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Partial differential operators depending analytically on a parameter


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PARTIAL DIFFERENTIAL OPERATORS
DEPENDING ANALYTICALLY ON A PARAMETER

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0. Introduction.

Consider a linear differential operator in $\mathbb{R}^n$,
\[ P(\lambda, D) = \sum_{|\alpha| \leq m} a_\alpha(\lambda) D^\alpha : D = -i \partial , \partial = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \]
where the coefficients $a_\alpha(\lambda)$ are constant with respect to the variable of differentiation $x$ and may depend analytically on a parameter $\lambda$ in a complex manifold $\Lambda$. We assume that $P(\lambda, D)$ is equally strong for each $\lambda \in \Lambda$.

In [H2], p. 59 L. Hörmander posed the question whether under these conditions there exists a fundamental solution $f_\lambda$ of $P(\lambda, D)$ which depends analytically on $\lambda$. In 1962 F. Treves [T2] had shown that this is true locally in $\Lambda$ and that the assumption of constant strength is necessary for this to hold [T1]. Recently the author could construct a global solution in the hypoelliptic case [M]. The proof of this result based on the fact that for each compact subset $\Lambda'$ of $\Lambda$ there exists an integration contour in $\mathbb{C}^n$ which yields fundamental solutions of $P(\lambda, D)$ simultaneously for all $\lambda \in \Lambda'$. In a second step we could apply a theorem of J. Leiterer [L] to obtain a global solution $f_\lambda$ by means of a Mittag-Leffler procedure.

The aim of the present paper is to eliminate the assumption of hypoellipticity. In section 1 we show that also in the general case one can

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always find a uniform integration contour $H_{\Lambda'}$ for all $\lambda$ in a compact subset $\Lambda'$ of $\Lambda$. As a consequence we obtain an explicit formula for $f_\lambda : \lambda \in \Lambda'$. Our proof uses some ideas of Hörmander [H2] concerning asymptotic properties of multivariate polynomials. The rest of this article is essentially an adaptation of the methods of [M]: in section 2 certain distribution spaces are introduced by means of the contours $H_{\Lambda'}$. These spaces constitute the setting for our application of the Leiterer theorem [L]. Section 3 contains the statements and proofs of our main results. We consider the equation $P(\lambda, D)f_\lambda = g_\lambda$ where $g_\lambda$ is a given analytic function of $\lambda$ with values in some distribution space and prove the existence of a solution $f_\lambda$ which also depends analytically on $\lambda$. In the special case $g_\lambda \equiv \delta$ (the Dirac distribution) we obtain a solution to the problem described above.


We begin by fixing some notations: for any $n, m \in \mathbb{N}$ let
\[
\text{Pol}(n, m) := \{ P \in \mathbb{C}[x_1, \ldots, x_n] \mid \deg P \leq m \};
\]
\[
\text{Pol}'(n, m) := \{ P \in \text{Pol}(n, m) \mid \deg P = m \}.
\]
If $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$ then we write
\[
\delta_P(\xi) := \text{dist}(\xi, \{ \zeta \in \mathbb{C}^n \mid P(\zeta) = 0 \}) : \quad \xi \in \mathbb{C}^n;
\]
\[
\tilde{P}(\xi, t) := \sum_\alpha t^{\vert \alpha \vert} \vert P^{(\alpha)}(\xi) \vert : \quad \xi \in \mathbb{C}^n, \quad t > 0,
\]
where $\vert \alpha \vert := \sum_{j=1}^n \alpha_j$ and $P^{(\alpha)} := \partial^\alpha P$;
\[
\tilde{P}(\xi) := \tilde{P}(\xi, 1);
\]
$P \prec Q : \iff \sup \{ \tilde{P}(\xi)/\tilde{Q}(\xi) \mid \xi \in \mathbb{R}^n \} < \infty$;
\[
P \sim W : \iff P < Q \land Q < P;
\]
\[
W(Q) := \{ P \in \mathbb{C}[x_1, \ldots, x_n] \mid P < Q \};
\]
\[
E(Q) := \{ P \in \mathbb{C}[x_1, \ldots, x_n] \mid P \sim Q \}.
\]

1.1. Remarks.
(i) Note that our definition of $\tilde{P}(\xi, t)$ differs from that of Hörmander [H2], §10.4, who used the notation $\tilde{P}(\xi, t) := \left( \sum_\alpha t^{2\vert \alpha \vert} \vert P^{(\alpha)}(\xi) \vert^2 \right)^{1/2}$. 


According to [H2], 10.4.3 we have
\[ P < Q \iff \sup\{ \tilde{P}(\xi, t)/\tilde{Q}(\xi, t) \mid \xi \in \mathbb{R}^n, \ t \geq 1 \} < \infty. \]
In this case we say that \( P \) is weaker than \( Q \). If \( P \sim Q \) then we say that \( P \) and \( Q \) are equally strong.

(ii) \( P < Q \implies \deg P \leq \deg Q \). This is clear by definition of \( \tilde{P} \). In particular, \( W(Q) \) is a finite-dimensional complex vector space (consequence of [H2], 10.4.1).

(iii) \( E(Q) \) is a linearly convex, open subset of \( W(Q) \) ([H2], 10.4.7). For our purposes it suffices to know that \( E(Q) \) is holomorphically convex (cf. [M]).

We assume the integers \( n, m \) to be fixed throughout this paper. The letters \( c, C \) denote positive constants which only depend on \( n \) and \( m \). We use the notations
\[
|\xi| := \sum |\xi_j|, \quad |\xi|_\infty := \max |\xi_j|: \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n.
\]
For \( K = \mathbb{R}, \mathbb{C} \) and \( \rho \geq 0 \) let
\[
\mathcal{B}_{K^n}(\rho) := \{ \xi \in \mathbb{K}^n \mid |\xi|_\infty \leq \rho \}.
\]
In the case \( \rho = 1 \) we simply write \( \mathcal{B}_{K^n} \). Further let
\[
T^r := \{ z \in \mathbb{C}^r \mid |z_1| = \cdots = |z_r| = 1 \} \text{ if } r \in \mathbb{N}.
\]

1.2. Theorem. — Let \( Q \in \text{Pol}'(n,m), \Pi \subseteq E(Q) \) be a compact set and \( \rho \geq 0 \). Then there exists \( A \geq 1 \) and a bounded measurable function \( \eta: \mathbb{R}^n \to \mathbb{R}^n \) such that
\[
(1.1) \quad \tilde{P}(\xi, t) \leq A|P(\xi + \zeta + z\eta(\xi))|: P \in \Pi, \xi, \zeta \in \mathbb{R}^n, \zeta \in \mathcal{B}_{\mathbb{C}^n}(\rho), z \in T^1.
\]

Our proof of this theorem is long and will occupy the rest of this section. First it requires a detailed study of the function \( \tilde{P}(\xi, t) \):

1.3. Lemma. — Let \( Q \in \text{Pol}'(n,m) \) and \( \Pi \subseteq E(Q) \) be compact. Then there exists \( B \geq 1 \) such that
\[
(1.2) \quad B^{-1} \leq \tilde{P}(\xi, t)/\tilde{Q}(\xi, t) \leq B: P \in \Pi, \xi \in \mathbb{R}^n, \ t \geq 1.
\]

Proof. — By 1.1 (i) the expression \( N_Q(P) := \sup\{ \tilde{P}(\xi, t)/\tilde{Q}(\xi, t) \mid \xi \in \mathbb{R}^n, \ t \geq 1 \} \) defines a norm on \( W(Q) \). Now let \( R \in \Pi \) be fixed. Since \( Q < R \) we have
\[
b_R := \inf\{ \tilde{R}(\xi, t)/\tilde{Q}(\xi, t) \mid \xi \in \mathbb{R}^n, \ t \geq 1 \} > 0.
\]
For any $P \in \omega_R := \{ P \in W(Q) \mid N_Q(R - P) < b_R/2 \}$ we get

\[
\frac{\tilde{P}(\xi, t)}{Q(\xi, t)} \geq \frac{\tilde{R}(\xi, t) - (R - P)^\sim(\xi, t)}{Q(\xi, \tau)} > b_R/2 : \quad \xi \in \mathbb{R}^n, \ t \geq 1.
\]

Since $\omega_R$ is an open neighborhood of $R$ it follows from the compactness of $\Pi$ that there exists $b_0 > 0$ with

\[
\tilde{P}(\xi, t) \geq b_0 \tilde{Q}(\xi, t) : \quad P \in \Pi, \ \xi \in \mathbb{R}^n, \ t \geq 1.
\]

On the other hand the boundedness of $\Pi$ implies that

\[
B_0 := \sup\{ N_Q(P) \mid P \in \Pi \} < \infty,
\]

hence

\[
\tilde{P}(\xi, t) \leq B_0 \tilde{Q}(\xi, t) : \quad P \in \Pi, \ \xi \in \mathbb{R}^n, \ t \geq 1.
\]

With $B := \max\{1/b_0, B_0\}$ the assertion follows .

\[\square\]

1.4. **Lemma** (cf. [H2], 11.1.4). — There exists $C \geq 1$ such that for any $P \in Pol'(n, m)$ the following holds :

\[
|P^{(\alpha)}(\xi)| \delta_P(\xi)^{\alpha} \leq C|P(\xi)| : \quad \xi \in \mathbb{C}^n, \ |\alpha| \leq m .
\]

(1.4) $C^{-1} \leq \delta_P(\xi) \sum_{\alpha \neq 0} |P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|} \leq C : \quad \xi \in \mathbb{C}^n, \ P(\xi) \neq 0 .

