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## ON EQUIVARIANT HARMONIC MAPS DEFINED ON A LORENTZ MANIFOLD

by MA LI

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### 1. Introduction.

It is interesting to study harmonic maps from a Lorentz manifold into a Riemannian manifold. In this case, the harmonic map equation is a Hyperbolic system of second order. In this paper, we look for equivariant harmonic maps defined on a specific Lorentz manifold ; namely, the Lorentz manifold  $M = M_0 \times R$  with the space-time metric

$$ds^2 = dt^2 - S^2(t) d\sigma^2$$

where  $(M_0, d\sigma^2)$  be the symmetric space for a compact Lie group  $G$  with a bi-invariant Riemannian metric  $d\sigma^2$  and  $S(t)$  is a smooth positive function defined on  $R$ . The target manifold is a compact Riemannian manifold  $(N, h)$  admitting an isometric group action of  $G$ . This kind of problem is called a  $\sigma$ -model in Physics literature and one may see [G] and [EL] for further datum. Without loss of generality, we may assume that  $N$  is a submanifold of some Euclidean space  $R^k$  by Nash's isometrical imbedding theorem, so we may think of  $G$  as  $\subset SO(k)$  with its Lie algebra  $LG \subset so(k)$  the Lie algebra of  $SO(k)$  whose elements are skew-matrices.

By definition, a smooth map  $u$  from  $M$  to  $N$  is called a harmonic map if it is a critical point of the following action integral

$$\begin{aligned} E_I(u) &= \int_{M_0 \times I} (Tr_{ds^2} u^* h) S^n(t) d\mu dt \\ &= \int_I \int_{M_0} (|\partial_t u|^2 - S^{-2}(t) |\nabla_0 u|^2) S^n(t) d\mu dt \end{aligned}$$

for every interval  $I = [a, b] \subset \mathbb{R}$  among all maps of its class, here  $n = \dim M_0$ ,  $d\mu$  is the invariant measure of  $(M_0, d\sigma)$ ,  $|\cdot|$  is the usual norm induced by  $\mathbb{R}^k$  and  $\nabla_0$  is the covariant derivative induced by  $d\sigma^2$  on  $M_0$ .

We will prove the following

**THEOREM.** — *Let  $M$  and  $N$  be the manifold defined above. Suppose  $S(t)$  is a smooth positive periodic function of period  $2\pi$ , then, there exist infinitely many  $G$ -equivariant harmonic maps which are of period  $2\pi$  in  $t$  from  $M$  to  $N$ .*

By equivariant, we mean that the map  $u : M \rightarrow N$  satisfies

$$u(g \cdot m, t) = g \cdot u(m, t)$$

for every  $g \in G$  and  $(m, t) \in M_0 \times \mathbb{R}$ . We denote the set of equivariant maps  $\mathcal{M}$  and it is non-empty by our assumptions on  $M$  and  $N$ . Select a basis  $\{e_j\}_{j=1}^n$  (note  $n = \dim M_0 = \dim_{\mathbb{R}} G$ ) of the Lie group  $G$  and let  $\{A_j\}_{j=1}^n$  denote the corresponding basis of its Lie algebra. Fix  $m \in M_0$  and write  $x(t) = u(m, t)$ . Because  $u$  is an equivariant map,  $u(\exp(sA_j)m, t) = \exp(sA_j)u(m, t)$ . Differentiating it w.r.t.  $s$  at  $s = 0$  we get that  $\nabla_0 u(m, t)(A_j) = A_j u(m, t)$  (matrix multiplication in  $\mathbb{R}^k$ ). From this and the invariance of the metric  $d\sigma^2$ , the action integral  $E_I(\cdot)$  for the  $G$ -equivariant map  $u$  becomes

$$\begin{aligned} E_I(u) &= \int_{M_0} d\mu \int_I (|u_t(m, t)|^2 - S^{-2}(t) \sum_{j=1}^n |A_j(u(m, t))|^2) S^n(t) dt \\ &= \text{Vol}(M_0) F_I(x), \end{aligned}$$

where the last integral factor  $F_I(x)$  will be written as  $F(x)$  when  $I = S^1$ .

It will be shown by the minimax principle that there exist infinitely many critical points of  $F(\cdot)$  just like closed geodesics in  $N$ . But here we should mention that it is conceptually different from the closed geodesic case because the Euler-Lagrange equation for our  $F(\cdot)$  is a non-autonomous one (see Lemma 2 below).

