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UNIQUE CONTINUATION FOR THE SOLUTIONS OF THE LAPLACIAN PLUS A DRIFT

by A. RUIZ & L. VEGA

1. Introduction.

We consider solutions of the inequality

$$|\Delta u(x)| \leq V(x)|\nabla u(x)| \quad x \in \Omega \tag{1}$$

where Ω is an open connected set contained in $\mathbf{R}^n$. We say that (1) has the unique continuation property if any solution which vanishes in an open subdomain of Ω must be identically zero in Ω.

Unique continuation properties for inequalities as (1) with singular drift V have been treated in several works. Hörmander [H] studied a general case

$$|Lu(x)| \leq V(x)|\nabla u(x)| + W(x)|u(x)|, \tag{2}$$

where L is an elliptic Lipschitz coefficient operator. In particular, concerning L^q_{loc} spaces he proved unique continuation if $q > \dfrac{3n-2}{2}$. This result was extended to $= \dfrac{3n-2}{2}$ in [BKRS] when L is the Laplacian. Both proofs are based on the so-called Carleman estimates which turn out to be false if $q < \dfrac{3n-2}{2}$ (see [Je]).

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In this work we substitute L^q_{loc} for the spaces of Morrey, also called the Fefferman-Phong class for its recent use in the study of the eigenvalues of Schrödinger operators [FeP]. (See also [CS], [K], [ChR], [ChF], and [RV] where these classes appear in the unique continuation context.)

We say that a function $v \geq 0$ locally in L^p is in $F^{\alpha,p}_{\text{loc}}$, $\alpha \leq n/p$, if there exists $r_0 > 0$ such that

$$(3) \quad |||v|||_{\alpha,p} = \sup \left\{ r^\alpha \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} v^p \right)^{1/p}, \ x \in \Omega, r < r_0 \right\} < \infty$$

where $B(x,r)$ denotes the ball of radius r centered at x and $|B(x,r)|$ its volume. This class corresponds to $L^{p,\lambda}$ in the classical notation for $\lambda = \alpha p - n$. (See [St], [C], [P].)

We prove the unique continuation property for solutions u in $H^2_{\text{loc}}(\Omega)$, the Sobolev space of functions in $L^2(\Omega)$, with two derivatives in $L^2(\Omega)$, for the inequality (1) when the drift term is in $F^{\alpha,p}_{\text{loc}}$ if $\alpha < 1$ and $p \geq (n-2)/2(1-\alpha)$.

Notice that it follows very easily from the definition of $F^{\alpha,p}_{\text{loc}}$ that $L^p_{\text{loc}} = F^{\frac{n}{p},p}_{\text{loc}}$ and therefore our result includes the previously known ones. On the other hand, it also applies to functions with worse singularities. For instance, assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and surjective and consider $w : \mathbb{R}^m \to \mathbb{R}$ a function in L^q for $q \geq (2m+n-2)/2$, then the function $v(x) = w(Tx)$ satisfies the above conditions and (1) has the unique continuation property. However, if $m < n$ this function does not need to be in L^p when $p \geq \dfrac{3n-2}{2}$. This result has interest in applications to physics, where very often the singularities of potentials and drifts are on manifolds of positive codimension.

The method we use is based on Carleman L^2 weighted inequalities as in [CS] and [ChR] and the diadic decomposition used in [RV]. There are some technical complications since the linear Carleman weight $\varphi(x) = \gamma \cdot x$ does not work and we need to use a parametrix, as in [Je], adapted to the quadratic weight $\varphi(x) = x_n + x_n^2/2$.

We would like to thank S. Chanillo for calling our attention to this type of lower dimensional potential.

We, lastly, have some comments about the notation.

$L^2(v)$: The set of functions f such that

$$\| f \|_{L^2(v)}= \left(\int |f(x)|^2 v(x)\, dx \right)^{1/2} < \infty.$$

$||| \quad |||_{\alpha,p}$: The norm of the Morrey space defined in (3).

$\wedge$: The Fourier Transform in $\mathbb{R}^n$.

$\wedge'$: The Fourier transform with respect to x' where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

χ_A : Characteristic function of the set A.

D : $-i(\partial/\partial x_1, \ldots, \partial/\partial x_n)$.

