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Remarks on the Lichnerowicz-Poisson cohomology

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REMARKS
ON THE LICHNEROWICZ-POISSON COHOMOLOGY
by Izu VAISMAN

The Lichnerowicz-Poisson (LP) cohomology of a Poisson manifold was defined in [L], and it provides a good framework to express deformation and quantization obstructions [L], [VK], [H], [V2]. The LP cohomology spaces are, generally, very large, and their structure is known only in some particular cases [VK], [X]. The homological algebraic place of these spaces was clarified in [H]. In the present note, we make a number of further remarks on the LP cohomology, most of them related with a certain natural spectral sequence which shows that, in the case of a regular Poisson manifold, the LP cohomology is connected with the cohomology of the sheaves of germs of foliated (i.e., projectable) forms of the symplectic foliation of the manifold (e.g., [V1]).

1. General remarks.

Let $M^m$ be a Poisson manifold with the Poisson bivector $\Pi$, and put $\mathcal{V}^0(M) \overset{\text{def}}{=} C^\infty(M)$, $\mathcal{V}(M) = \mathcal{V}^1(M) = \text{the space of } C^\infty \text{ vector fields of } M$, $\mathcal{V}^k(M) = \text{the space of } k\text{-vector fields (i.e., antisymmetric } k\text{-contravariant tensor fields of } M)$, $\mathcal{V}^*(M) = \text{the space of Pfaff forms of } M$, and, finally $\mathcal{L}(M) = \bigoplus_{k=0}^m \mathcal{V}^k(M) = \text{the contravariant Grassmann algebra of } M$. The bivector $\Pi$ has an associated morphism $\#: T^* M \to TM$, defined by $\beta(\alpha^*) = \Pi(\alpha, \beta)$, $\forall \alpha, \beta \in T^* M$, and it yields the Poisson bracket of functions $\{f, g\} = \Pi(df, dg)$, as well as Hamiltonian vector fields $X_f$, $\forall f \in \mathcal{V}^0(M)$, given by $X_f g = \{f, g\}$. These fields define a generalized foliation with symplectic leaves called the symplectic foliation of $(M, \Pi)$ (i.e., $\{X_f\}$ generate the tangent spaces of

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the leaves). It is important to remember that the Poisson bracket induces a bracket of Pfaff forms which is the unique natural extension of the formula \(\{df, dg\} = d\{f, g\}\), and is given by

\[
\{\alpha, \beta\} = L_\alpha \# \beta - L_\beta \# \alpha - d(\Pi(\alpha, \beta)).
\]

The basic Poisson condition \([\Pi, \Pi] = 0\), where \([ , ]\) denotes the Schouten-Nijenhuis bracket, ensures that \((\mathcal{V}^0(M), \{ , \})\) and \((\mathcal{V}^*(M), \{ , \})\) are Lie algebras. The same condition also shows that the operator \(\sigma Q = -[\Pi, Q]\) is a coboundary on \(\mathcal{L}(M)\) (i.e., \(\sigma^2 = 0\)), and the cohomology of the cochain complex \((\mathcal{L}, \sigma)\) is, by definition, the LP cohomology of \((M, \Pi)\). Its spaces will be denoted by \(H_{\mathcal{L}P}^k(M, \Pi)\). It is also important to remind that, for \(Q = \mathcal{V}^k(M)\), one has \([BV]\]

\[
\sigma Q(\alpha_0, \ldots, \alpha_k) = \sum_{i=1}^k \alpha_i^\wedge \mathcal{Q}(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_k)
+ \sum_{i<j=0}^k (-1)^{i+j} \mathcal{Q}(\{\alpha_i, \alpha_j\}, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_k),
\]

where \(\alpha_i \in \mathcal{V}^*(M)\), and \(^\wedge\) denotes the absence of an argument.

