Q = 0$ is equivalent to $d_f \lambda^{p,k-p} = 0$ ($k = 0, \ldots, p$). But $d_f$ satisfies a local Poincaré lemma [VI], p. 215, hence, there are local forms $\mu$ such that $\lambda^{p,k-p} = d_f \mu^{p,k-p-1}$ for $k-p > 0$, while $\lambda^{k,0}$ is a foliate form. The conclusion follows by using again (3.3). Q.e.d

**Bibliography**


Manuscrit reçu le 6 novembre 1990.

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