2. Statement of the results.

THEOREM 1. — *Let u be a solution of (1) in $H^2_{\rm loc}(\Omega)$ and V a nonnegative function in $F^{\alpha,p}_{\rm loc}$ for $p \geq (n-2)/2(1-\alpha)$, $\alpha < 1$, with the further hypothesis of $|||V|||_{\alpha,p}$ being sufficiently small if $p = (n-2)/2(1-\alpha)$. Then if u vanishes in an open subset of Ω, u must be zero in Ω.*

The proof of Theorem 1 relies on the following Carleman estimate.

THEOREM 2. — *There exist constants $c > 0$, depending only on n, and $\lambda_0 > 0$ depending only on n and r_0 in (3), such that for $\varphi(x) = x_n + x_n^2/2$ and V in $F^{\alpha,p}_{\rm loc}$, $\alpha < 1$,*

$$\|e^{\lambda\varphi}\nabla u\|_{L^2(V)} \leq \lambda^\beta C |||V|||_{\alpha,p} \|e^{\lambda\varphi}\Delta u\|_{L^2(V^{-1})} \tag{4}$$

holds for any C^∞ function u supported on $\mathbb{R}^{n-1} \times [-1/2, 1/2]$ and any $\lambda > \lambda_0$, where

$$\beta = ((\alpha - 1) + (n-2)/2p)/2.$$

3. The Proofs.

Proof of Theorem 1 from Theorem 2. — We follow well-known arguments.

1. As in [KRS], [BKRS], we may reduce by reflection, rotations, dilations and translations to the case where $u = 0$ out of the unit ball

$B((0,\dots,0,-1),1)$ centered at the point $(0,\dots,0,-1)$ and it suffices to prove that in this case u must be zero in a neighborhood of the origin.

2. Take $u(x)\eta(|x|)=g(x)$ where η is a C^∞ function such that $\eta=1$ in $[0,\varepsilon/2]$, $\eta=0$ in $[\varepsilon,\infty)$, for $\varepsilon>0$ to be chosen later on. We may substitute V by

$$W(x)=\chi_{B(0,\varepsilon)}(x)\left(V(x)+\frac{\delta}{|x|^\gamma}\right),\quad \delta>0,\gamma<\alpha,$$

and so we deal with a drift bounded below.

Since W is in $F^{\alpha,p}_{\mathrm{loc}}$ we use (4) and write :

$$\begin{aligned}\| e^{\lambda\varphi}\nabla g\|_{L^2(W)} &\le C\lambda^\beta(|||V|||_{\alpha,p}+\delta)\|e^{\lambda\varphi}\Delta g\|_{L^2(W^{-1})}\\ &\le C\lambda^\beta(|||V|||_{\alpha,p}+\delta)(\|e^{\lambda\varphi}\Delta g\|_{L^2(W^{-1}\chi_{B(0,\varepsilon)\setminus B(0,\varepsilon/2)})}\\ &\qquad+\| e^{\lambda\varphi}V\nabla u\|_{L^2(W^{-1})})\end{aligned}$$

where the last term on the right-hand side of the above inequality is bounded using known Sobolev inequalities (see [FeP], [ChF]). If x is in $\mathrm{supp}g\cap(B(0,\varepsilon)\setminus B(0,\varepsilon/2))$ there exists $\xi(\varepsilon)<0$ such that $\varphi(x)<\xi$. Then if $\beta<0$ and λ big or $|||V|||_{\alpha,p}<1/2C$, we obtain

$$\| e^{\lambda\varphi}\nabla u\|_{L^2(W\chi_{\{x:\varphi(x)>\xi/2\}})}\le C\lambda^\beta(|||V|||_{\alpha,p}+\delta)e^{\xi\lambda}\|\Delta g\|_{L^2(W^{-1})}.$$

The proof would be finished if the last term is finite, but

$$\Delta g=(\Delta\eta)u+\nabla\eta\cdot\nabla u+\eta\Delta u,\quad \eta\in C_0^\infty,$$

hence

$$\|\Delta g\|_{L^2(W^{-1})}\le C\sum_{|\alpha|\le 2}\|D^\alpha u\|_{L^2(W^{-1})}\le C'\|u\|_{H^2},$$

since $W^{-1}(x)\le M$.

The following elementary lemma is the fundamental tool which allows us to go beyond the L^p_{loc} class and to prove the Carleman inequality for Morrey spaces.