Now, the definitions given above have some easy consequences such as

a) \([X], [VK]\). \(H^0_{\mathcal{L}P}(M, \Pi) = \{f \in C^\infty(M) | \forall g \in C^\infty(M), X_g f = 0\}\). (Since \(\sigma f = -X_f\).)

b) \([X], [VK]\). \(H^1_{\mathcal{L}P}(M, \Pi) = \mathcal{V}^1(M)/\mathcal{V}_1^*(M)\), where

\(\mathcal{V}_1^*(M) = \{X \in \mathcal{V}(M) | L_X \Pi = 0\}\), \(\mathcal{V}^1(M) = \{X_f | f \in \mathcal{V}^0(M)\}\). (Since \(\sigma X = -L_X \Pi \Pi [L]\).)

c) \([L]\), \(\Pi = 0\), and \(\Pi\) defines a fundamental class \([\Pi] \in H^2_{\mathcal{L}P}(M, \Pi)\).

d) The LP cohomology satisfies the Mayer-Vietoris exact sequence property i.e., if \(U, V\) are open subsets of \(M\), there is an exact sequence of the form

\[
\ldots \to H^k_{\mathcal{L}P}(U \cup V, \Pi) \to H^k_{\mathcal{L}P}(U, \Pi) \oplus H^k_{\mathcal{L}P}(V, \Pi) \\
\to H^k_{\mathcal{L}P}(U \cap V, \Pi) \to H^{k+1}_{\mathcal{L}P}(U \cup V, \Pi) \to \ldots
\]

The definition of the arrows and the proof of the exactness are the same as for the de Rham cohomology (e.g., \([BT]\)).
e) [L], [K]. Natural homomorphisms \( \rho : H^k(M, \mathbb{R}) \to H^k_{LP}(M, \Pi) \), which are isomorphisms in the symplectic case, exist. Namely, \( \rho \) is defined by the extension of \( \# \) to \( k \)-forms \( \lambda \) by

\[
(1.4) \quad \lambda^\#(\alpha_1, \ldots, \alpha_k) = (-1)^k \lambda(\alpha_1^\#, \ldots, \alpha_k^\#),
\]

since (1.2) shows that \( \sigma(\lambda^\#) = (-1)^k (d\lambda)^\# \).

Because of \( e) \), it is natural to ask for a covariant interpretation of the whole \( LP \) cohomology via a Riemannian metric, and such an interpretation can be obtained by using Koszul’s generating operators of the Schouten-Nijenhuis bracket. If we change signs such as to agree with [L], Koszul’s formula for \([A, B] \) where \( A \in \mathcal{V}^{-1}(M) \), \( B \in \mathcal{V}^{-1}(M) \) is [K]

\[
(1.5) \quad [A, B] = D_v(A \wedge B) - (D_vA) \wedge B - (-1)^i A \wedge (D_vB),
\]

where \( V \) is a torsionless linear connection on \( M \), and \( D_v \) is defined by the coordinatewise formula

\[
(1.6) \quad (D_vA)^{h_1 \ldots h_i} = \nabla_k A^{kh_2 \ldots h_i}.
\]

If \( V \) is the Riemannian connection of a metric \( g \), (1.6) means \( D_v = -\#_g \delta_g \#_g^{-1} \), where \( \#_g : T^*M \to TM \) is the well known musical isomorphism, and \( \delta_g \) is the codifferential of \( (M, g) \). Now, if we denote \( \pi = \#_g^{-1} \Pi \), \( B = \#_g \lambda \), and take \( A = \Pi \) in (1.5), we obtain \( \sigma(\#_g \lambda) = \#_g \delta_\pi \), where, if \( e(i) \) denotes the exterior (interior) multiplication by a form, one has

\[
(1.7) \quad \delta_\pi = \delta_g e(\pi) - e(\pi) \delta_g - e(\delta_g \pi).
\]

Hence, \( H^k_{LP}(M, \Pi) \) are isomorphic to the cohomology spaces of the Grassmann complex \( \Lambda M \) endowed with the coboundary \( \delta_\pi \).

Of course, \( \pi \) must satisfy the condition \( \delta_\pi \pi = 0 \), which is equivalent to \( [\Pi, \Pi] = 0 \) i.e., we must have

\[
(1.8) \quad \delta_g (\pi \wedge \pi) = 2\pi \wedge (\delta_g \pi),
\]

and this is a new characterization of a Poisson structure which may have some usefulness. For instance, it shows that the parallel 2-forms of a Riemannian manifold (if any) and the harmonic 2-forms of a compact Riemannian symmetric space (where the exterior product of two harmonic forms is again a harmonic form) define Poisson structures. Formulas (1.7), (1.8) may also be used if we are looking for compatible
Poisson structures on a given symplectic manifold $M$ with symplectic form $\omega$ i.e., Poisson bivectors $\Pi$ such that $[\omega^{-1}, \Pi] = 0$ (e.g., [G]). After the choice of a metric $g$ on $M$, this problem amounts to solving the equations