(1.5) $|P(\xi)| \leq \tilde{P}(\xi, \delta_P(\xi)) \leq C|P(\xi)| : \quad \xi \in \mathbb{C}^n .

\[\text{Proof. — (1.4) is due to Hörmander [H2], 11.1.4. (1.5) is a consequence of (1.3) which follows from (1.4).} \square\]

1.5. **Lemma** (cf. [H2], 11.1.9). — There exists $c > 0$ such that for any $P, Q \in Pol'(n, m)$ and $\xi \in \mathbb{C}^n$ we have : if

\[
B^{-1} \leq \frac{\tilde{P}(\xi, t)/\tilde{Q}(\xi, t)}{B} : \quad t \geq 1
\]

holds with some $B \geq 1$ then

\[
\frac{c}{1 + B^2} \leq \frac{1 + \delta_P(\xi)}{1 + \delta_Q(\xi)} \leq \frac{1 + B^2}{c} .
\]

\[\text{Proof. — If } \delta_Q(\xi) \geq 1 \text{ then}

\[
\sum_{\alpha} |P^{(\alpha)}(\xi)| \delta_Q(\xi)^{\alpha} \leq B \sum_{\alpha} |Q^{(\alpha)}(\xi)| \delta_Q(\xi)^{\alpha} \leq C_1 B |Q(\xi)| \leq C_1 B^2 \sum_{\alpha} |P^{(\alpha)}(\xi)| .
\]

(1.5) (1.6) (1.7)
When \( \delta_Q(\xi) \geq 2C_1B^2 =: D \) (hence \( \frac{1}{2}\delta_Q(\xi)^{|\alpha|} \leq \delta_Q(\xi)^{|\alpha|} - \frac{D}{2}, \alpha \neq 0 \)) this yields
\[
\sum_{\alpha} |P^{(\alpha)}(\xi)|\delta_Q(\xi)^{|\alpha|} \leq D|P(\xi)|.
\]
In particular then \( P(\xi) \neq 0 \) and
\[
|P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|}\delta_P(\xi) \leq D\delta_P(\xi)/\delta_Q(\xi) : \alpha \neq 0.
\]
Summing up we get
\[
C_2B^2\delta_P(\xi)/\delta_Q(\xi) \geq \delta_P(\xi) \sum_{\alpha \neq 0} |P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|} \geq C_3^{-1},
\]
\[
1 + \frac{\delta_P(\xi)}{\delta_Q(\xi)} \geq \frac{1}{2} \frac{\delta_P(\xi)}{\delta_Q(\xi)} \geq (2C_2C_3B^2)^{-1} \text{ if } \delta_Q(\xi) \geq D.
\]
In the case \( \delta_Q(\xi) \leq D \) we have
\[
1 + \frac{\delta_P(\xi)}{\delta_Q(\xi)} \geq \frac{1}{1 + 2C_1B^2}.
\]
With suitable \( c > 0 \) we obtain the lefthand side of (1.7). The second inequality follows from this one by interchanging the roles of \( P \) and \( Q \).

1.6. **Lemma** (cf. [H2], 10.4.2). — There exists \( C \geq 1 \) such that for any \( P \in \text{Pol}(n, m), \xi \in \mathbb{C}^n \) and \( \tau > 0 \):
\[
C^{-1} \tilde{P}(\xi, \tau) \leq \max\{|P(\xi + \eta)| \mid \eta \in \mathbb{B}_{K'}(\tau)\} \leq C\tilde{P}(\xi, \tau);
\]
\[
C^{-1}\tau \leq \max\{|\delta_P(\xi + \eta)| \mid \eta \in \mathbb{B}_{K'}(\tau)\} \text{ if } P \text{ is nonconstant}.
\]
This holds for \( K = \mathbb{R} \) and \( K = \mathbb{C} \).

**Proof.** Assertion (1.8) corresponds to [H2], 10.4.2. (Our use of the \( \ell_1 \)-norm in the definition of \( \tilde{P}(\xi, t) \) only results in a change of the constants.)

Ad (1.9) : first we note that for \( \tau > 0 \) and \( \eta \in \mathbb{B}_{K'}(\tau) \),
\[
|P^{(\alpha)}(\xi + \eta)| \leq \sum_{\beta} |P^{(\alpha+\beta)}(\xi)||\tau|^{\beta} \leq \tau^{-|\alpha|} \tilde{P}(\xi, \tau)
\]
by Taylor’s formula. As a consequence we have the estimate
\[
\tilde{P}(\xi + \eta, \tau) \leq C_1\tilde{P}(\xi, \tau) : P \in \text{Pol}(n, m), \xi \in \mathbb{C}^n, \eta \in \mathbb{B}_{K'}(\tau),
\]
which will be used later. By (1.8) there exists for fixed \( \xi \in \mathbb{C}^n \) and \( \tau > 0 \) an \( \eta \in \mathbb{B}_{K'}(\tau) \) such that
\[
\tilde{P}(\xi, \tau) \leq C_2|P(\xi + \eta)|.
\]
In particular then \( P(\xi + \eta) \neq 0 \) and
\[
\sum_{\alpha \neq 0} |P^{(\alpha)}(\xi + \eta)/P(\xi + \eta)|^{1/|\alpha|} \leq \sum_{1 \leq |\alpha| \leq m} (C_2 \tau^{-|\alpha|})^{1/|\alpha|} \leq C_3 \tau^{-1} .
\]
From (1.4) it follows that \( \delta_P(\xi + \eta) \geq C_4^{-1} \tau \), hence the assertion. \( \square \)

Now we can already prove a preliminary version of Theorem 1.2:

1.7. COROLLARY. — Let \( Q \in \text{Pol}'(n,m) \) and \( \Pi \subseteq E(Q) \) compact. Then there exist \( A, \mu \geq 1 \) such that
\[
(1.11) \quad \forall \tau \geq \mu, \; \xi \in \mathbb{R}^n \; \exists \eta \in B_{\mathbb{R}^n}(\tau) \; \forall P \in \Pi : \tilde{P}(\xi, \tau) \leq A|P(\xi + \eta)| .
\]

Proof. — By Lemma 1.3 there exists \( B \geq 1 \) such that
\[
B^{-1} \leq \tilde{P}(\xi, t)/\tilde{Q}(\xi, t) \leq B : \quad P \in \Pi, \; \xi \in \mathbb{R}^n, \; t \geq 1 .
\]
With \( A_1 := (1 + B^2)/c \geq 1 \) we get from (1.7),
\[
A_1^{-1}(1 + \delta_Q(\xi)) \leq 1 + \delta_P(\xi) : \quad P \in \Pi, \; \xi \in \mathbb{R}^n .
\]
By (1.9) we have
\[
(1.12) \quad \max\{\delta_Q C\xi + \eta) \mid \eta \in B_{\mathbb{R}^n}(\tau)\} \geq C_0^{-1} \tau : \quad \xi \in \mathbb{R}^n, \; \tau > 0 .
\]
Choose \( A_2 \geq 1 \) with \( C_0^{-1} - A_1/A_2 > 0 \) and put
\[
\mu := \max\{1, (A_1 - 1)/(C_0^{-1} - A_1/A_2)\} .
\]
If \( \tau \geq \mu \) then \((1 + C_0^{-1})/A_1 \geq 1 + \tau/A_2 \). For such a \( \tau \) and arbitrary \( \xi \in \mathbb{R}^n \)
we may now choose \( \eta \in B_{\mathbb{R}^n}(\tau) \) with \( \delta_Q(\xi + \eta) \geq C_0^{-1} \tau \) according to (1.12).
For any \( P \in \Pi \) we then obtain
\[
1 + \delta_P(\xi + \eta) \geq A_1^{-1}(1 + \delta_Q(\xi + \eta)) \geq A_1^{-1}(1 + C_0^{-1}) \geq 1 + \tau/A_2 ,
\]
i.e. \( \tau \leq A_2 \delta_P(\xi + \eta) \). Because of (1.5) this yields
\[
\tilde{P}(\xi + \eta, \tau) \leq \tilde{P}(\xi + \eta, A_2 \delta_P(\xi + \eta)) \leq A_2^n \tilde{P}(\xi + \eta, \delta_P(\xi + \eta)) \leq A_3|P(\xi + \eta)| .
\]
Finally, replacing in (1.10) \( \eta \) by \(-\eta\) and \( \xi \) by \( \xi + \eta \), we obtain
\[
\tilde{P}(\xi, \tau) \leq C_1 \tilde{P}(\xi + \eta, \tau) \leq C_1 A_3|P(\xi + \eta)| : \quad P \in \Pi . \quad \square
\]

For any \( R \in \mathbb{C}[x_1, \ldots, x_n] \) and \( k \in \mathbb{N}_0 \) we put
\[
(\Phi_k R)(\xi) := \sum_{|\alpha| = k} R^{(\alpha)}(\xi)R^{(\alpha)}(\xi) ,
\]
where $\bar{R}$ is obtained from $R$ by taking complex conjugates of the coefficients. Note that $\Phi_k R \in \mathbb{R}[x_1, \ldots, x_n]$ and $(\Phi_k R)(\xi) \geq 0$ for $\xi \in \mathbb{R}^n$. With the notation

$$(\Psi_k R)(\xi) := \sum_{|\alpha| = k} |R^{(\alpha)}(\xi)|$$

we have

$$\tilde{R}(\xi, t) = \sum_{k=0}^{m} t^k (\Psi_k R)(\xi) : \quad R \in \text{Pol}(n, m).$$

1.8. LEMMA. — There exists $C \geq 1$ such that for any $P \in \text{Pol}(n, m)$, $k \in \mathbb{N}_0$, $\xi \in \mathbb{R}^n$ and $t > 0$:

$$C^{-1}(\Phi_k P)^\sim(\xi, t) \leq \left( \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi) \right)^2 \leq C(\Phi_k P)^\sim(\xi, t).$$