LEMMA 3. — *For $x\in\mathbb{R}^n$ and $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define the operator $\mathcal{K}$ by*

$$\mathcal{K}f(x)=\int K(x,x-y)f(y)\,dy$$

where $\mathrm{supp}K(x,\cdot)$ is contained in a box $R=\prod_{i=1}^n[a_i,b_i]$. Using R as a fundamental piece, construct by translations of R a grid of boxes $\{R_\nu\}_{\nu\in\mathbf{Z}^n}$ such that the interiors of R_ν are mutually disjoint and $\mathbb{R}^n=\bigcup_\nu R_\nu$. Then

$$\|\mathcal{K}f\|_{L^2(V)}\le c\|K\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^n)}\left(\sup_\nu\int_{R_\nu}V\right)\|f\|_{L^2(V^{-1})}.$$

Proof. — Define $f_\nu = \chi_{R_\nu} f$. Because of the assumption about the support of K we trivially have

$$\sum_\nu \chi_{\operatorname{supp} K(x,\cdot)}(x-y)\,\chi_{R_\nu}(y) \le 5^n.$$

Then

$$\begin{aligned}
\|\mathcal{K}f\|^2_{L^2(V)} &= \left\|\mathcal{K}\left(\sum_\nu f_\nu\right)\right\|^2_{L^2(V)} \\
&\le 5^{2n}\sum_\nu \int_{R_\nu} |\mathcal{K}f_\nu|^2 V(x)dx \\
&\le 5^{2n}\|K\|^2_{L^\infty(\mathbf{R}^n\times\mathbf{R}^n)}\left(\sup_\nu \int_{R_\nu} V\right)^2 \sum_\nu \|f_\nu\|_1^2
\end{aligned}$$

and by Cauchy-Schwarz inequality

$$\le 5^{2n}\|K\|^2_{L^\infty(\mathbf{R}^n\times\mathbf{R}^n)}\left(\sup_\nu \int_{R_\nu} V\right)^2 \sum_\nu \|f_\nu\|^2_{L^2(V^{-1})}.$$

Proof of Theorem 2. — As we said, we use a parametrix of the Carleman perturbation as in [Je] and [BKRS]. In the sequel see these references for the claimed properties.

The substitution $u = ve^{-\lambda\varphi}$ reduces (4) to

$$\|(D+i\lambda(1+x_n)N)v\|_{L^2(V)} \le c(p,n,r_0,\alpha)|||V|||_{\alpha,p}\lambda^\beta \|\,|D+i\lambda(1+x_n)N|^2 u\|_{L^2(V^{-1})}$$

where $N = (0,\dots,0,1)$, supp $V \subset \mathbb{R}^{n-1}\times(-1/2,1/2)$ and $D = -i(\partial/\partial x_1,\dots,\partial/\partial x_n)$. Our aim is to take a left inverse of

$$|D+i\lambda(1+x_n)N|^2 = \sum_{j=1}^{n-1} D_j^2 - \left(\frac{\partial}{\partial x_n} - (1+x_n)\lambda\right)^2,$$

which is a differential operator with variable coefficients with respect to the last variable x_n. Denote by $\wedge'$, the Fourier transform with respect to the $n-1$ first variables. Thus

$$\begin{aligned}
\left(|D+i\lambda(1+x_n)N|^2 v\right)^{\wedge'}(x_n,\xi') &= \left[|\xi'|^2 - \left(\frac{\partial}{\partial x_n} - (1+x_n)\lambda\right)^2\right]\overset{\wedge'}{v}(\xi',x_n) \\
&= \Omega_1(x_n,\xi')\Omega_2(x_n,\xi')\overset{\wedge'}{v}(x_n,\xi')
\end{aligned}$$

where $\Omega_j(x_n,\xi') = |\xi'| + (-1)^j\left(\frac{\partial}{\partial x_n} - (1+x_n)\lambda\right)$, $j=1,2$. A simple change of variable reduces the problem to finding an inverse to the one-variable differential operator $\dfrac{d}{dz} - z$. (The details can be found in [Je] p. 124.) Therefore a left inverse of $|D+i\lambda(1+x_n)N|^2$ is given by $B_2(x_n, D)B_1(x_n, D)$ where $B_j(x_n, D)$, $j = 1, 2$, is the pseudodifferential operator with symbol

$$(5) \qquad p_{j,s}(x,\xi) = (-1)^j s^{-1} b(s(1+x_n) + (-1)^j s^{-1}|\xi'|, s^{-1}\xi_n)$$

with $\lambda^2 = s$, $x = (x', x_n)$, $\xi = (\xi', \xi_n)$, $x \in \mathbb{R}^{n-1} \times [-1/2, 1/2]$, $\xi \in \mathbb{R}^n$ and

