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Metric properties of eigenfunctions of the Laplace operator on manifolds


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METRIC PROPERTIES OF EIGENFUNCTIONS 
OF THE LAPLACE OPERATOR ON MANIFOLDS 

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In this note, we prove two theorems which express a quasi-symmetry relation between the positive and the negative part of the distribution function of an eigenfunction of the Laplace operator on a Riemannian manifold. 

1. An estimate of the volume of a domain on which an eigenfunction of the Laplace operator on a Riemannian surface has constant sign. 

Let $M$ be a two-dimensional compact real analytic Riemannian manifold, $u_1, u_2, \ldots$ -eigenfunctions of the Laplace operator on $M$, $\Delta u_i = \lambda_i u_i$. 

THEOREM. — There exists a positive constant $C$ which depends on $M$ such that, for every $i = 1, 2, \ldots$, 

$$ \text{vol}\{ x \in M, u_i(x) > 0 \} > C. $$ 

The proof is based on the two following lemmas. 

Let $f$ be a bounded function, continuous on $[0, 1]$. Let us denote by $N(f)$ the number of changes in the sign of the function $f$ on $[0, 1]$. 

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Lemma 1. — Let $f_n$ be a sequence of non-zero continuous functions, defined on $\mathbb{R}$, with values in $\mathbb{R}$, with support in $[0,1]$ and assume that $N(f_n)$ is bounded by some fixed number $N$.

Then there exists a subsequence $n_i$ of $\mathbb{N}$ ($n_i \to \infty$), real numbers $\alpha_{n_i}$, such that $\alpha_{n_i} f_{n_i}$ converges as $i \to \infty$, for the (weak-)topology of the space of distributions $\mathcal{D}'$ to a non-zero distribution of order less than $N$.

Remark. — First, we recall that a distribution is said of order less than $N$ if it is a sum of derivatives of order less than $N$ of Radon measures. Moreover, if $P$ is a polynomial of degree $N$ and $\mu$ a Radon measure, the set of all $T \in \mathcal{D}'$ satisfying $PT = \mu$ is an $N$-dimensionnal affine subspace of the space of all distributions of order less than $N$.

Proof of Lemma 1 (suggested by Y. Colin de Verdière). — Let $P_n = \prod_{k=1}^{N}(x-x_k)$ be a sequence of polynomials of degree $N$ such that $P_n.f_n$ is $\geq 0$. By renormalisation and taking a subsequence, we may assume that $\int_0^1 P_n.f_n = 1$, that $P_n.f_n$ converges to a probability measure $\mu$ and that $P_n$ converges to a polynomial $P$ of degree exactly $N$.

Let $T_0 \in \mathcal{D}'$ be such that $PT_0 = \mu$.

Let $T_n = f_n - T_0$, then we get :

$$\lim P_n.T_n = 0.$$ 

We introduce now the following decomposition of the space of distributions :

$$\mathcal{D}' = Z_P \oplus W,$$

where $Z_P = \{ T \in \mathcal{D}' | PT = 0 \}$, and $W$ is a topological complement of $Z_P$.

$W$ is a complement to $Z_{P_n}$ if $n$ is big enough. Now we can write uniquely :

$$T_n = z_n + w_n,$$

where $z_n \in Z_{P_n}$ and $w_n \in W$. Now $P_n.w_n \to 0$ and we deduce that $w_n \to 0$, because the multiplication by $P_n$ is uniformly invertible in $W$.

Now $z_n \in Z_{P_n}$ and $Z_{P_n}$ converges to $Z_P$.

Two cases are possible :

First case : $z_n$ is bounded and we can extract a convergent subsequence converging to $T_1$ in $Z_P$. Then $T_0 + T_1$ is not zero and we get the conclusion.
Second case: \( z_n \) is unbounded; then there exists a sequence \( \beta_{n_i} \to 0 \) such that \( \beta_{n_i} z_{n_i} \) converges to a non-zero distribution \( T_1 \) and then:

\[ \beta_{n_i} f_{n_i} \]

converges to \( T_1 \).

Let us denote by \( B \) the unit disk in \( \mathbb{R}^2 \), \( S = \partial B \), if \( f \) is a continuous function on \( S \) then \( N(f) \) is the number of changes of sign of the function \( f \) on \( S \).

**Lemma 2.** — Let \( u \) be a harmonic function in \( B \) which is continuous in \( \overline{B} \), \( u|_S = f \), \( u(0) = 0 \). Let \( N(f) = k < \infty \). Define

\[ G_u = \{ x \in B \mid u(x) > 0 \}. \]

Then \( \text{mes } G_u > C \), where constant \( C > 0 \) is dependent on \( k \).