## 2. Some well-known facts.

Since the  $A_j$  is a skew-symmetric matrix, there exists a non-negative symmetric matrix  $A$  such that

$$\sum_{j=1}^n A_j^2 = - \sum_{j=1}^n A_j A_j^* = -A^2.$$

So

$$(1) \quad F(x) = \int_0^{2\pi} (|x'(t)|^2 - S^{-2}(t)|Ax(t)|^2) S^n(t) dt.$$

Think of  $M_0 \times S^1$  as a Riemannian manifold with the metric  $dt^2 + S^2(t) d\sigma^2$ , we may define a Hilbert manifold  $H = W^{l,2}(M_0 \times S^1, N)$  for  $l$  large enough. Now,  $H$  admits an isometric group action  $(u, g) \rightarrow g^{-1} \cdot u \cdot g$  of  $G$ . Applying the theorem in page 23 of R. S. Palais [P2] to  $F$  on  $H$  and to the fixed point set of the map  $u \rightarrow g^{-1} \cdot u \cdot g$ , we find

LEMMA 1. — *If  $u \in \mathcal{M}$ , then  $u$  is harmonic if and only if  $x(t) = u(m, t)$  is a critical point for  $F_I(x)$  for all intervals  $I \subset R$ .*

Let  $\mathcal{O}$  be an open uniform tubular neighborhood of  $N$  in  $R^k$  such that the  $P: \mathcal{O} \rightarrow N$  given by  $P(y) =$  the nearest point in  $N$  to  $y$ , is a smooth fibration.

LEMMA 2. — *The Euler-Lagrange equations for an equivariant harmonic map from  $M$  to  $N$  are*

$$(2) \quad S^{-n}(t)(S^n(t)x')' - D^2P(x', x') + S^{-2}(t)A^2x = 0,$$

which is a non-autonomous system except if  $S(t) = \text{const}$ .

*Proof.* — Suppose  $x$  is the critical point of  $F(\cdot)$  which corresponds to the equivariant harmonic map we consider. For  $\eta \in W^{1,2}(S^1, R^k)$ , if  $\varepsilon > 0$  is small enough, we have that  $P(x(\cdot) + s\eta(\cdot))$  is a smooth curve in  $W^{1,2}(S^1, N) := \{y \in W^{1,2}(S^1, R^k); y(t) \in N\}$  passing through  $x$  for  $s \in (-\varepsilon, \varepsilon)$ . Hence

$$\begin{aligned} 0 &= 2^{-1} d/ds |_{s=0} F(P(x + s\eta)) \\ &= \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \int_{S^1} |DP_{x+s\eta} \cdot (x'(t) + s\eta'(t))|^2 S^n(t) dt \\ &\quad - \int_{S^1} |AP(x(t) + s\eta(t))|^2 S^{n-2}(t) dt \\ &= \int_{S^1} (\langle DP_x \cdot x'(t), D^2P_x(x'(t), \eta(t)) + DP_x \cdot \eta'(t) \rangle) S^n(t) dt \\ &\quad - \int_{S^1} \langle A^2x(t), \eta(t) \rangle S^{n-2}(t) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_{S^1} \langle x'(t), \eta'(t) \rangle S^n(t) dt \\
 &+ \int_{S^1} (\langle D^2 P_x(x'(t), x'(t)) - S^{-2}(t) A^2 x(t), \eta(t) \rangle) S^n(t) dt.
 \end{aligned}$$

Since  $P(x) = x$ , we have that  $DP_x(x') = x'$ . So by integration by part we get (2). Since, for  $S(\cdot) \neq \text{const.}$ ,

$$(|x'(t)|^2 + S^{-2}(t)|Ax(t)|^2) S^n(t)$$

is not conserved, (2) is a non-autonomous system. □

Define

$$\Lambda^1 = \Lambda^1(N) = W^{1,2}(S^1, N).$$

It is well-known that  $\Lambda^1(N)$  is a Hilbert manifold [P1]. Since  $N$  is compact, there exist constants  $c_i > 0 (i=1, 2, 3)$  such that

$$(3) \quad c_1 D(y) - c_3 \leq F(y) \leq c_2 D(y) - c_3,$$

here  $D(y) := |y|_1^2 = \int_{S^1} |y'|^2$  for every  $y \in \Lambda^1$ . We will also need the following inequality

$$(4) \quad |y|_\infty \leq |y(0)| + c_4 |y|_1,$$

for every  $y \in \Lambda^1$  and the Sobolev imbedding  $W^{1,2}(S^1, R^k) \rightarrow C^0(S^1, R^k)$  is compact.

LEMMA 3. — i)  $F(\cdot)$  satisfies Palais and Smale condition C;

ii) For every  $c > 0$ , there exists an integer  $\bar{n} = \bar{n}(c)$  such that

$$H^n(I_c) = 0$$

for  $n > \bar{n}$ , where  $I_c = D^{-1}(-\infty, c]$ .