$$\delta_{\#_g^{-1}(\omega^{-1})} \pi = 0, \quad \delta_g (\pi \wedge \pi) = 2\pi \wedge \delta_g \pi),$$

where also, if we ask $g$ to be almost Hermitian $\omega$-compatible, then $\#_g^{-1}(\omega^{-1}) = \omega$. For instance, (1.9) shows that, if $M$ is a compact Hermitian symmetric space, and $\omega$ is its Kähler form, then any harmonic form of $M$ defines an $\omega$-compatible Poisson structure. On the other hand, we shall notice that, in case $M$ is compact and oriented, $\delta_\pi$ has the formal adjoint

$$d_\pi = i(\pi) d - di(\pi) - i(\delta_\pi \pi),$$

and we may expect to be able to apply the abstract Hodge decomposition theorem of [LT]. (From the expression of the Schouten-Nijenhuis bracket [L], it follows easily that the complex

$$\ldots \rightarrow \mathcal{V}^k(M) \xrightarrow{\sigma} \mathcal{V}^{k+1}(M) \rightarrow \ldots$$

is elliptic along the leaves of the symplectic foliation of $(M, \Pi)$.)

Finally, we make a remark which will be important for the next sections of this paper. Namely, that there is a Serre-Hochschild spectral sequence associated with the LP cohomology. Let $\mathcal{V}^*_\#(M) = \ker \# = \text{the space of conormal 1-forms}$ of the symplectic foliation of $(M, \Pi)$. Since the bracket (1.1) satisfies $\{ [\alpha, \beta] \# = [\alpha \#, \beta \#] [BV], \mathcal{V}^\#_\#(M)$ is an abelian ideal of $(\mathcal{V}^*(M), \{ , \}, )$, and we may define the filtration degree of $Q \in \mathcal{V}^k(M)$ to be $h$ if $Q(\alpha_1, \ldots, \alpha_k) = 0$ as soon as $k - h + 1$ of the arguments are conormal. This yields a differential filtration of the LP complex $\mathcal{L}(M)$, where $S^k_\#(M)$ is the space of $k$-vector fields of filtration degree $h$ is equal to the locally finite span of $\{ f_0 X_{f_1} \wedge \ldots \wedge X_{f_h} \wedge Y_1 \wedge \ldots \wedge Y_{k-h}, \mathcal{V}_0^0(M), Y_j \in \mathcal{V}^{k-1}(M) \}$. Now, the spectral sequence which we have in mind, and which we shall denote by $E_{pq}^p(M, \Pi)$, is the one associated with this filtration i.e., the Serre-Hochschild sequence of the pair of Lie algebras $(\mathcal{V}^*_\#, \mathcal{V}^*_\#(M), \{ , \})$. This sequence converges to $H^*_\#(M, \Pi)$, and one has (e.g., [F])

$$E_{2}^{pq}(M, \Pi) = H^p(V^*(M)/\mathcal{V}^*(M), H^q(\mathcal{V}^*_\#(M); \mathcal{C}^\infty(M))).$$
2. The regular case.

In the remaining part of this paper we assume that \( \Pi \) is of the constant rank \( 2n \), and \( m = 2n + s \). This is the regularity condition, and then the symplectic foliation of \((M, \Pi)\), hereafter to be denoted by \( \mathcal{F} \), is regular. Hence, we can and shall define a transversal distribution \( \mathcal{F}' \), and \( TM = \mathcal{F}' \oplus T \mathcal{F} \), \( T^*M = \mathcal{F}'^* \oplus T^* \mathcal{F} \) induce a bigrading of the covariant and contravariant tensors of \( M \). A tensor whose transversal degree is \( p \) and whose leafwise degree is \( q \) is said to be of the type \((p,q)\). We shall denote by \( \mathcal{V}^{p,q}(M) \) and \( \Lambda^{p,q}(M) \) the spaces of \( k \)-vector fields and \( k \)-forms (\( k = p + q \)) of the type \((p,q)\) of \( M \), respectively. For instance, it is easy to understand that \( \ker \# \) (i.e., \( \mathcal{V}^*(M) \)) is just \( \mathcal{F}'^* \), the space of the 1-forms of type \((1,0)\), and that \( \Pi = (0,2) \). (E.g., see [VI] for details on the bigrading of differential forms.)