(6)
$$b(z,\eta) = \sqrt{2}\left(\int_0^\infty e^{-s^2-2sz} ds\right) e^{-iz\eta-(z^2+\eta^2)/2} - \int_0^\infty e^{-s^2/2-s(z-i\eta)} ds,$$
$$z, \eta \in \mathbb{R}.$$

From (6) and the property $b(-z,\eta) = \bar{b}(z,\eta)$ it is easy to obtain

$$(7) \qquad \left|\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\gamma}{\partial \xi^\gamma} p_{j,s}\right| \le c_{\beta\gamma}(s + |s^2(1+x_n) + (-1)^j|\xi'| - i\xi_n)^{-1-|\gamma|-|\beta|}$$

for any multi-index $\alpha, \beta \in \mathbb{N}^n$.

Therefore, it is enough to prove

$$(8) \qquad \|T_j B_2(x_n, D) B_1(x_n, D) u\|_{L^2(V)} \le c\lambda^\beta \|u\|_{L^2(V^{-1})}$$

with supp $u \subset \mathbb{R}^{n-1} \times (-1/2, 1/2)$, $T_j = \partial/\partial x_j$, $1 \le j \le n$ and $T_{n+1}u = \lambda(1+x_n)u$.

From (7) it follows very easily that if $1 \le j \le n+1$ then $T_j B_2$ is a Calderòn - Zygmund operator with constants which depend only on λ_0. (See [Jo], theorem on p. 68.)

Now for $V \in F^{\alpha,p}$ $0 < \alpha < n, 1 < p \le n/\alpha$ consider $\widetilde{V} = (MV^{p_1})^{1/p_1}$ where M is the Hardy -Littlewood maximal function and $1 < p_1 < p$. Then $V(x) \le \widetilde{V}(x)$, $\widetilde{V}$ is an A_1 weight and $\widetilde{V} \in F^{\alpha,p}$ with $|||\widetilde{V}|||_{\alpha,p} \le c|||V|||_{\alpha,p}$. (See [ChF], Lemma 1. Observe the notation there is different and that the proof still holds for the case $0 < p < n$.)

Therefore we have

$$\|T_j B_2 u\|_{L^2(V)} \le c\|u\|_{L^2(\widetilde{V})}.$$

It will be enough then to prove

$$(9) \qquad \|B_1 u\|_{L(V)} \le c\lambda^\beta |||V|||_{\alpha,p} \|u\|_{L^2(V^{-1})}$$

for $V \in F^{\alpha,p}$ as in Theorem 2 (notice that $\widetilde{V}^{-1} \le V^{-1}$) and

$$B_1 u(x', x_n) = s^{-1} \int b(s(1+x_n) - s^{-1}|\xi'|, s^{-1}\xi_n) e^{ix\cdot\xi} \widehat{f}(\xi) d\xi$$

with $x = (x', x_n)$.

Consider now a bump function $\psi_0 \in C_0^\infty(\mathbb{R})$ such that $\psi_0(x) = 1$ if $|x| < 1$ and vanishes if $|x| \geq 2$. For $k = 1, 2, \ldots, [\lg s] + 1$ define $\psi_k(x) = \psi_0(\frac{x}{2^k}) - \psi_0(\frac{x}{2^{k-1}})$ and $\psi_\infty(x) = 1 - \psi_0(\frac{x}{2^{[\lg s]+1}})$. Call for $k = 0, \ldots, [\lg s] + 1$ or $k = \infty$

$$b_k(z, \eta) = \psi_k(|z - i\eta|)b(z, \eta), \tag{10}$$

(11) $p_{1s}^k(x, \xi) = s^{-1}b_k(s(1 + x_n) - s^{-1}|\xi'|, s^{-1}\xi_n)\, x = (x', x_n), \xi = (\xi', \xi_n)$

and B_{1s}^k the pseudodifferential operator with symbol p_{1s}^k. Notice that by construction

$$\sum_{k=0}^{[\lg s]+1} p_{1s}^k + p_{1s}^\infty = p_1,$$

and therefore (9) will follow from

$$\begin{aligned} &\|B_{1s}^k u\|_{L^2(V)} \leq c2^{-\varepsilon k}\lambda^\beta |||V|||_{\alpha,p}\|u\|_{L^2(V^{-1})} \\ &\|B_{1s}^\infty u\|_{L^2(V)} \leq c\lambda^\beta |||V|||_{\alpha,p}\|u\|_{L^2(V^{-1})} \end{aligned} \tag{12}$$

with $\varepsilon > 0$ and $k = 0, 1, \ldots, [\lg s] + 1$.