*Proof.* — i) Suppose  $\{x_m\} \subset \Lambda^1$  is a sequence such that

$$(5) \quad F(x_m) \rightarrow c$$

and

$$(6) \quad dF(x_m) \rightarrow 0, \quad \text{in } H^{-1}.$$

Since  $N$  is compact, we may assume that  $x_m(0) \rightarrow p$ . By inequality (3), we get that, there exist a constant  $C > 0$  such that

$$(7) \quad D(x_m) \leq C.$$

So we may assume that  $x_m \rightarrow x$  in  $C^0(S^1, N)$ . Now,

$$\langle dF(x_m), \eta \rangle = 2 \int_{S^1} (\langle x', \eta' \rangle + \langle D^2 P(x', x'), \eta \rangle - S^{-2}(t) \langle A^2 x, \eta \rangle) S^n(t).$$

Take  $\eta = x_m - x_n$  and  $x = x_m, x_n$  in (6), we get by (7) and (4) that

$$\begin{aligned} o(1) &= 2^{-1} \langle dF(x_m) - dF(x_n), x_m - x_n \rangle \\ &\geq c_1 D(x_m - x_n) - 2C |x_m - x_n|_\infty \\ &\quad - \int_{S^1} S^{n-2}(t) \langle A^2(x_m - x_n), x_m - x_n \rangle \\ &\geq c_1 D(x_m - x_n) - 2C_5 o(1). \end{aligned}$$

Here we implicitly used boundness of the positive function  $S(t)$ . Hence,  $D(x_m - x_n) = o(1)$ .

ii) This is borrowed from Milnor's book (see theorem 16.2 in [M]). Since  $I_c$  is a strong deformation retract of a finite dimensional manifold, whose dimension  $n$  depends on  $c$ , then, we get the conclusion if we let  $\bar{n}(c) = n$ . □

Now, let us recall a result of M. Vigue-Poirrier and D. Sullivan [V-PS] about the topology of  $\Lambda^1$ .

PROPOSITION 4. — *If  $N$  is compact and simply connected, then there exists an infinite set of positive integers  $\mathbb{M} \subset \mathbb{N}$  such that*

$$H^q(\Lambda^1) \neq 0$$

for  $q \in \mathbb{M}$ .

### 3. Final argument.

Consider a non-trivial  $\alpha \in H^*(\Lambda^1)$  and set

$$(8) \quad \bar{\alpha} = \{B \subset \Lambda^1; i_B^*(\alpha) \neq 0\},$$

where

$$i_B^* : H^*(\Lambda^1) \rightarrow H^*(B)$$

is the homomorphism induced by the inclusion

$$i_B : B \rightarrow \Lambda^1.$$

*Remark 5.* —  $\bar{\alpha}$  defined in (8) is non-empty and contains the compact support of a  $k$ -chain  $a \in \alpha$ ,  $k = \deg \alpha$ , which is not homologous to constant by the nontrivial property of  $\alpha$ .

LEMMA 6. — Let  $\alpha \in H^*(\Lambda^1)$ ,  $\alpha \neq 0$  and define

$$(9) \quad c_\alpha = \inf_{B \in \bar{\alpha}} \sup F(B).$$

Then,  $c_\alpha$  is a critical value of  $F$  on  $\Lambda^1$ ; moreover, if we assume that  $H^q(\Lambda^1) \neq 0$  for infinitely many  $q$ , there exists a sequence  $\{c_\alpha\}$  of critical values of  $F$  defined as in (9) which satisfies that

$$(9') \quad c_\alpha \rightarrow +\infty, \quad \text{as } \deg \alpha \rightarrow +\infty.$$

*Proof.* — By our Remark 5 we have

$$c_\alpha < +\infty.$$

Suppose some  $c_\alpha$  is not a critical value of  $F$ , then by lemma 3 i) and a well-known deformation lemma in page 125 of R. S. Palais [P1], we know that there exists a positive number  $\varepsilon$  and a homeomorphism  $\eta$  on  $\Lambda^1$  such that

$$(10) \quad \eta(F_{c_\alpha + \varepsilon}^{-1}) \subset F_{c_\alpha - \varepsilon}^{-1}.$$

Since

$$\eta^* : H^q(\eta(\Lambda^1)) \rightarrow H^q(\Lambda^1)$$

is an isomorphism, we have that

$$i_{\eta(B)}^*(\alpha) = (\eta^*)^{-1} \cdot i_B(\alpha) \neq 0$$

for all  $B \in \bar{\alpha}$ . Hence  $\eta$  leaves  $\bar{\alpha}$  invariant. But, by the definition of  $c_\alpha$ , there exists  $B \in \bar{\alpha}$  such that

$$\sup F(B) < c_\alpha + \varepsilon.$$

So by (10) and  $\eta(B) \in \bar{\alpha}$  we have

$$\sup F(\eta(B)) < c_\alpha - \varepsilon.$$

It is absurd.