Now, if \( Q \in \mathcal{V}^k(M) \) is of type \((p,q)(p+q=k)\), and if we use bihomogeneous arguments \( \alpha_i \) in (1.2), we see that \( \sigma = \sigma' + \sigma'' \) where type \( \sigma' = (1,0) \), type \( \sigma'' = (0,1) \), and, for arguments \( \alpha \) of type \((1,0)\) and \( \beta \) of type \((0,1)\), one has

\[
\begin{align*}
(1.1) \quad (\sigma')Q(\alpha_0, \ldots, \alpha_{p-2}, \beta_0, \ldots, \beta_q) &= \sum_{i < j = 0}^{q+1} (-1)^{i+j}Q(\beta_i, \beta_j), \\
(1.2) \quad (\sigma'')Q(\alpha_0, \ldots, \alpha_{p-1}, \beta_0, \ldots, \beta_q) &= \sum_{i = 0}^{q} (-1)^{p+i}\beta_i(\alpha_0, \ldots, \alpha_{p-1}, \\
&+ \sum_{i = 0}^{p-1} \sum_{j = 0}^{q} (-1)^{p+i+j}Q(\{\alpha_i, \beta_j\}, \alpha_0, \ldots, \alpha_{p-1}, \\
&+ \sum_{i < j = 0}^{q} (-1)^{p+i+j}Q(\alpha_0, \ldots, \alpha_{p-1}, \\
&\{\beta_i, \beta_j\}'', \beta_0, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_q).
\end{align*}
\]

Remember that type \( \alpha = (1,0) \) means \( \alpha \in \mathcal{V}^*(M) \), and that the latter is an ideal of \( \mathcal{V}^*(M) \). On the other hand, we denoted by \( \{, \}'', \{, \}'' \) the type \((1,0)\) and \((0,1)\) components of \((, , )\). Particularly, if type \( X = (1,0) \), we get easily

\[
(2.3) \quad (\beta_1, \beta_2)'(X) = (L_X\pi)(\beta_1, \beta_2).
\]
In this section we use the type decomposition of $\sigma$ in order to indicate a recurrent computational process of the LP cohomology which, in fact, is similar to the one used in [VK] for the case where $\mathcal{P}$ is a fibration. Take $Q \in \mathcal{V}^k(M)$, and decompose it as

$$(2.4) \quad Q = Q^{k,0} + Q^{k-1,1} + \cdots + Q^{0,k},$$

where the indices denote the type of the components. Then, $\sigma Q = 0$ means

$$(2.5) \quad \sigma Q^{i,k-i} + \sigma' Q^{i+1,k-i-1} = 0 \quad (i = 0, \ldots, k).$$

For $i = k$, (2.5) gives $\sigma Q^{k,0} = 0$, and, on the other hand, $(Q + \tilde{Q})^{k,0} = Q^{k,0}$, $\forall \tilde{Q} \in \mathcal{V}^{k-1}(M)$. Therefore, there exist homomorphisms

$$(2.6) \quad p_{k,0} : H^k_{LP}(M,\Pi) \to \mathcal{V}^0_{0}(M),$$

where $\mathcal{V}^0_{0}(M)$ is the space of $\sigma''$-closed $k$-vectors of type $(k,0)$, and, furthermore, (2.5) shows that $\text{im} \ p_{k,0}$ consists of $k$-vectors $Q^{k,0} \in \mathcal{V}^0_{0}(M)$ which satisfy the following sequence of existence conditions of $k$-vectors $Q^{k-1,1}, \ldots, Q^{0,k}$ such that

$$(c_1) \quad \sigma' Q^{k,0} = \sigma''\text{-exact} \overset{\text{def}}{=} - \sigma'' Q^{k-1,1},$$

$$(c_2) \quad \sigma' Q^{k-1,1} = \sigma''\text{-exact} \overset{\text{def}}{=} - \sigma'' Q^{k-2,2},$$

$$(c_k) \quad \sigma' Q^{1,k-1} = \sigma''\text{-exact} \overset{\text{def}}{=} - \sigma'' Q^{0,k}.$$  

In this case we shall say that $\sigma' Q^{k,0}$ satisfies $k$ times the $\sigma''$-exactness condition, and we shall denote by $\mathcal{V}^0_{0}(M)$ the space of such $Q^{k,0}$. If we also denote $\ker p_{k,0} = 0H^k_{LP}(M,\Pi)$ = the space of $k$-dimensional LP cohomology classes whose cocycles are (2.4) with $Q^{k,0} = 0$, we obtain the result of the first recurrence step

$$(2.7) \quad H^k_{LP}(M,\Pi) \approx 0H^k_{LP}(M,\Pi) \oplus \mathcal{V}^0_{0}(M).$$