Proof. — First we have by (1.8) (note that $\Phi_k P \in \text{Pol}(n, 2m)$),

$$C^{-1}(\Phi_k P)^\sim(\xi, t) \leq \max_{\eta \in \mathbb{B}_R^n} (\Phi_k P)(\xi + t\eta) \leq C_1(\Phi_k P)^\sim(\xi, t)$$

and

$$C^{-1}_1 \sum_{|\alpha| = k} (P^{(\alpha)})^\sim(\xi, t) \leq \sum_{|\alpha| = k} \max_{\eta \in \mathbb{B}_R^n} |P^{(\alpha)}(\xi + t\eta)| \leq C_1 \sum_{|\alpha| = k} (P^{(\alpha)})^\sim(\xi, t).$$

Furthermore an easy calculation shows that

$$C_2^{-1} \sum_{|\alpha| = k} (P^{(\alpha)})^\sim(\xi, t) \leq \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi) \leq C_2 \sum_{|\alpha| = k} (P^{(\alpha)})^\sim(\xi, t),$$

hence

$$C_3^{-1} \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi) \leq \sum_{|\alpha| = k} \max_{\eta \in \mathbb{B}_R^n} |P^{(\alpha)}(\xi + t\eta)|$$

$$\leq C_3 \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi).$$

Now let $M(n, k) = \{\alpha \in \mathbb{N}_0^n | |\alpha| = k\}$. Obviously the expressions

$$N_1((R_\alpha)_{\alpha \in M(n, k)}) := \left( \max_{\eta \in \mathbb{B}_R^n} \sum_{|\alpha| = k} R_\alpha(\eta)\bar{R}_\alpha(\eta) \right)^{1/2},$$

$$N_2((R_\alpha)_{\alpha \in M(n, k)}) := \sum_{|\alpha| = k} \max_{\eta \in \mathbb{B}_R^n} |R_\alpha(\eta)|$$
define norms on the finite-dimensional vector space $\text{Pol}(n, m)$, hence they are equivalent. On replacing $R_{\alpha}(\eta)$ by $P^{(\alpha)}(\xi + \eta)$ we get
\[
C_{4}^{-1} \sum_{|\alpha|=k} \max_{\eta \in \mathbb{R}^{n}} |P^{(\alpha)}(\xi + \eta)| \leq \left( \max_{\eta \in \mathbb{R}^{n}} (\Phi_{k} P)(\xi + \eta) \right)^{1/2} \\
\leq C_{4} \sum_{|\alpha|=k} \max_{\eta \in \mathbb{R}^{n}} |P^{(\alpha)}(\xi + \eta)| .
\]

With (1.14) and (1.15) we obtain the assertion. \(\square\)

1.9. **LEMMA.** — There exist $0 < c < 1 \leq C$ such that for any $P, Q \in \text{Pol}'(n, m)$ and $\xi \in \mathbb{R}^{n}$ the following holds: let $0 \leq k \leq m - 1$ and $B \geq 1$ with
\[
(1.16) \quad B^{-1} \leq \left( \sum_{j=k}^{m} \nu^{j-k}(\Psi_{j} P)(\xi) / \left( \sum_{j=k}^{m} \nu^{j-k}(\Psi_{j} Q)(\xi) \right) \right) \leq B : \quad t \geq 1.
\]
Further let $\nu \geq 1$ such that $\tilde{\nu} := \left( \frac{c \nu}{1 + B^4} - 1 \right) / C \geq 1$. Then we have with $\tilde{\nu} := C(1 + \nu)(1 + B^4)$:

(i) $(\Psi_{k} Q)(\xi) \geq \sum_{j=k+1}^{m} \nu^{j-k}(\Psi_{j} Q)(\xi) \Rightarrow (\Psi_{k} P)(\xi) \geq \sum_{j=k+1}^{m} \nu^{j-k}(\Psi_{j} P)(\xi),$

(ii) $(\Psi_{k} Q)(\xi) \leq \sum_{j=k+1}^{m} \nu^{j-k}(\Psi_{j} Q)(\xi) \Rightarrow (\Psi_{k} P)(\xi) \leq \sum_{j=k+1}^{m} \nu^{j-k}(\Psi_{j} P)(\xi) .$

**Proof.**

(i) Let $\nu \geq 1$ with $(\Psi_{k} Q)(\xi) \geq \sum_{j=k+1}^{m} \nu^{j-k}(\Psi_{j} Q)(\xi)$. Then we have
\[
|Q^{(\alpha)}(\xi)| \leq \nu^{-|\alpha|-k}(\Psi_{k} Q)(\xi) \leq C_{1} \nu^{-|\alpha|-k} \sqrt{(\Phi_{k} Q)(\xi)} : \quad |\alpha| \geq k .
\]
This implies by Leibniz’ rule,
\[
|(\Phi_{k} Q)^{\beta}(\xi)| = \mid \sum_{|\alpha|=k} \sum_{\gamma \leq \beta} \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) Q^{(\alpha+\gamma)}(\xi) \sqrt{Q^{(\alpha+\beta-\gamma)}(\xi)} \mid \\
\leq C_{2} \nu^{-|\beta|}(\Phi_{k} Q)(\xi)
\]
for any multiindex $\beta$ $(C_{2} \geq 1)$. In particular then $(\Phi_{k} Q)(\xi) \neq 0$ and
\[
|(\Phi_{k} Q)^{\beta}(\xi)/(\Phi_{k} Q)(\xi)|^{1/|\beta|} \leq C_{2} \nu^{-1} : \quad \beta \neq 0 .
\]
An application of (1.4) yields
\[
C_{3}^{-1} \leq \delta_{\Phi_{k} Q}(\xi) \sum_{\beta \neq 0} |(\Phi_{k} Q)^{\beta}(\xi)/(\Phi_{k} Q)(\xi)|^{1/|\beta|} \leq C_{4} \nu^{-1} \delta_{\Phi_{k} Q}(\xi) .
\]
By (1.13) and (1.16) we also have
\[(C_5B^2)^{-1} \leq (\Phi_kP)^\sim(\xi, t)/(\Phi_kQ)^\sim(\xi, t) \leq C_5B^2: \quad t \geq 1.\]

Using (1.7) we obtain
\[
\frac{1 + \delta_{\Phi_kP}(\xi)}{1 + C_3^{-1}C_4^{-1} \nu} \geq \frac{1 + \delta_{\Phi_kP}(\xi)}{1 + \delta_{\Phi_kQ}(\xi)} \geq \frac{c_1}{1 + C_5^2 B^4},
\]

\[
\delta_{\Phi_kP}(\xi) \geq \frac{c_1(1 + C_3^{-1}C_4^{-1} \nu)}{1 + C_5^2 B^4} - 1 \geq \frac{c_2 \nu}{1 + B^4} - 1 =: \tilde{\nu}
\]

with \(0 < c_2 \leq 1\). Let \(\nu\) be so large that \(\tilde{\nu} \geq 1\). Then
\[
(\Phi_kP)(\xi) \geq C_6^{-1}(\Phi_kP)^\sim(\xi, \delta_{\Phi_kP}(\xi)) \geq C_6^{-1}(\Phi_kP)^\sim(\xi, \tilde{\nu}) \geq C_7^{-1}\left(\sum_{j=k}^{m} \tilde{\nu}^{-k}(\Psi_jP)(\xi)\right)^2
\]

with \(C_7 \geq 1\), hence
\[
(\Psi_kP)(\xi) \geq \sqrt{(\Phi_kP)(\xi)} \geq C_7^{-1/2} \sum_{j=k}^{m} \tilde{\nu}^{-k}(\Psi_jP)(\xi)
\]

\[
\geq \sum_{j=k+1}^{m} \left(\tilde{\nu}/C_7\right)^{-k}(\Psi_jP)(\xi).
\]

With \(c := c_2, C \geq C_7\) we obtain the first assertion.

(ii) Now assume that \((\Psi_kQ)(\xi) \leq \sum_{j=k+1}^{m} \nu^{-k}(\Psi_jQ)(\xi)\). If then
\[
(\Psi_kP)(\xi) \geq \sum_{j=k+1}^{m} \mu^{-k}(\Psi_jP)(\xi) \text{ and } \tilde{\mu} := \frac{c_2 \mu}{1 + B^4} - 1 \geq 1
\]

with some \(\mu \geq 1\) we obtain as above (on interchanging the roles of \(P\) and \(Q\)):
\[
(\Psi_kQ)(\xi) \geq \sum_{j=k+1}^{m} (\tilde{\mu}/C_7)^{-k}(\Psi_jQ)(\xi), \text{ hence}
\]

\[
\sum_{j=k+1}^{m} (\tilde{\mu}/C_7)^{-k}(\Psi_jQ)(\xi) \leq \sum_{j=k+1}^{m} \nu^{-k}(\Psi_jQ)(\xi).
\]

This implies \(\tilde{\mu}/C_7 \leq \nu\), i.e.
\[
\mu \leq (1 + C_7 \nu)(1 + B^4)/c_2 \leq C_7(1 + \nu)(1 + B^4)/c_2.
\]

Thus, with \(C := C_7/c_2\) the second assertion also holds. \(\square\)
Proof of Theorem 1.2. — The subsequent procedure will yield a decomposition of \( \Omega_0 := \mathbb{R}^n \) into \( m+1 \) disjoint subsets, \( \Omega_0 = \Omega_0' \cup \Omega_1' \cup \cdots \cup \Omega_m' \), such that the following holds:

\[ \exists A \geq 1 \forall k = 0, \ldots, m \exists \tau_k \geq 1 \forall \xi \in \Omega_k' \exists \eta_k \in B_{\mathbb{R}^n} (\tau_k) : \]

\[ |P(\xi + z \eta_k)| \geq \frac{1}{2A} \tilde{p}(\xi, \tau_k) : P \in \Pi, \ z \in T^1. \quad (1^k) \]