**Proof.** — Let us assume the contrary. This means that there exists a sequence of harmonic functions \( u_n \) in \( B \), \( u_n|_S = f_n \), \( u_n(0) = 0 \), \( N(f_n) \leq k \), \( \text{mes } G_{u_n} \to 0 \), \( n \to \infty \). According to lemma 1 there exists a real valued sequence \( \alpha_m \) and a subsequence \( f_{n_m} \) such that, \( \alpha_m f_{n_m} \to \tilde{f} \neq 0 \) in the sense of distributions. From the convergence of the distributions \( \alpha_m f_{n_m} \) on \( S \) it follows that in an arbitrary compact subdomain of \( B \) the convergence of functions \( \alpha_m u_{n_m} \) is uniform. Let \( \alpha_m u_{n_m} \to U \) in \( B \). From [1] it follows that \( U \neq 0 \) in \( B \) if \( \tilde{f} \neq 0 \) on \( S \). We have \( U(0) = 0 \). From the assumption \( \text{mes } G_{u_n} \to 0 \), \( n \to \infty \), it follows that \( U \leq 0 \) in \( B \). Equality \( U(0) = 0 \) and inequality \( U \leq 0 \) in \( B \) contradicts the maximum principle for harmonic functions.

**Proof of the theorem.**

1. Let us denote by \( B^r_x \), \( x \in M \), \( r \), the geodesic circle on \( M \) with centre \( x \) and radius \( r \).

There is a constant \( C_0 > 0 \), such that for every \( \varepsilon > 0 \) there exists points \( x_1 \ldots x_N \in M \), \( N > C_0/\varepsilon^2 \), such that the circles \( B^r_{\varepsilon^2} \ldots B^r_{\varepsilon^N} \) mutually have no intersections.

2. There exists a constant \( r_0 \), such that for every \( x \in M \), \( 0 < r < r_0 \), \( B^r_x \) is diffeomorphic to a disk.

3. There is a constant \( C_1 > 0 \), such that for all \( x \in M \), \( \lambda > 0 \) in the circle \( B^r_{1/C_1\sqrt{\lambda}} \) there exists a positive solution of the equation \( \Delta u + \lambda u = 0 \).

4. Let \( x \in M \), \( \lambda > 0 \), \( r = 1/C_1\sqrt{\lambda} < r_0 \), \( u \) is a solution of the equation \( \Delta u + \lambda u = 0 \) in \( B^r_x \). Then there exists a diffeomorphism \( h \) of the
unit disk $B$ on $B^x$, $h(0) = x$, and a function $s$ in $B$, $0 < s < \infty$, such that $s.u(h)$ is a harmonic function in $B$ (by a representation theorem in quasiconformal mapping theory, [2]). From the compactness of $M$ it follows that the Jacobian of the mapping $h$ is uniformly bounded.

5. There is a constant $C_2 > 0$, such that for all $x \in M$, $\lambda > 0$ in the circle $B^{x_{1/C_2 \sqrt{\lambda}}}$ every solution $u \neq 0$ of the equation $\Delta u + \lambda u = 0$ changes its sign [3].

6. Let $u_i$ be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on $M$, $\gamma$ is a nodal line of the function $u_i$. For a two-dimensional real analytic manifold the following estimate is true, [4],

$$\text{length } \gamma \leq C_3 \sqrt{\lambda_i}$$

where constant $C_3 > 0$ is dependent on $M$.

7. Let $u_i$ be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on $M$. According to 1 we can choose circles $B_{\varepsilon}^{x_1}, \ldots, B_{\varepsilon}^{x_n}$ with $\varepsilon = 2/C_2 \sqrt{\lambda_i}$. We have $N > C_0 C_2^2 \lambda_i/4$.

According to 5 there exist points $y_n \in B_{\varepsilon/2}^{x_n}$, $n = 1 \ldots N$, such that $u_i(y_n) = 0$.

According to 6 at least $N/2$ points $y_{k_1}, \ldots, y_{k_J}$, $J > N/2$, from the set $\{y_n\}$ have the following property: for all $j = 1 \ldots J$ there exist $r_j$,

$$\frac{1}{2C_1 \sqrt{\lambda_i}} < r_j < \frac{1}{C_1 \sqrt{\lambda_i}},$$

such that restriction of the function $u_j$ on $\partial B_{r_j}^{y_{k_j}}$ has no more than

$$8C_1 C_3 \frac{C_2^3}{C_0^2}$$

zeros.

According to 4 and lemma 2 for all $j = 1 \ldots J$

$$\text{mes}\{x \in B_{r_j}^{y_{k_j}}, \ u_i(x) > 0\} > C_4 \varepsilon^2.$$

We have $J > C_0/2\varepsilon^2$ and so the theorem is proved.