Now, in the next step we have to compute $0H^k_{LP}(M,\Pi)$, and for this purpose we take the subcomplex $0\mathcal{L}(M)$ of $\mathcal{L}(M)$ consisting of multivectors $Q$ with a vanishing $(.,0)$ component, and denote by $H^k(0\mathcal{L}(M))$ its cohomology spaces. Then $0H^k_{LP}(M,\Pi)$ is the image of $H^k(0\mathcal{L}(M))$ with respect to the inclusion $0\mathcal{L}(M) \subseteq \mathcal{L}(M)$. It is clear that the complex $\mathcal{L}(M)/0\mathcal{L}(M)$ has coboundary zero, therefore, $H^k(\mathcal{L}/0\mathcal{L}) = (\mathcal{L}/0\mathcal{L})^k = \mathcal{V}^k(0\mathcal{L}(M))$. This gives us the exact sequence
\[ \psi^{-k-1,0}(M) \xrightarrow{\sigma} H^k(\mathcal{L}(M)) \xrightarrow{i^*} H^k(\mathcal{L}(M)), \text{ and we get} \]

\[ 0^\circ H^1_{\mathcal{P}}(M,\Pi) \approx H^k(\mathcal{L}(M))/\sigma(\psi^{-k-1,0}(M)). \]

Hence, the second step will have to consist of an analysis of \( H^k(\mathcal{L}(M)) \), which can be made in the same way as in step 1, and resulting in a formula similar to (2.7), and so on.

For \( k = 1 \), we get easily

\[ 0^1 H^1_{\mathcal{P}}(M,\Pi) = \{X \in \psi^{-0,1}(M)/\sigma''X = 0\}/\sigma''(\psi^{-0}(M)). \]

For \( k = 2 \), we have first

\[ H^2(0^1 \mathcal{L}(M)) = \frac{\{Q^{1,1} + Q^{0,2}/\sigma''Q^{1,1} = 0, \sigma''Q^{0,2} + \sigma'Q^{1,1} = 0\}}{\sigma''x^{0,1}} \]

and the analysis which gave (2.7) now yields

\[ \psi^{-1,1}_{0(1)}(M) = \{Q^{1,1}/\sigma''Q^{1,1} = 0 \text{ and } \sigma'Q^{1,1} = \sigma''\text{-exact}\}. \]

(We shall see in Section 3 that, if the foliation \( \mathcal{S} \) is either transversally Riemannian or transversally symplectic, then

\[ \psi^{i}(\mathcal{L}^{0,0}(M)) \approx H^i(M,\Phi^0(\mathcal{S})), \]

where \( \Phi^0(\mathcal{S}) \) is the sheaf of germs of functions which are constant along the leaves of \( \mathcal{S} \).) Summing up the results we get

\[ H^1_{\mathcal{P}}(M,\Pi) \approx \left( \psi^{1,0}_{0(1)}(M) \right) \oplus \left( \psi^{-1,0}_{0(2)}(M) \right), \]

Etc.

### 3. The spectral sequence.

In this section we continue to refer to a regular Poisson manifold \((M,\Pi)\), and use the notation introduced in Section 2, while we are focussing on the spectral sequence \( E^{2}_{pq}(M,\Pi) \) defined at the end of Section 1. We have:
Proposition 3.1. - The first terms of the LP Serre-Hochschild spectral sequence of a regular Poisson manifold \((M,\Pi)\) are given by

\[
\begin{align*}
E_0^{pq}(M,\Pi) &= E_1^{pq}(M,\Pi) = \varphi^{-q,p}(M), \\
E_2^{pq}(M,\Pi) &= H^p(\oplus \varphi^{-q,*},\sigma^*).
\end{align*}
\]

The reader can prove this by noticing that the \(h\)-filtering subcomplex of \(\mathcal{L}(M)\) as defined in Section 1 is equal to \(S_h(M) = \bigoplus_{i\geq h} p_i(M)\),
and then following the usual definition of \(E_0^{pq}\). Here, we just prefer to observe that \(\mathcal{L}(M) = \bigoplus \omega^{-i,j}(M), \sigma = \Sigma d_{hk}\), where \(\omega^{-i,j}(M) = \varphi^{-j,i}(M)\),
and the terms of \(\sigma\) are \(d_{01} = 0, d_{10} = \sigma''\), \(d_{2,-1} = \sigma'\), is a double semipositive cochain complex in the sense of [VI], p. 76-77, and then (3.1) follows from this reference.