Now note that the set

\[ \Pi_{\rho} := \{ P(\cdot + \zeta) \mid P \in \Pi, \ \zeta \in B_{\mathbb{C}^+} (\rho + 1) \} \]

is a compact subset of \( E(Q) \) since for fixed \( \zeta \) the polynomial \( P(\cdot + \zeta) \) is equally strong as \( P \). So we may assume that \( (1^0), \ldots, (1^m) \) is already proved for \( \Pi_{\rho} \) instead of \( \Pi \). It follows that for any \( \vartheta \in \mathbb{Z}^n \) there exists \( \eta_{\vartheta} \in B_{\mathbb{R}^n} (\tau) \), where \( \tau := \max \{ \tau_0, \ldots, \tau_m \} \), such that if \( |\xi - \vartheta|_{\infty} \leq 1 \) we have for each \( P \in \Pi, \ \zeta \in B_{\mathbb{C}^+} (\rho) \) and \( z \in T^1 \):

\[ |P(\xi + \zeta + z \eta_{\vartheta})| = |P(\vartheta + z \eta_{\vartheta} + (\xi - \vartheta + \zeta))| \geq \frac{1}{2A} \tilde{p}(\vartheta) \geq \frac{1}{2CA} \tilde{p}(\xi). \]

In particular we may choose \( \eta(\xi) \equiv \eta_{\vartheta} \) in any cube \( \{ \xi \mid \vartheta_j \leq \xi_j < \vartheta_j + 1 \} \), where \( \vartheta_1, \ldots, \vartheta_n \) are integers, such that (1.1) holds and \( |\eta(\xi)|_\infty \leq \tau \).

This completes the proof. The sets \( \Omega'_k \) will be defined inductively as follows:

\[ \Omega'_k := \{ \xi \in \Omega_k \mid (\Psi_k Q)(\xi) \geq \sum_{j=k+1}^{m} \nu_k^{j-k}(\Psi_j Q)(\xi) \} \quad (0 \leq k \leq m - 1) \]

with suitable constants \( \nu_k \geq 1 \), and

\[ \Omega_{k+1} := \Omega_k \setminus \Omega'_k ; \ \Omega'_m := \Omega_m. \]

In what follows the statements \( (2^k) \) \( (0 \leq k \leq m) \) will be needed:

\[ \exists B_k \geq 1 \forall P \in \Pi, \ \xi \in \Omega_k, \ t \geq 1 : \]

\[ B_k^{-1} \leq \left( \sum_{j=k}^{m} \psi_{j-k}(\Psi_j P)(\xi) \right) / \left( \sum_{j=k}^{m} \psi_{j-k}(\Psi_j Q)(\xi) \right) \leq B_k. \quad (2^k) \]

With the constants \( c, C \) in Lemma 1.9 we set

\[ \hat{\psi}_k := \left( \frac{c \nu_k}{1 + B_k^4} - 1 \right) / C \quad \text{and} \quad \hat{\nu}_k := C(1 + \nu_k)(1 + B_k^4). \]

Then for each \( 0 \leq k \leq m - 1 \) we have by \( (2^k) \) and Lemma 1.9, if \( \hat{\nu}_k \geq 1, \)

\[ (3^k) \quad (\Psi_k P)(\xi) \geq \sum_{j=k+1}^{m} \hat{\psi}_k^{j-k}(\Psi_j P)(\xi) : P \in \Pi, \ \xi \in \Omega'_k. \]

In particular we may choose \( \eta(\xi) \equiv \eta_{\vartheta} \) in any cube \( \{ \xi \mid \vartheta_j \leq \xi_j < \vartheta_j + 1 \} \), where \( \vartheta_1, \ldots, \vartheta_n \) are integers, such that (1.1) holds and \( |\eta(\xi)|_\infty \leq \tau \).
Now the proof of \((1^k), (2^k)\) proceeds by induction on \(k\). Recall that by Corollary 1.7 there exist \(A, \mu \geq 1\) such that

\[
\forall \tau \geq \mu, \xi \in \mathbb{R}^n \exists \eta \in \mathcal{B}_{\mathbb{R}_n}(\tau) \forall P \in \Pi : \hat{P}(\xi, \tau) \leq A|P(\xi + \eta)|.
\]

Without loss of generality we may assume that \(Q \in \Pi\).

**Case \(k = 0\).** — Lemma 1.3 yields the existence of \(B_0\) satisfying (2°). Choose \(\nu_0 \geq 1\) such that \(\nu_0 \geq 1\) and define \(\Omega_0, \Omega_1\) as above. Let \(\tau_0 := \nu_0\) and for any \(\xi \in \Omega_0\) choose \(\eta_0 := 0 \in \mathcal{B}_{\mathbb{R}_n}(\tau_0)\). We obtain

\[
2|P(\xi + z\eta_0)| = 2(\Psi_0 P)(\xi) \geq \sum_{j=0}^{m} \hat{\nu}_0^j(\Psi_j P)(\xi) = \hat{P}(\xi, \tau_0)
\]

for \(P \in \Pi, z \in \mathbb{T}^1\), i.e. (1°) is satisfied.

**Case \(1 \leq k \leq m\).** — The inductive assumption yields \((2^{k-1})\) and \((4^0), \ldots, (4^{k-1})\). Since \(\Omega_k \subseteq \Omega_{k-1}\) this implies for \(\xi \in \Omega_k, t \geq \nu_{k-1}\):

\[
(2B_{k-1})^{-1} \sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi) \leq (2B_{k-1})^{-1} \frac{1}{t} \sum_{j=k}^{m} t^{j-(k-1)}(\Psi_j Q)(\xi)
\]

\[
\leq \frac{1}{2t} \sum_{j=k}^{m} t^{j-(k-1)}(\Psi_j P)(\xi)
\]

\[
\leq \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi).
\]

For \(1 \leq t \leq \nu_{k-1}\) this yields

\[
(2B_{k-1})^{-1} \sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi) \leq \sum_{j=k}^{m} \hat{\nu}_{k-1}^{j-k}(\Psi_j P)(\xi)
\]

\[
\leq \hat{\nu}_{k-1}^{m-k} \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi).
\]

Analogous estimates hold with \(P\) and \(Q\) interchanged. Setting \(B_k := 2B_{k-1} \hat{\nu}_{k-1}^{m-k}\) we obtain \((2^k)\). Now let

\[
\mu_k := \max\{\mu, \hat{\nu}_0, \ldots, \hat{\nu}_{k-1}\} \geq 1.
\]

For \(P \in \Pi, \xi \in \Omega_{j+1}\) \((j = 0, \ldots, k-1), \tau \geq \mu_k\) it follows from \((4^j)\):

\[
(\Psi_j P)(\xi) \leq \sum_{i=j+1}^{m} \left(\frac{\mu_k}{\tau}\right)^{i-j} t^{i-j}(\Psi_j P)(\xi) \leq \frac{\mu_k}{\tau} \sum_{i=j+1}^{m} t^{i-j}(\Psi_j P)(\xi).
\]
Multiplying by $\tau^j$ and summing up this yields (note that $\Omega_k \subseteq \Omega_{j+1}$):

\begin{equation}
\sum_{j=0}^{k-1} \tau^j(\Psi_j P)(\xi) \leq \frac{k\mu_k}{\tau} \tilde{P}(\xi, \tau) : \quad P \in \Pi, \xi \in \Omega_k, \tau \geq \mu_k .
\end{equation}

In the case $k \leq m - 1$ we choose $\tau_k, \nu_k \geq 1$ such that

\begin{equation}
\mu_k \leq \tau_k \leq \nu_k , \quad A^{-1} - \frac{2k\mu_k}{\tau_k} - \frac{2\tau_k}{\nu_k} \geq \frac{1}{2A} .
\end{equation}

and define $\Omega'_k, \Omega_{k+1}$ as above. By $\left(3^k\right)$ (consequence of $\left(2^k\right)$) we have

\begin{equation}
\sum_{j=0}^{m} \tau^j_k(\Psi_j P)(\xi) \leq \frac{\tau_k}{\nu_k} \tau^k(\Psi_k P)(\xi) \leq \frac{\tau_k}{\nu_k} \tilde{P}(\xi, \tau_k) : \quad P \in \Pi, \xi \in \Omega'_k .
\end{equation}

Now let $\xi \in \Omega'_k$ be fixed and choose $\eta_k \in B_{R^m}(\tau_k)$ such that

\begin{equation}
\tilde{P}(\xi, \tau_k) \leq A|P(\xi + \eta_k)| : \quad P \in \Pi \quad (\text{cf.}\ (5)) .
\end{equation}

An application of Taylor’s formula gives for $P \in \Pi, z \in T^1$:

\begin{equation}
|P(\xi + z\eta_k)| \geq \left| \sum_{|\alpha|=k} \frac{P^{(\alpha)}(\xi)}{\alpha!} \eta_{\alpha}^2 \right| - \sum_{j \neq k} \tau^j(\Psi_j P)(\xi) \geq \sum_{j=0}^{m} \left| \sum_{|\alpha|=j} \frac{P^{(\alpha)}(\xi)}{\alpha!} \eta_{\alpha}^2 \right| - 2 \sum_{j \neq k} \tau^j(\Psi_j P)(\xi)
\end{equation}

\begin{equation}
\geq \left\{ A^{-1} - \frac{2k\mu_k}{\tau_k} - \frac{2\tau_k}{\nu_k} \right\} \tilde{P}(\xi, \tau_k)
\geq \frac{1}{2A} \tilde{P}(\xi, \tau_k) .
\end{equation}

This yields $\left(1^k\right)$.