2. An estimate of the relation between the positive and the negative extrema of an eigenfunction.

Let $M$ be a $n$-dimensional compact smooth Riemannian manifold, $u_1, u_2, \ldots$-eigenfunctions of the Laplace operator on $M$, $\Delta u_i = \lambda_i u_i$. 
**THEOREM 2.** There exists a positive constant $C$ which depends only on $n$ and a positive constant $N$ which depends on $M$ such that, for every $i > N$,

$$\frac{1}{C} < \frac{\sup M u_i(x)}{\inf M u_i(x)} < C.$$  

We denote by $B_r \subset \mathbb{R}^n$ the ball centered at 0 of radius $r$.

In $B_r$ we consider a uniformly elliptic second order operator $L$ defined by

$$L = \sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) ,$$

where $a_{ij}$ is a symmetric positive definite matrix in $B_r$. If the eigenvalues of the matrix $\|a_{ij}(x)\|$ lie on the segment $[e^{-1}, e]$, $e \geq 1$ we say that the operator $L$ has an ellipticity constant $e$.

**LEMMA 3.** Let $u$ be a solution of the equation

$$a(x)Ly + \lambda u = 0$$

in the ball $B_1$, $1/A < a(x) < A$, $A > 0$, $L$ is an elliptic operator with the ellipticity constant $e$, $\lambda$ is a constant such that $|\lambda| < C$. Let us assume that $u(x_0) > 0$ and that there exists $x_0 \in B_{1/2}$ with $u(x_0) = 0$. Then

$$\inf_{B_1} u > \delta u(0) ,$$

where the constant $\delta > 0$, $\delta = \delta(n, A, e, C)$.

**Proof.**

1. We shall prove Lemma 2 under the assumption that $\lambda = 0$. Denote

$$\varphi_1 = \sup\{0, u |_{\partial B_1}\}$$
$$\varphi_2 = \inf\{0, u |_{\partial B_1}\} .$$

Let $u_1, u_2$ be the solutions of the following Dirichlet problems:

$$Lu_1 = 0 \text{ in } B_1 , \ u_1 |_{\partial B_1} = \varphi_1,$$
$$Lu_2 = 0 \text{ in } B_2 , \ u_2 |_{\partial B_1} = \varphi_2 .$$

Then, $u = u_1 + u_2$, $u_1 > 0$ in $B_1$, $u_1(0) \geq u(0)$. From the Harnack inequality [5] it follows that there exists a constant $\delta > 0$, $\delta = \delta(n, e)$ such that

$$u_1 |_{B_{1/2}} > \delta u_1(0) .$$
Since \( u(x_0) = 0, \ x_0 \in B_{1/2} \), then
\[
\inf \varphi_2 < -\delta u_1(0) < -\delta u(0).
\]

2. Let \( \lambda \neq 0 \). Let us make a cylindric extension of the functions \( u(x), a(x) \) and the operator \( L \) in the new coordinate \( x_{n+1} \). After this extension we shall keep the notations \( u, a, L \). Denote
\[
v = u e^{{\sqrt{\lambda} x_{n+1}}}
\]
clearly the function \( v \) is a solution of the elliptic equation
\[
aLv + \frac{\partial^2 v}{\partial x_{n+1}^2} = 0.
\]
Now the statement of Lemma 3 follows from the assertion 1 to the function \( v \) in the unit ball in \( \mathbb{R}^{n+1} \).

**Proof of Theorem 2.**

1. There are constant \( C_1 = C_1(M) > 0, \ C_2 = C_2(M) > 0 \) such that for all \( x \in M, \ \lambda > C_2 \) any solution of the equation \( \Delta u + \lambda u = 0 \) in the ball \( B^{x}_{C_1/\sqrt{\lambda}} \) change its sign.

2. There exists a constant \( N > C_2, \ N = N(M) \), such that for all \( x \in M \) there exists a diffeomorphism
\[
d : B^{x}_{2C_1/\sqrt{\lambda}} \subset M \rightarrow B_1 \subset \mathbb{R}^n
\]
such that the equation \( \Delta u + \lambda u = 0 \) in \( B^{x}_{2C_1/\sqrt{\lambda}} \) viewed in the ball \( B_1 \) has the form
\[
a(x)Lu + \lambda' u = 0 \tag{2.2}
\]
where \( L \) is an elliptic operator of the type (2.1), \( e = 2, A = 2, |\lambda| < C = C(n) > 0 \). We can obtain such a diffeomorphism \( d \) if we introduce in the ball \( B^{x}_{2C_1/\sqrt{\lambda}} \) a normal coordinate system. Applying Lemma 3 to the solution \( u \) of the equation (2.2) we obtain the statement of Theorem 2.

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