Let us first deal with B_{1s}^∞. From (7) and (11) we have

$$\left|\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\gamma}{\partial \xi^\gamma}p_{1s}^\infty\right| \leq \frac{c_{\beta\gamma}}{(s^2 + |\xi|)^{1+|\alpha|+|\gamma|}}.$$

Consider $\widetilde{p}_{1s}^\infty = (s^2 + |\xi|)p_{1s}^\infty$ and $\widetilde{B}_{1s}^\infty$ the associated pseudodifferential operator. Then $\widetilde{B}_{1s}^\infty$ is as T_jB_2 and can be treated in the same way. Therefore we have to prove

$$\|J_1^s u\|_{L^2(V)} \leq cs^{2\beta}|||V|||_{\alpha,p}\|u\|_{L^2(V^{-1})} \tag{13}$$

for $V \in F^{\alpha,p}$, $2\beta = (\alpha - 1) + \dfrac{n-2}{2p}$, $\alpha < 1$ and $(J_1^s)^\wedge(\xi) = s^{-2}\widehat{\phi}(s^{-2}\xi)\widehat{u}(\xi)$ where $\widehat{\phi}(\xi) = (1 + |\xi|)^{-1}$. Then it is well known that ϕ behaves as $|x|^{1-n}$ when x is close to the origin and decays faster than any polynomial at infinity. Therefore we can write $\phi(x) = \sum_{j=-\infty}^{\infty} \phi_j(x)$ with $\operatorname{supp}\phi_j \subset B(0, 2^j)$ and

$$\begin{cases} \|\phi_j\|_{L^\infty} \leq c2^{-j(n-1)} & \text{if } j \leq 0 \\ \|\phi_j\|_{L^\infty} \leq c_m 2^{-mj} & \text{if } j \geq 0. \end{cases}$$

Thus we have

$$\|J_1^s u\|_{L^2(V)} \leq cs^{2(n-1)}\sum_j \|\Phi_j(s^2\cdot) * u\|_{L^2(V)}$$

and by Lemma 3 and the estimate for $\|\phi_j\|_{L^\infty}$

$$\leq cs^{2(n-1)}\Big(\sum_{j\leq 0} 2^{-j(n-1)}\Big(\sup_\nu \int_{Q_\nu^j} V\Big)+\sum_{j\geq 0} c_m 2^{-mj}\Big(\sup_\nu \int_{Q_\nu^j} V\Big)\Big)\|u\|_{L^2(V^{-1})}$$

where $\{Q_\nu^j\}_{\nu\in\mathbf{Z}}$ is a grid made by cubes of volume $(s^{-2}2^j)^n$. On the other hand

$$\sup_\nu \int_{Q_\nu^j} V \leq (s^{-2}2^j)^{n-\alpha}|||V|||_{\alpha,1}.$$

Putting all the estimates together and choosing $m > n-\alpha$ we obtain, if $\alpha < 1$

$$\|J_1^s u\|_{L^2(V)} \leq cs^{2(\alpha-1)}|||V|||_{\alpha,1}\|u\|_{L^2(V^{-1})}$$

which is better than (13).

Let us consider now p_{1s}^k for $k=0,1,\ldots,[\lg s]+1$. Recall that $B_{1s}^k f$ is given by

$$B_{1s}^k f(x) = \int K(x_n, x-y)f(y)\,dy \quad x=(x',x_n)$$

where

$$K(x_n,y) = \int p_{1s}^k(x_n,\xi)e^{iy\xi}\,d\xi \tag{14}$$

and p_{1s}^k as in (11). Notice that K depends on k and s but for simplicity of the notation we have dropped the index.

First we need an upper bound of the kernel K.

LEMMA 4. — *Define K as in (14). Then for any natural number m there exists a constant $c_m > 0$ such that*

(15)

$$|K(x_n,y)| \leq c_m s^{2n-3}2^k(1+|2^k sy'|^m)^{-1}(1+|s^2y'|)^{-(n-2)/2}(1+|2^k sy_n|^m)^{-1}$$

where $y=(y',y_n)\in\mathbf{R}^n$ and $|x_n|<1/2$.