In the case $k = m$ we choose $\tau_m \geq 1$ such that

\begin{equation}
A^{-1} - \frac{2m\mu_m}{\tau_m} \geq \frac{1}{2A} .
\end{equation}

Let $\xi \in \Omega'_m := \Omega_m$ be fixed and choose $\eta_k \in B_{R^m}(\tau_m)$ such that

\begin{equation}
\tilde{P}(\xi, \tau_m) \leq A|P(\xi + \eta_k)| : \quad P \in \Pi \quad (\text{cf.}\ (5)) .
\end{equation}

Using (6), (10) and (11) an analogous computation as above yields $\left(1^m\right)$:

\begin{equation}
|P(\xi + z\eta_k)| \geq \left\{ A^{-1} - \frac{2m\mu_m}{\tau_m} \right\} \tilde{P}(\xi, \tau_m) \geq \frac{1}{2A} \tilde{P}(\xi, \tau_m) : \quad P \in \Pi, z \in T^1 .
\end{equation}
2. Some distribution spaces.

We adopt the standard notations for spaces of test functions and distributions (cf. [H1], [H2]) :

\[ \mathcal{D} = C_c^\infty(\mathbb{R}^n) = C^\infty \text{-functions with compact support;} \]
\[ \mathcal{D}' = \mathcal{D}'(\mathbb{R}^n) = \text{space of all distributions;} \]
\[ \mathcal{S} = \mathcal{S}(\mathbb{R}^n) = \text{space of rapidly decreasing } C^\infty \text{-functions;} \]
\[ \mathcal{S}' = \mathcal{S}'(\mathbb{R}^n) = \text{space of tempered distributions.} \]

Recall that each of these spaces carries a natural locally convex vector space topology. The scalar product of two vectors \( \langle C, D \rangle \) will be denoted by

\[ \langle \xi, \zeta \rangle := \sum_{\nu=1}^n \xi_\nu \zeta_\nu. \]
If \( \varphi \in \mathcal{S} \) then the Fourier transform \( \hat{\varphi} \) of \( \varphi \) is the function

\[ \hat{\varphi}(\zeta) := \int_{\mathbb{R}^n} \exp(-i[\zeta, \cdot]) \varphi(x) dx : \zeta \in \mathbb{R}^n. \]

The Fourier transform \( \hat{u} \) of \( u \in \mathcal{S}' \) is defined by the formula

\[ \langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle : \varphi \in \mathcal{S}, \]
where \( \langle \cdot, \cdot \rangle \) denotes the distribution pairing. The following definitions and results are taken from Hörmander [H2], §10.1.

2.1. Definition.

(a) A function \( k : \mathbb{R}^n \to (0, \infty) \) will be called a temperate weight function if there exist constants \( a, b > 0 \) such that

\[ k(\xi + \zeta) \leq (1 + a|\xi|)^b k(\zeta) : \xi, \zeta \in \mathbb{R}^n. \]

The set of all such functions will be denoted by \( \mathcal{K} \).

(b) If \( k \in \mathcal{K} \) and \( 1 \leq p \leq \infty \) we denote by \( \mathcal{B}_{p,k} \) the set of all distributions \( u \in \mathcal{S}' \) such that \( \hat{u} \) is a function and

\[ \|u\|_{p,k} := \left( (2\pi)^{-n} \int_{\mathbb{R}^n} |k(\xi)u(\xi)|^p d\xi \right)^{1/p} < \infty. \]

In the case \( p = \infty \) this expression has to be interpreted as \( \text{ess.sup } |k(\xi)u(\xi)| \).

By [H2], 10.1.7 we have

\[ \mathfrak{S} \hookrightarrow \mathcal{B}_{p,k} \hookrightarrow \mathcal{S}', \]

where \( \mathfrak{S} \hookrightarrow \mathcal{S} \) means a continuous embedding of topological vector spaces \( \mathfrak{S}, \mathcal{S} \). The spaces \( \mathcal{B}_{p,k} \) are Banach spaces which for \( 1 \leq p < \infty \) contain \( \mathcal{D} \).
as a dense subset. In this case the dual $(B_{p,k})'$ of $B_{p,k}$ is (isometrically) isomorphic to $B_{p',k'}$, where
\[ 1/p + 1/p' = 1 \quad k'(\xi) := 1/k(-\xi). \]
Any continuous linear form on $B_{p,k}$ is given by continuous extension of a form $\varphi \mapsto \langle v, \varphi \rangle$, defined for $\varphi \in \mathcal{D}$ with $v \in B_{p',k'}$. The norm of this functional equals $\|v\|_{p',k'}$ ([H2], 10.1.14). Let
\[ B_{p,k}^{loc} := \{u \in \mathcal{D}' \mid \psi \cdot u \in B_{p,k}, \psi \in \mathcal{D} \} \]
denote the local space associated with $B_{p,k}$. This is a Fréchet space with the system of seminorms $u \mapsto \|\psi \cdot u\|_{p,k}$, $\psi \in \mathcal{D}$.

In the following we shall consider certain subspaces of $B_{p,k}^{loc}$:

2.2. Definition. — Let $\sigma : [0, \infty) \to \mathbb{R}$ be a $C^\infty$-function satisfying\
\[ \lim_{\rho \to +\infty} \sigma(\rho) = +\infty \quad \text{and} \quad \sigma^{(j)} \text{ is bounded for all } j \geq 1. \]
Further let $\tilde{\sigma}(x) := \exp(\sigma([x,x] \cdot \sqrt{1 + [x,x]}))$, $x \in \mathbb{R}^n$. For $1 \leq p \leq \infty$ and $k \in \mathcal{K}$ we consider the distribution spaces
\[ B_{p,k}^{+\sigma} := \{u/\tilde{\sigma} \mid u \in B_{p,k}\} ; \quad B_{p,k}^{-\sigma} := \{\tilde{\sigma} \cdot v \mid v \in B_{p,k}\}. \]
Obviously these are Banach spaces with the norms
1) $\|u/\tilde{\sigma}\|_{p,k}^{+\sigma} := \|u\|_{p,k}$
2) $\|\tilde{\sigma} \cdot v\|_{p,k}^{-\sigma} := \|v\|_{p,k}$.

Remarks.
(i) Since $\tilde{\sigma}$, $1/\tilde{\sigma} \in C^\infty(\mathbb{R}^n)$ we have $B_{p,k}^{\pm\sigma} \subseteq B_{p,k}^{loc}$ by [H2], 10.1.23.
(ii) It is our intention to keep the spaces $B_{p,k}^{\sigma}$ as small as possible. This can be achieved by letting the function $\sigma$ tend to $+\infty$ very slowly. For example, choose $\sigma_0 \in C^\infty(\mathbb{R})$ with $\sigma_0(\rho) = \begin{cases} 0, & \rho \leq 0 \\ 1, & \rho \geq 1 \end{cases}$ and put
\[ \sigma(\rho) := \sum_{j=1}^{\infty} \sigma_0(\rho/a_j - a_j), \]
where the sequence $(a_j)$ tends to $+\infty$ very fast (e.g. $a_1 := 2, a_{j+1} := a_j^{a_j'}$).

2.3. Lemma. — Let $1 \leq p \leq \infty$, $k \in \mathcal{K}$ and $\sigma$ as in Definition 2.2. Then we have
\[ B_{p,k}^{-\sigma} \hookrightarrow B_{p,k}^{loc} \]
Proof. — Let $\psi \in \mathcal{D}$ and $v \in \mathcal{B}^{-\sigma}_{p,k}$ arbitrary. Since $\psi \cdot \tilde{\sigma} \in \mathcal{D} \subseteq \mathcal{S}$ it follows from [H2], 10.1.15 that
\[ \|\psi \cdot v\|_{p,k} = \|\psi \cdot \tilde{\sigma} \cdot v / \tilde{\sigma}\|_{p,k} \leq K \|v / \tilde{\sigma}\|_{p,k} = K \|v\|_{p,k}^\sigma, \]
with $K < \infty$ depending only on $\sigma, k$ and $\psi$. Since the topology of $\mathcal{B}^{\text{loc}}_{p,k}$ is given by the seminorms $v \mapsto \|\psi \cdot v\|_{p,k}$ the proof is complete. \Box

The same proof shows that if $\sigma_1, \sigma_2$ are such that $\tilde{\sigma}_1 / \tilde{\sigma}_2 \in \mathcal{S}$ (e.g. if $\limsup_{\rho \to \infty} \sigma_1(\rho) - \sigma_2(\rho) < 0$) then $\mathcal{B}^{-\sigma_1}_{p,k} \hookrightarrow \mathcal{B}^{-\sigma_2}_{p,k}$.