To get (9'), we take  $k \in \mathbb{N}$ . By lemma 3 ii), there exists  $\bar{n} = \bar{n}(k) \in \mathbb{N}$  such that  $H^q(I_k) = 0$  for  $q > \bar{n}$ . By our assumption on  $H^*(\Lambda^1)$  we may take  $q_k > \bar{n}$  with  $H^{q_k}(\Lambda^1) \neq 0$  and consider  $\alpha \in H^{q_k}(\Lambda^1)$ ,  $\alpha \neq 0$ . Denote

$$I^k = \{x \in \Lambda^1; D(x) > k\},$$

we claim that

$$(11) \quad \forall B \in \bar{\alpha}, B \cap I^k \neq \emptyset.$$

Suppose it is not true, then, there exists  $B \in \bar{\alpha}$  such that

$$B \subset \Lambda^1 \setminus I^k := I_k,$$

then

$$(12) \quad H^{q_k}(\Lambda^1) \xrightarrow{i_2^*} H^{q_k}(I_k) \xrightarrow{i_1^*} H^{q_k}(B),$$

where  $i_2^*$ ,  $i_1^*$  are the homomorphisms induced by the inclusion maps

$$i_2: I_k \rightarrow \Lambda^1, \quad i_1: B \rightarrow I_k.$$

Then, by  $B \in \bar{\alpha}$  we have that

$$(13) \quad i_1^* \cdot i_2^*(\alpha) = i_B^*(\alpha) \neq 0.$$

From (12) and (13) we obtain that  $H^{q_k}(I_k) \neq 0$ , a contradiction to our assumption on  $q_k$ . So (11) is true.

By (11) and our choices of  $c_\alpha$  we have that

$$c_\alpha \geq c_1 k - C$$

which implies our conclusion. □

*Proof of Theorem.* – 1) If  $N$  is simply-connected, then the result follows from Proposition 4 and Lemma 6.

2) If  $\pi_1(N) \neq 0$  and finite. Then the universal covering  $(\tilde{N}, \Pi)$  is compact. By 1) we have infinitely many critical points  $\tilde{x}_n: S^1 \rightarrow \tilde{N}$  of  $F$  such that

$$F(\tilde{x}_n) \rightarrow +\infty, \quad \text{as } n \rightarrow \infty.$$

Therefore, set  $x_n = \Pi(\tilde{x}_n)$ , we obtain the existence of infinitely many critical points of  $F$ , and infinitely distinct harmonic maps of periodic  $2\pi$  in  $t$  from  $M$  to  $N$  by Lemma 1.



3) If  $\pi_1(N) = \infty$ . We may get a minimizer of  $F$  in each homotopy class by the Palais-Smale condition in lemma 3 i).  $\square$

*Remark 7.* – (1) Suppose  $S(t)$  is not periodic in  $t$ . Take  $I = [0, 1]$ ,  $x(0)$  and  $x(1)$  two point in  $N$ , we can prove as in our theorem that there are infinitely many geometrical distinct critical points of  $F$ . It is interesting to consider the behavior of the orbit of some critical point of  $F$  just like that of the geodesic in  $N$ .

(2) It is an open question to obtain our theorem when  $S(t) = 1 - \cos(t)$ . In this case, the Lorentz manifold  $M$  is called Friedman-Robertson-Walker space-time in general relativity.

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#### BIBLIOGRAPHY

- [E] J. EELLS JR, Proc 1981 Shanghai-Hefei Symps. Diff. Geom. Diff. Eq., Sci. Press, Beijing, (1984), 55-73.
- [EL] J. EELLS JR and L. LEMAIRE, Another Report on Harmonic Maps, Bull. London Math. Soc., 20 (1988), 385-524.
- [G] GU Chao-hao, On the Two-dimensional Minkowski space, Comm. Pure and Appl. Math., 33 (1980), 727-738.
- [M] J. MILNOR, Morse Theory, Princeton, 1963.
- [P1] R. S. PALAIS, Lusternik-Schnirelmann theory on Banach Manifold, Topology, 5 (1966), 115-132.
- [P2] R. S. PALAIS, The Principle of Symmetric Criticality, Comm. Math. Phys., 69 (1979), 19-30.
- [V-PS] M. VIGUE-POIRRIER, D. SULLIVAN, The Homology Theory of the Closed Geodesic Problem, J. Diff. Geom., 11 (1976), 633-644.

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