Now, let \(G\) be a metric of the vector bundle \(\mathcal{S}'\) of Section 2, and let \(\# = \#_G \oplus \#: \mathcal{S}' \oplus T^* \mathcal{S} \rightarrow \mathcal{S}' \oplus T^* \mathcal{S}\) be the corresponding musical isomorphism also extended to \(\Lambda^k(M) \rightarrow \varphi^{-k}(M)\). Then, if \(\lambda\) is a differential form of type \((p,q)\), \(\lambda^{\#}\) is a multivector of the same type, and we have

\[
(3.2) \quad (\#^{-1}\sigma''\lambda^{\#})(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q)
= (-1)^{q+1}(\sigma''\lambda^{\#})(\#_G^{-1}X_0, \ldots, \#_G^{-1}X_{p-1}, \#^{-1}Y_0, \ldots, \#^{-1}Y_q).
\]

In this relation, and in the sequel, we agree that type \(X = (1,0)\) and type \(Y = (0,1)\). Furthermore, in order to compute \(\sigma''\lambda^{\#}\) by (2.2) we establish first

\[
\{\#^{-1}Y_i, \#^{-1}Y_j\}^{\#} = \{\#^{-1}Y_i, \#^{-1}Y_j\}^{\#} = [Y_i, Y_j]
\]
(remember that \(\{\alpha,\beta\}^\# = [\alpha^\#, \beta^\#] [BV]\), and using (1.1))

\[
\{\#_G^{-1}X_i, \#^{-1}Y_j\}(X) = -(L_{Y_j}G^*)(X_i, X) - G^*([Y_i, X], X),
\]
where \(G^*\) is the dual metric of \(G\) on \(\mathcal{S}'\). If these formulas are used, and the result is compared with the formula of the \(\mathcal{S}'\)-leafwise exterior differential \(d_f\) [VI], p. 184, one gets

\[
(3.3) \quad (\#^{-1}\sigma''\lambda^{\#})(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q)
= -(d_f\lambda)(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q)
+ \sum_{i=0}^{p-1} \sum_{j=0}^{q} (-1)^{p+i+j}\lambda_{ij}([L_{Y_j}G^*](X_i, X))^{\#G},
\]

\[
X_0, \ldots, \hat{X}_i, \ldots, X_{p-1}, Y_0, \ldots, \hat{Y}_j, \ldots, Y_q).
\]
Remark. — The same result holds if $G$ is a symplectic structure on $\mathcal{S}^*$. This computation leads to

**PROPPOSITION 3.2.** — If the symplectic foliation $\mathcal{S}$ of the regular Poisson manifold $(M,\Pi)$ is either transversally Riemannian or transversally symplectic, one has

$$E^p_q(M,\Pi) = E^p_q(\mathcal{S}) = H^p(M,\Phi^q(\mathcal{S}))$$

where $E^p_q(\mathcal{S})$ is the spectral sequence of the foliation $\mathcal{S}$ (e.g., [KT]), and $\Phi^q(\mathcal{S})$ is the sheaf of germs of $\mathcal{S}$-foliated $q$-forms of $M$ (e.g., [V1]). Particularly, (3.4) holds if $\mathcal{S}$ is a fibration.

Indeed, under the hypotheses, $L_Y G = 0$ in (3.3), and in view of (3.1) we get an isomorphism $E^p_q(M,\Pi) = H^p(\oplus \Lambda^r(M),d)$. But then (3.4) is known [V1], p. 216, 222, 77. (Remember that an $\mathcal{S}$-foliated $q$-form is a $q$-form which, locally, is the pull-back of a form of a local transversal manifold of the foliation $\mathcal{S}$.)