2.4. Remark. — Let $Q \in \text{Pol}'(n, m)$ be fixed and $\Pi \subseteq \mathcal{E}(Q)$ a compact set. By Theorem 1.2 there is a bounded measurable function $\eta : \mathbb{R}^n \to \mathbb{R}$ such that
\[ \tilde{P}(-\xi) \leq A|P(-\xi - z\eta(\xi))| : P \in \Pi, \xi \in \mathbb{R}^n, z \in T^1. \]
Using this we can for every $P \in \Pi$ define a distribution $\tilde{f}_P \in \mathcal{D}'$ through
\begin{equation}
(2.2) \quad (\tilde{f}_P, \varphi) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{z \in T^1} \frac{\varphi(\xi + z\eta(\xi))}{\tilde{P}(-\xi - z\eta(\xi))} \frac{dz}{2\pi i z} d\xi : \varphi \in \mathcal{D}. \tag{2.2}
\end{equation}
This type of formula has been introduced by L. Hörmander. Similarly as in [T2] we could now show that $\tilde{f}_P$ is an analytic function of $P \in \Pi$ with values in $\mathcal{B}^{-\sigma}_{\infty,\tilde{Q}}$ and $\tilde{f}_P$ is a fundamental solution of $P(D)$ for each $P$. (In fact, $\tilde{f}_P$ takes its values in the smaller space $\mathcal{B}^{-1}_{\infty,\tilde{Q}}$ defined below, where $H^1 = (\eta).$) We shall not do so since it is our aim to prove a more general result (Theorem 3.1 below). However, formula (2.2) serves as a motivation for the following

2.5. Definition. — In order to simplify notations we introduce the measure $|dz| := |dz_1| \cdots |dz_r|$ on the torus $T^r$ ($r \in \mathbb{N}$). Let $1 \leq p \leq \infty$, $k \in \mathbb{K}$ and $H^r = (\eta_s)_{s=1}^r : \mathbb{R}^n \to (\mathbb{R}^n)^r$ a bounded measurable function. For any $\varphi \in \mathcal{D}$ we set
\[ \|\varphi\|_{p,k}^r := \left( (2\pi)^{-n-r} \int_{\mathbb{R}^n} \int_{T^r} |k(\xi) \varphi(\xi + \tilde{H}^r(\xi, z))|^p |dz| d\xi \right)^{1/p} (p < \infty), \]
where $\tilde{H}^r(\xi, z) := \sum_{s=1}^r z_s \cdot \eta_s(\xi)$,
\[ \|\varphi\|_{\infty,\tilde{Q}}^r := \sup \{ |k(\xi) \varphi(\xi + \tilde{H}^r(\xi, z))| : \xi \in \mathbb{R}^n, z \in T^r \}. \]
The theorem of Paley-Wiener-Schwartz ([H1], §7.3) ensures that $\|\varphi\|_{p,k}^r$ is finite for each $\varphi \in \mathcal{D}$. Obviously, $(\mathcal{D}, \| \cdot \|_{p,k}^r)$ is a normed space. Its “dual space”
\[ \mathcal{B}^{\text{loc}}_{p,k}^r := \{ v \in \mathcal{B}^{\text{loc}}_{p,k} : \|v\|_{p,k}^r := \sup \{ |\langle v, \varphi \rangle| / \|\varphi\|_{p,k}^r : \varphi \neq \varphi \in \mathcal{D} \} < \infty \} \]
will be endowed with the norm \( \| \cdot \|^*_{p',k'} \). Here \( p' := 1 \) if \( p = \infty \).

The reason why we have introduced the space \( \mathcal{B}^\sigma_{q,k} \) is that it contains each \( \mathcal{B}^{*H'}_{q,k} \), yet it is small enough to give quite precise information on the growth at infinity of solutions of the equation \( P(D)f_P = \delta \) when \( P \) runs through \( E(Q) \) and \( f_P \) depends analytically on \( P \) (cf. the remark at the end of [M]).

2.6. LEMMA. — Let \( H^{r+1} = (\eta_s)_{s=1}^{r+1} \) as in Definition 2.5. With \( H' := (\eta_s)_{s=1}^{r+1} \) we then have

\[
\|\varphi\|_{p,k} \leq \|\varphi\|_{p,k}^{H'} \leq \|\varphi\|_{p,k}^{H'+1} , \quad \varphi \in \mathcal{D} ,
\]

hence

\[
\mathcal{B}_{p',k'} \hookrightarrow \mathcal{B}^{*H'}_{p',k'} \hookrightarrow \mathcal{B}^{*H'+1}_{p',k'} .
\]

Proof. — By Cauchy’s formula and the Hölder inequality we have, if \( p < \infty \),

\[
|\hat{\varphi}(\xi + \bar{H}'(\xi,z'))|^p \leq \int_{z_{r+1} \in \mathbb{T}^1} |\hat{\varphi}(\xi + \bar{H}'(\xi,z))|^p |dz_{r+1}| ,
\]

where \( z = (z',z_{r+1}) \). Inserting this in the definition of \( \|\varphi\|_{p,k}^{H'+1} \) yields the second inequality in (2.3). In the case \( p = \infty \) we can argue similarly using the maximum principle. Choosing \( H^0 \equiv 0 \) we also get \( \|\varphi\|_{p,k} = \|\varphi\|_{p,k}^{H^0} \leq \|\varphi\|_{p,k}^{H'} \). The embedding (2.4) is a direct consequence of these estimates. \( \square \)

2.7. LEMMA. — Let \( \sigma \) as in Definition 2.2 and \( H' \) as in Definition 2.5. Then there exists a constant \( K < \infty \) such that

\[
\|\varphi\|_{p,k}^{H'} \leq K\|\varphi\|_{p,k}^{*H'} : \quad \varphi \in \mathcal{D} .
\]

Proof. — Let \( \rho := 1 + \sup\{|\bar{H}'(\xi,z)| \infty | \xi \in \mathbb{R}^n , z \in \mathbb{T}^r \} \). For any \( \varphi \in \mathcal{D} \) and fixed \( \xi \in \mathbb{R}^n , z \in \mathbb{T}^r \) we have

\[
|\hat{\varphi}(\xi + \bar{H}'(\xi,z))|^p \leq \left( \frac{\rho^p}{2\pi} \right)^n \int_{\mathbb{T}^n} |\hat{\varphi}(\xi + \rho \zeta)|^p |d\zeta| \quad \text{if} \ p < \infty .
\]

This implies

\[
(\|\varphi\|_{p,k}^{H'})^p \leq \frac{\rho^p}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{T}^n} |k(\xi) \cdot \hat{\varphi}(\xi + \rho \zeta)|^p |d\zeta| |d\xi|
\]

\[
= \left( \frac{\rho^p}{2\pi} \right)^n \int_{\mathbb{T}^n} (2\pi)^{-n} \int_{\mathbb{R}^n} |k(\xi) \cdot \exp(-i[\rho \zeta, \cdot]) \varphi(\xi)|^p |d\xi| |d\zeta|
\]

\[
= \left( \frac{\rho^p}{2\pi} \right)^n \int_{\mathbb{T}^n} (\|\exp(-i[\rho \zeta, \cdot]) \varphi\|_{p,k}^p |d\zeta| .
\]

Now consider the functions
\[ \Phi_\zeta(x) := \exp(-i[p\zeta, x]) / \sigma(x) : \ \zeta \in T^n. \]
It is not hard to check that \( \{ \Phi_\zeta \} \) is a bounded subset of \( S \). With the weight function \( M_k \in \mathcal{K} \) (cf. [H2], §10.1),
\[ M_k(\xi) := \sup_{\xi' \in \mathbb{R}^n} k(\xi + \xi') / k(\xi') : \ \xi \in \mathbb{R}^n, \]
we have \( S \hookrightarrow B_{1,M_k} \) ([H2], 10.1.7), hence
\[ \sup \{ \| \Phi_\zeta \|_{1,M_k} \ | \ \zeta \in T^n \} =: K < \infty. \]
It follows from [H2], 10.1.15 that
\[ \sup \{ \| \Phi_\zeta \cdot \psi \|_{p,k} \ | \ \zeta \in T^n \} \leq K \psi \|_{p,k} : \ \psi \in D. \]
From (2.6) we thus obtain with \( \psi = \sigma \cdot \varphi \):
\[ \| \varphi \|_{p,k} \leq \left( \left( \frac{1}{2\pi} \right)^n \int_{T^n} (\| \Phi_\zeta \cdot \sigma \cdot \varphi \|_{p,k})^p |d\zeta| \right)^{1/p} \leq K \rho^n \| \sigma \cdot \varphi \|_{p,k} = K' \| \varphi \|_{p,k}. \]
The case \( p = \infty \) can be treated analogously.

2.8. COROLLARY. — Under the assumptions of Lemma 2.7 the mapping \( v \mapsto \langle v, \cdot \rangle \) identifies \( B_{p',k'}^\ast \) isometrically with the dual of the normed space \( (D, \| \cdot \|_{p,k}) \). In particular, \( B_{p',k'}^\ast \) is complete. Furthermore we have
\[ (2.7) \ B_{p',k'}^\ast \hookrightarrow B_{p',k'}^{-\sigma}. \]

Proof. — Clearly, \( v \mapsto \langle v, \cdot \rangle \) defines an isometric embedding of \( B_{p',k'}^\ast \) into \( (D, \| \cdot \|_{p,k})' \). We have to show that it is onto. So let \( \ell \) be a continuous linear form on \( (D, \| \cdot \|_{p,k}) \). By Lemma 2.7 we have
\[ (2.8) \ |(\ell / \sigma, \varphi)| \leq \| \ell \|_{p,k} \| \sigma \cdot \varphi \|_{p,k} \leq K \| \ell \|_{p,k} \| \varphi \|_{p,k} : \ \varphi \in D. \]
If \( p < \infty \) then \( B_{p',k'}^\ast \) is the dual space of \( B_{p,k} \), so \( \ell \in B_{p',k'}^{-\sigma} \subseteq B_{p,k}^{\text{loc}} \). Hence \( \ell \in B_{p',k'}^{\ast H'} \) and \( \| \ell \|_{p',k'}^{-\sigma} = \| \ell / \sigma \|_{p',k'} \leq K \| \ell \|_{p,k}^{H'} \) by (2.8).

In the case \( p = \infty \) we can analogously derive (2.8) with \( \sigma \) replaced by \( \sigma_j(\rho) := \sigma(\rho) - 1. \) Since \( S \hookrightarrow B_{\infty,k} \) the functional \( \ell_1 := \ell / \sigma_1 \) can be extended such that \( |\langle \ell_1, \varphi \rangle| \leq K \| \ell \|_{p,k} \| \varphi \|_{\infty,k} \) holds for all \( \varphi \in S \). Hence \( \ell_1 \in S' \) and the Fourier transform of \( \ell_1 \) is a continuous linear form on \( S \) equipped with the norm sup \( |k(-\xi) \varphi(\xi)| \). But then \( \langle \ell_1, \varphi \rangle = \int \varphi(\xi) d\mu(\xi) \) with a measure \( d\mu \) in \( \mathbb{R}^n \) of total mass \( \int |d\mu(\xi)| / k(-\xi) < \infty. \) Noting that \( \tau := \sigma_1 / \sigma \in S \) we obtain \( \ell / \sigma = \tau \cdot \ell_1 \in S' \) and \( (\ell / \sigma)^\wedge = (2\pi)^{-n} \hat{\tau} * d\mu \) which
is a $C^\infty$-function satisfying $\int |(\ell/\bar{\sigma})^\alpha(\xi)|/k(-\xi)\,d\xi < \infty$, i.e. $(\ell/\bar{\sigma}) \in B_{1,k'}$. As in the case $p < \infty$ we conclude that $\ell \in B_{1,k'}^{H^r}$ and $\|\ell\|_{1,k'}^{H^r} \leq K\|\ell\|_{1,k'}^{H^r}$ by the closed graph theorem.

Now we shall investigate how a differential operator with constant coefficients acts in the spaces $B_{q,k}^{H^r}$ ($1 \leq q \leq \infty$, $k \in \mathbb{K}$). If $P(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha$ is a polynomial in $x \in \mathbb{R}^n$ we consider the differential expression

$$
P(D) := \sum_{|\alpha| \leq m} a_\alpha D^\alpha \quad \text{where} \quad D := -i\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).
$$

2.9. PROPOSITION. — Let $P, Q \in \text{Pol}'(n,m)$ with $P < Q$ and $H^r = (\eta_s)_{s=1}^r$ as in Definition 2.5. Then the operator $P(D)$ maps $B_{q,k}^{H^r}$ continuously into $B_{q,k}^{H^r}$.

Proof. — Let $\rho := \sup\{|\tilde{\mathbb{H}}^r(\xi, z)|_\infty \mid \xi \in \mathbb{R}^n, z \in \mathbb{T}^r\}$ and $\xi \in \mathbb{R}^n$, $z \in \mathbb{T}^r$ fixed. With $\zeta := \tilde{\mathbb{H}}^r(\xi, z)$ we have for any $\varphi \in D$:

$$
|((k\tilde{Q})'(-\xi) \cdot (P(-D)\varphi)^\alpha(\xi + \zeta))| = |((k\tilde{Q})'(-\xi) \cdot P(-\xi - \zeta) \cdot \tilde{\varphi}(\xi + \zeta))| \leq |((k\tilde{Q})'(-\xi) \cdot \tilde{\varphi}(-\xi, \rho) \cdot \tilde{\varphi}(\xi + \zeta))| \leq (1 + \rho)^m \frac{\tilde{P}(-\xi)}{\tilde{Q}(-\xi)}|k'(\xi) \cdot \tilde{\varphi}(\xi + \zeta)|.
$$

Since $\sup_{\xi \in \mathbb{R}^n} \frac{|\tilde{P}(-\xi)|}{|\tilde{Q}(-\xi)|} < \infty$ we obtain

$$
(2.9) \quad \|P(-D)\varphi\|_{q',(k\tilde{Q})'} \leq K\|\varphi\|_{q',k'} : \varphi \in D.
$$

Now, if $v \in B_{q,k}^{H^r} \subseteq B_{q,k}^{\text{loc}}$ it follows from [H2], 10.1.22 that $P(D)v \in B_{q,k}^{\text{loc}}$. Furthermore, (2.9) implies that

$$
\|(P(D)v, \varphi)| = \langle v, P(-D)\varphi \rangle \leq \|v\|_{q',(k\tilde{Q})'}\|P(-D)\varphi\|_{q',(k\tilde{Q})'} \leq K\|v\|_{q',(k\tilde{Q})'}\|\varphi\|_{q',k'}
$$

for any $\varphi \in D$. In particular this means that $P(D)v \in B_{q,k}^{H^r}$ and

$$
\|P(D)v\|_{q,k}^{H^r} \leq K\|v\|_{q,k}^{H^r}.
$$

2.10. PROPOSITION. — Let $P, Q \in \text{Pol}'(n,m)$ with $P \sim Q$, $H^r = (\eta_s)_{s=1}^r$ as in Definition 2.5 and $\rho := \sup\{|[\mathbb{H}^{r-1}(\xi, z')]|_\infty \mid \xi \in \mathbb{R}^n$, $z' \in \mathbb{T}^r\}$.
$z' \in T^{r-1}$ ($\rho := 0$ if $r = 1$). Assume that with some constant $A > 0$ we have
\[ \tilde{P}(-\xi) \leq A|P(-\xi - \xi - z_r \eta_r(\xi))| : \xi \in \mathbb{R}^n, \xi \in B_{C^\infty}(\rho), \ z_r \in T^1. \]
Then the operator $P(D) : B^{*H^r}_{q,k\tilde{Q}} \rightarrow B^{*H^r}_{q,k}$ is surjective.

Proof. — Since $\tilde{Q}(-\xi) \leq B\tilde{P}(-\xi)$ the assumption implies that
\begin{equation}
\|P(-D)\varphi\|_{q',(k\tilde{Q})'} \geq (AB)^{-1}\|\varphi\|_{q',k'} : \varphi \in \mathcal{D}.
\end{equation}
Now let $w \in B^{*H^r}_{q,k}$ be given. Then by (2.10) the mapping
\[ P(-D)\varphi \mapsto \langle w, \varphi \rangle \]
is a well-defined continuous linear form on the subspace $P(-D)\mathcal{D}$ of $E := (D, \| \cdot \|_{q',(k\tilde{Q})'})$. By the Hahn-Banach theorem there exists a continuous extension $v$ of this form to the whole of $E$ and Corollary 2.8 implies that $v \in B^{*H^r}_{q,k\tilde{Q}}$. Finally it is clear that
\[ \langle P(D)v, \varphi \rangle = \langle v, P(-D)\varphi \rangle = \langle w, \varphi \rangle : \varphi \in \mathcal{D}, \]
i.e. $P(D)v = w$.

3. Parameter depending differential operators.

We come back to the main topic of this article. Let $Q \in \text{Pol}'(n, m)$ be fixed. Consider a family of differential operators
\begin{equation}
P(\lambda, D) = \sum_{|\alpha| \leq m} a_\alpha(\lambda)D^\alpha,
\end{equation}
where the coefficients $a_\alpha$ (constant with respect to $x$) are analytic functions of a parameter $\lambda$ varying in a complex manifold $\Lambda$. The only assumption we make is that for each value of $\lambda$ the polynomial $P(\lambda, \cdot)$ is equally strong as $Q$. Denoting by \{R_1, \ldots, R_\nu\} any fixed basis of the vector space $W(Q)$ we can write
\begin{equation}
P(\lambda, D) = \sum_{\mu=1}^\nu b_\mu(\lambda)R_\mu(D)
\end{equation}
with analytic functions $b_\mu : \Lambda \rightarrow \mathbb{C}$. Recall (1.1 (iii)) that the set $E(Q)$ is a holomorphically convex open submanifold of $W(Q)$. Hence we may take in (3.2) $\Lambda = E(Q)$ and \{b_\mu\} as the coordinate functions of $P$ with respect to the basis \{R_\mu\}. 
It $E$ is a locally convex vector space we denote by $H(\Lambda, E)$ the set of all analytic functions $e : \Lambda \to E$. Further let $\sigma \in C^\infty[0, \infty)$ be any fixed weight function as in Definition 2.2. Recall that $B_{q,k}^{\sigma,k} \hookrightarrow B^{\text{loc}}_{q,k}$ for $1 \leq q \leq \infty$, $k \in \mathcal{K}$.

3.1. Theorem. — Let $1 \leq q \leq \infty$ and $k \in \mathcal{K}$. Assume that $\Lambda$ is a Stein manifold. Then for any $g \in H(\Lambda, B_{q,k})$ there exists $f \in H(\Lambda, B^{-\sigma}_{q,k})$ such that

(i) $P(\lambda, D)f(\lambda) = 0$, $\lambda \in \Lambda$;

(ii) $R(D)f \in H(\Lambda, B^{-\sigma}_{q,k})$ for any $R \in W(Q)$.

In the following corollaries we do not make any assumptions concerning $\Lambda$:

3.2 Corollary. — Let $1 \leq q \leq \infty$ and $k \in \mathcal{K}$. Then for any $g_0 \in B_{q,k}$ there exists $f \in H(\Lambda, B^{-\sigma}_{q,k})$ such that $P(\lambda, D)f(\lambda) \equiv g_0$, and 3.1 (ii) holds.

Proof. — By our above remark we may take $P$ itself as a parameter varying in the Stein manifold $E(Q)$. Theorem 3.1 yields a function $\hat{f} \in H(E(Q), B^{-\sigma}_{q,k})$ such that $P(D)f(P) = g_0$, $P \in E(Q)$. Since the mapping $\lambda \mapsto \hat{f}(\lambda) := P(\lambda, \cdot)$ is analytic with values in $E(Q)$ we have $f := \hat{f} \circ P \in H(\Lambda, B^{-\sigma}_{q,k})$ and $P(\lambda, D)f(\lambda) \equiv g_0$. □

By $\delta$ we denote the Dirac distribution at 0, $\langle \delta, \varphi \rangle := \varphi(0)$. The next corollary answers a question of L. Hörmander ([H2], p. 59):

3.3. Corollary. — There exists $f \in H(\Lambda, B^{-\sigma}_{\infty,k})$ such that $P(\lambda, D)f(\lambda) \equiv \delta$, and 3.1 (ii) holds with $q = \infty$, $k \equiv 1$.