Now, let us define an interesting special class of Poisson manifolds. A vector field $V$ of $M$ is $\mathcal{S}$-foliated if it sends leaves to leaves or, equivalently, $[V,Y] \in T\mathcal{S}$, $[V,Y] \in T\mathcal{S}$. For instance, this happens if $V$ is an infinitesimal automorphism of $\Pi$ i.e., $L_Y \Pi = 0$, a condition which is easily seen to be equivalent to each of the following two conditions, where $f$, $g \in C^\infty(M)$,

$$V\{f,g\} = [V,X_f](g) - [V,X_g](f),$$

(3.5)

$$[V,X_f] = X_{V(f)}. $$

(3.6)

A regular Poisson structure $\Pi$ of $M$ will be called transversally constant if $\mathcal{S}$ has a transversal distribution $\mathcal{S}'$ such that every local foliate vector field $V \in \mathcal{S}'$ is a local infinitesimal automorphism of $\Pi$. For instance, if $M = S \times N$, and $\Pi$ is defined by a symplectic structure of $S$, the distribution $\mathcal{S}' = TN$ has this property. Particularly, the existence of the local canonical coordinates of $\Pi$ in the sense of [L] p. 256-257, shows that every regular Poisson manifold is locally transversally constant. Another example is the Dirac bracket defined as follows. Let $(M,\omega)$ be a symplectic manifold endowed with a foliation $\mathcal{F}$ such that $\omega$ induces symplectic structures of its leaves. These induced structures yield a Poisson bivector $\Pi$ such that $\mathcal{F}(\Pi) = \mathcal{F}$, and $\{ \ , \ \}_\Pi$ is the Dirac bracket of $(M,\omega,\mathcal{F})$. It follows that every $\mathcal{F}$-foliate vector
field $V$ which is $\omega$-orthogonal to $\mathcal{F}$ is an infinitesimal automorphism of $\Pi$. Indeed, for such $V$, (3.5) is equivalent to $(L_{\omega}(\Phi)(X_f, X_\eta) = 0$, and this is an easy consequence of $d\omega = 0$. Using this definition, we have

**Proposition 3.3.** - If $\Pi$ is transversally constant, $\sigma = 0$, and

\[
H^k_{LP}(M, \Pi) = \bigoplus_{q=0}^k E^k_{L^q}(M, \Pi).
\]

**Proof.** - Of course, the proposition refers to $\sigma'$ of (2.1) taken with respect to the distribution $\mathcal{F}'$ involved in the definition of a transversally constant Poisson structure. Let us use the notation of (2.1), and evaluate there $\{ \beta_1, \beta_j \} (X_p) (p \in M, X_p \in \mathcal{F}'_p)$. This may be done by extending $X_p$ to a local foliate $(1,0)$-vector field $X$, and using (2.3). Since $\Pi$ is transversally constant, $L_{\omega}\Pi = 0$ and we get $\sigma = 0$. Then, (3.7) follows from (3.1). Q.e.d.

We shall finish by giving various corollaries of Propositions 3.1, 3.2, 3.3.

**Corollary 3.1.** - If $(M, \Pi)$ is a transversally constant Poisson manifold whose symplectic foliation is either transversally Riemannian or transversally symplectic, one has

\[
H^k_{LP}(M, \Pi) = \bigoplus_{q=0}^k E^k_{\omega}(\mathcal{F}') = \bigoplus_{q=0}^k H^q(M, \Phi^{k-q}(\mathcal{F}')).
\]

**Corollary 3.2.** - Let $\Pi$ be a Dirac bracket of a symplectic manifold $(M, \omega)$ endowed with a leafwise symplectic foliation $\mathcal{F}$, and its $\omega$-orthogonal distribution $\mathcal{F}'$. Assume that the bihomogeneous components of $\omega$ with respect to the decomposition $TM = \mathcal{F}' \oplus T\mathcal{F}$ are closed. Then, again, formula (3.8) holds good.

**Proof.** - Being a Dirac bracket, $\Pi$ is transversally constant. On the other hand, if $\omega = \omega_{(2,0)} + \omega_{(0,2)}$; the hypothesis $d\omega_{(2,0)} = 0$ implies $(L_{\omega}(\mathcal{F}'))(X_1, X_2) = 0$ for $(Y \in T\mathcal{F}, X_1, X_2 \in \mathcal{F}')$, and we see that $\omega_{(2,0)}$ defines a transversal symplectic structure of $\mathcal{F}$. Q.e.d.