Proof. — This is a special case of Corollary 3.2 since with $k \equiv 1$ we have $\delta = g_0 \in B_{\infty,k}$. □

3.4. Remark. — If $\Lambda$ is an open subset of $\mathbb{R}^d$ (or a real analytic manifold) then the analogues of Theorem 3.1 and its corollaries hold with “analytic” replaced by “real analytic”.

Proof. — By a result of Grauert [G] there exists a neighborhood basis of $\Lambda$ in $\mathbb{C}^d$ consisting of holomorphically convex open sets. Using this
the real analytic case can be reduced to the analytic one (cf. [M]). □

It remains to prove Theorem 3.1. If \( \mathcal{F} \), \( \mathcal{G} \) are Banach spaces we denote by \( \mathcal{L}(\mathcal{F}, \mathcal{G}) \) the space of all bounded linear operators from \( \mathcal{F} \) to \( \mathcal{G} \) equipped with the operator norm topology. In the proof of 3.1 we shall make use of the following result of J. Leiterer [L].

3.5. Theorem. — Let \( \mathcal{F} \), \( \mathcal{G} \) be Banach spaces and \( \Lambda \) a complex Stein manifold. Let \( \mathcal{T} \in \mathcal{H}(\Lambda, \mathcal{L}(\mathcal{F}, \mathcal{G})) \) such that \( \mathcal{T}(\lambda)\mathcal{F} = \mathcal{G} \) for each \( \lambda \in \Lambda \). Then

(a) There exists for each function \( g \in \mathcal{H}(\lambda, \mathcal{G}) \) a function \( f \in \mathcal{H}(\Lambda, \mathcal{F}) \) such that \( \mathcal{T}(\lambda)f(\lambda) = g(\lambda) \), \( \lambda \in \Lambda \).

(b) For any open subset \( \Lambda' \) of \( \Lambda \) let \( \mathcal{N}(\Lambda') := \{ f \in \mathcal{H}(\Lambda', \mathcal{F}) \mid \mathcal{T}(\lambda)(f(\lambda)) \equiv 0 \} \). If \( \Lambda' \) is holomorphically convex then the set \( \mathcal{N}(\Lambda)|_{\Lambda'} \) of restrictions to \( \Lambda' \) of functions in \( \mathcal{N}(\Lambda) \) is dense in \( \mathcal{N}(\Lambda') \).

Proof of Theorem 3.1. — Let \( \{\Lambda_r\}_{r \in \mathbb{N}} \) be an exhausting sequence of open submanifolds of \( \Lambda \) such that each \( \Lambda_r \) is holomorphically convex, \( \overline{\Lambda}_r \) is compact and \( \overline{\Lambda}_r \subseteq \Lambda_{r+1} \). For each \( r \in \mathbb{N} \) we inductively choose a bounded measurable function \( H^r = (\eta_r)^{z=1} : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^r \) in the following way : set \( \rho_r := \sup\{|H^{r-1}(\xi, z')|_{\infty} \mid \xi \in \mathbb{R}^n, z' \in T^{r-1}\} \) (\( \rho_1 := 0 \)). Then by Theorem 1.2 there exist \( A_r \geq 1 \) and a bounded measurable function \( \eta_r : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that for all \( \lambda \in \overline{\Lambda}_r \), \( \xi \in \mathbb{R}^n \), \( \zeta \in B_{C^\alpha}(\rho_r) \), \( z_r \in T^1 \) we have

\[
(3.3) \quad \tilde{P}(\lambda, -\xi) \leq A_r|P(\lambda, -\xi - \zeta - z_r \eta_r(\xi))|.
\]

Thus, \( H^r \) is defined for each \( r \in \mathbb{N} \). Now consider the spaces

\[
\tilde{\mathcal{F}}_r := B^{H^r}_{q,k \tilde{Q}}, \quad \mathcal{G}_r := B^{H^r}_{q,k} : r \in \mathbb{N}.
\]

By (2.1), (2.4) and (2.7) we have the embeddings

\[
(3.4) \quad \tilde{\mathcal{F}}_r \hookrightarrow \tilde{\mathcal{F}}_{r+1} \hookrightarrow \mathcal{F} := B^{-\sigma}_{q,k \tilde{Q}} \hookrightarrow B^{\text{loc}}_{q,k \tilde{Q}},
\]

\[
(3.5) \quad B_{q,k} \hookrightarrow \mathcal{G}_r \hookrightarrow \mathcal{G}_{r+1} \hookrightarrow \mathcal{G} := B^{-\sigma}_{q,k} \hookrightarrow B^{\text{loc}}_{q,k}.
\]

Consider the representation (3.2) of \( P(\lambda, D) \). From Proposition 2.9 we know that each \( R_n(D) \) induces a bounded linear operator from \( \tilde{\mathcal{F}}_r \) into \( \mathcal{G}_r \). Hence the mapping \( \lambda \mapsto P(\lambda, D) \) is analytic with values in \( \mathcal{L}(\tilde{\mathcal{F}}_r, \mathcal{G}_r) \). From (3.3) and Proposition 2.10 we conclude that \( P(\lambda, D)\tilde{f}_r = \mathcal{G}_r \) for each \( \lambda \in \overline{\Lambda}_r \).

Furthermore, \( g \in \mathcal{H}(\Lambda, \mathcal{G}_r) \) by (3.5). It follows from part (a) of Theorem 3.5 that there exists for each \( r \in \mathbb{N} \) a function \( \tilde{f}_r \in \mathcal{H}(\Lambda_r, \mathcal{G}_r) \) such that

\[
P(\lambda, D)\tilde{f}_r(\lambda) = g(\lambda) : \lambda \in \Lambda_r.
\]
We construct a sequence of functions $f_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r)$ as follows. Put $f_1 := \tilde{f}_1$ and assume that $f_1, \ldots, f_r$ are already defined. Consider then

$$\delta_{r+1}(\lambda) := \tilde{f}_{r+1}(\lambda) - f_r(\lambda) : \lambda \in \Lambda_r.$$ 

By (3.4) we have $\delta_{r+1} \in \mathcal{H}(\Lambda_r, \mathfrak{F}_{r+1})$ and we may assume inductively that

$$P(\lambda, D)\delta_{r+1}(\lambda) = 0 : \lambda \in \Lambda_r.$$ 

By part (b) of Theorem 3.5 there exists for arbitrary $\varepsilon_{r+1} > 0$ a function $\varepsilon_{r+1} \in \mathcal{H}(\Lambda_{r+1}, \mathfrak{F}_{r+1})$ with the properties

$$P(\lambda, D)\varepsilon_{r+1}(\lambda) = 0 : \lambda \in \Lambda_{r+1} ; \sup_{\lambda \in \Lambda_{r-1}} ||\delta_{r+1}(\lambda) - \varepsilon_{r+1}(\lambda)||_{\mathfrak{F}_{r+1}} \leq \varepsilon_{r+1},$$

where for convenience we put $\Lambda_0 := \emptyset$. Since $\mathfrak{F}_{r+1} \hookrightarrow \mathfrak{F}$, $\mathfrak{F}_{r+1} \hookrightarrow \mathfrak{F}$ and the operators $R_{\mu}(D) : \mathfrak{F}_{r+1} \rightarrow \mathfrak{F}_{r+1}$ ($\mu = 1, \ldots, \nu$) are continuous (Proposition 2.9) one can choose $\varepsilon_{r+1}$ so small that

$$\sup_{\lambda \in \Lambda_{r-1}} ||\delta_{r+1}(\lambda) - \varepsilon_{r+1}(\lambda)||_{\mathfrak{F}} \leq 2^{-r},$$

$$\sup_{\lambda \in \Lambda_{r-1}} ||R_{\mu}(D)(\delta_{r+1}(\lambda) - \varepsilon_{r+1}(\lambda))||_{\mathfrak{F}} \leq 2^{-r} : \mu = 1, \ldots, \nu.$$ 

With this choice of $\varepsilon_{r+1}$ we set

$$f_{r+1}(\lambda) := \tilde{f}_{r+1}(\lambda) - \varepsilon_{r+1}(\lambda) : \lambda \in \Lambda_{r+1}.$$ 

We obtain a sequence of functions $f_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r) \subseteq \mathcal{H}(\Lambda_r, \mathfrak{F})$ with the properties

$$P(\lambda, D)f_r(\lambda) = g(\lambda) : \lambda \in \Lambda_r,$$

$$\sup_{\lambda \in \Lambda_{r-1}} ||f_{r+1}(\lambda) - f_r(\lambda)||_{\mathfrak{F}} \leq 2^{-r},$$

$$\sup_{\lambda \in \Lambda_{r-1}} ||R_{\mu}(D)(f_{r+1}(\lambda) - f_r(\lambda))||_{\mathfrak{F}} \leq 2^{-r} : \mu = 1, \ldots, \nu.$$ 

By (3.7) the limit

$$f(\lambda) := \lim_{r \to \infty} f_r(\lambda)$$

exists in $\mathfrak{F}$ for each $\lambda \in \Lambda$, and $f \in \mathcal{H}(\Lambda, \mathfrak{F})$. Since $\{R_{\mu}\}$ is a basis of $W(Q)$ we conclude from (3.8) that $R(D)f \in \mathcal{H}(\Lambda, \mathfrak{F})$ for any $R \in W(Q)$. Finally it is clear by (3.6) that $P(\lambda, D)f(\lambda) \equiv g(\lambda)$ since for fixed $\lambda \in \Lambda$ the sequence $\{f_r(\lambda)\}$ converges in $B_{q, k}^{\text{loc}}$ and the operator $P(\lambda, D) : B_{q, k}^{\text{loc}} \rightarrow B_{q, k}^{\text{loc}}$ is continuous ([H2], 10.1.22). The proof is complete.

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