**Corollary 3.3 [X].** - Let $\Pi$ be the Poisson structure defined on $M = S \times N$ by a fixed symplectic structure of $S$, and assume that $S$ has finite Betti numbers. Then one has

\[
H^k_{LP}(M, \Pi) = \bigoplus_{q=0}^k [H^q(S, \mathcal{R}) \otimes \Lambda^{k-q}(N)].
\]
This result follows from (3.8) and from

**Proposition 3.4.** — Let \( \mathcal{F} \) be the foliation of \( M = F \times N \) by the leaves \( F \times \{x\} (x \in N) \), and assume that \( F \) has finite Betti numbers. Then

\[
H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N).
\]

**Proof.** — For \( q = 0 \) the result was proven in [E] by a spectral sequence argument. Generally, we have the following straightforward argument. By the foliated de Rham theorem [V1], p. 216, we have

\[
H^q(M, \Phi^p(\mathcal{F})) = \ker [d_f: \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)] \overline{\text{im} [d_f: \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M)]}.
\]

In our case, \( \Lambda^{p,q}(M) \) is isomorphic to the space \( \Lambda^q(F, \Lambda^p(N)) \) of \( \Lambda^p(N) \)-valued \( q \)-forms on \( F \) by the mapping which sends \( \lambda \in \Lambda^{p,q}(M) \) to \( \tilde{\lambda} \in \Lambda^p(F, \Lambda^p(N)) \) defined by

\[
(\tilde{\lambda}_y(Y_1, \ldots, Y_q))_x(X_1, \ldots, X_p, y) = (-1)^p \lambda_{(x,y)}(X_1, \ldots, X_p, Y_1, \ldots, Y_q),
\]

\( y \in F, \ x \in N, \ Y_i \in T_xF, \ X_j \in T_xN \). Moreover, this isomorphism sends \( d_f \) to the exterior differential of \( \Lambda^p N \)-valued forms. Hence (3.11) becomes

\[
H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \Lambda^p(N)) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N),
\]

where the last equality follows from the hypothesis on \( F \). Q.e.d.

**Remark.** — If \( M = S \times N \) of Corollary 3.3 is given a Poisson structure \( \Pi \) which has the symplectic foliation \( S \times \{x\} (x \in N) \), but where each leaf has a different symplectic structure (e.g., the structure studied in [X]), \( \Pi \) is no more transversally constant, but we may use Propositions 3.2. and 3.4, and get

\[
E^{p,q}_2(M, \Pi) = H^p(S, \mathbb{R}) \otimes \Lambda^q(N).
\]

**Corollary 3.4.** — Let \( (M, \Pi) \) be an arbitrary regular Poisson manifold. Then every \( x \in M \) has a connected open neighbourhood \( Y \) such that

\[
H^k_{\text{Lie}}(U, \Pi|_U) = \Gamma(\Phi^k(\mathcal{F}|_U)),
\]

i.e., the space of the \( \mathcal{F} \)-foliated \( k \)-forms over \( U \).
Indeed, we may take \( U = S \times N \) where \( S \) is contractible, and such that the product coordinates are canonical for \( \Pi \) in the sense of [L], p. 257. Then Corollary 3.3 holds on \( U \), and we get (3.13). We shall say that such a neighbourhood \( U \) is LP-simple.

**Corollary 3.5 (The LP Poincaré Lemma [L]).** Let \( (M, \Pi) \) be a regular Poisson manifold, and \( x \in M \). Then, there exists an open neighbourhood \( U \) of \( x \) in \( M \) such that, if \( Q \in \mathcal{V}^k(U) \) and \( \sigma Q = 0 \), one has \( Q = A + \sigma B \) for some \( B \in \mathcal{V}^{k-1}(U) \) and a \( k \)-vector field \( A \) over \( U \) which is projectable to a \( k \)-vector field of a local transversal submanifold of \( \mathcal{S} \) in \( U \).

**Proof.** Take \( U \) LP-simple, and with \( \Pi \)-canonical coordinates. The latter define a bigrading, and we may write \( Q = \sum_{p=0}^k (\lambda^{p,k-p}) \), where \( \lambda \) are differential forms, and \( \# \) is like in (3.2). The use of the canonical coordinates makes \( \Pi_{/U} \) transversally constant and transversally Riemannian hence, by Proposition 3.3 and formula (3.3), \( \sigma = \sigma^{n} \), and \( \sigma Q = 0 \) is equivalent to \( d_{\tau} \lambda^{p,k-p} = 0 \) \( (k = 0, \ldots, p) \). But \( d_{\tau} \) satisfies a local Poincaré lemma [VI], p. 215, hence, there are local forms \( \mu \) such that \( \lambda^{p,k-p} = d_{\tau} \mu^{p,k-p-1} \) for \( k - p > 0 \), while \( \lambda^{k,0} \) is a foliate form. The conclusion follows by using again (3.3). Q.e.d

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