Proof. — Assume first that $|y_n| \leq (2^{k-1}s)^{-1}$ and $|y'| \leq (2^{k-1}s)^{-1}$. Then we can write using cylindrical coordinates $\xi=(|\xi'|\omega,\xi_n)$, $\omega\in S^{n-2}$

$$\begin{aligned} K(x_n,y) = \int_{|\xi_n|\leq 2^{k+2}s} e^{iy_n\xi_n}\Big(\int_{2^{k-3}s<||\xi'|-s^2(1+x_n)|\leq 2^{k+3}s} |\xi'|^{n-2} \\ \cdot\int_{S^{n-2}} p_{1s}^k(x_n,\xi)e^{i(y'|\xi'|\omega)}\,d\sigma_\omega d|\xi'|\Big)\,d\xi_n. \end{aligned} \tag{16}$$

Therefore

$$\begin{aligned} |K(x_n,y)| &\leq c2^{2k}s^{2(n-1)}\sup_{\xi_n}\sup_{s^2/100\leq|\xi'|\leq 100s^2}\int_{S^{n-2}} p_{1s}^k(x_n,\xi)e^{i(y'|\xi'|\omega)}\,d\sigma_\omega \\ &\leq c2^{2k}s^{2(n-1)}(2^ks)^{-1}(1+s^2|y'|)^{-(n-2)/2} \end{aligned}$$

where the last step is a consequence of (7), (11) and the well-known estimates of the Fourier transform of a smooth measure on the sphere (see [Se] p. 233).

The general case is proved by integration by parts in (16) with respect to the variables ξ_n and $|\xi'|$ and using again (7).

Hereafter we shall follow very closely the argument in [T]. The result will be obtained by interpolation between two estimates. The first one will be an $L^2(\omega^{-1}) - L^2(\omega)$ estimate (ω will be eventually V^p) and the second one an $L^2 - L^2$ bound. In order to obtain the first part we shall need first an upper bound of the kernel. This suggests to cut K into pieces according to its level values. Consider as before the bump function $\psi_0 \in C_0^\infty(\mathbf{R})$ such that $\psi_0(x) = 1$ if $|x| < 1$ and vanishes if $|x| \geq 2$. Then for $j = 1, 2, \ldots$ we define $\psi_j(x) = \psi_0(\frac{x}{2^j}) - \psi_0(\frac{x}{2^{j-1}})$ and

$$K_j(x_n, y', y_n) = \psi_j(s^2 y') K(x_n, y', y_n). \tag{17}$$

Then $K = \sum_{j=0}^{\infty} K_j$ and from the Lemma 4 we have

(18)
$$\begin{cases} |K_j| \leq c_m s^{2n-3} 2^{k-j(\frac{n-2}{2})} (1 + |2^k s y_n|^m)^{-1} & \text{if } 0 \leq j \leq [\lg s] + 1 - k \\ |K_j| \leq c_m s^{2n-3+3m} 2^{k-j(\frac{n-2}{2})} 2^{-m(k+j)} (1|2^k s y_n|^m)^{-1} & \text{if } j > [\lg s] + k \end{cases}$$

and supp $K_j \subset \{(y', y_n), |y'| \leq 2^{j+2} s^{-2}\}$. Then we have the following lemma.

LEMMA 5. — *For $j = 0, 1, \ldots$, define K_j as in (17). Then*

$$\left\| \int K_j(x_n, x - y) f(y) dy \right\|_{L^2(V^p)} \leq \tau_j |||V|||_{\alpha,p}^p \|f\|_{L^2(V^{-p})} \tag{19}$$

where

$$\tau_j = \begin{cases} c s^{2\alpha p - 2} 2^{j(n-2\alpha p)/2} & \text{if } 0 \leq j \leq [\lg s] + 1 - k \\ c_m s^{2\alpha p - 2} 2^{j(n-2\alpha p)/2} (2^{k+j} s^{-1})^{-m} & \text{if } j \geq [\lg s] + 1 - k. \end{cases}$$

Proof. — We want to apply Lemma 3 but notice that the kernel K_j has compact support only in y' variables. Therefore we have to cut K_j according to level sets with respect to the y_n variable. Define for $d = 0, 1, \ldots$

$$K_{jd}(x_n, y', y_n) = K_j(x_n, y', y_n) \chi_{I_d}(y_n)$$

where χ_{I_d} is the characteristic function of the set I_d and $I_d = \{y_n, 2^d \leq 2^k s y_n \leq 2^{d+1}\}$ if $d \neq 0$ and $I_0 = \{y_n, |y_n| \leq (2^k s)^{-1}\}$. Notice that $K_j = \sum_{d=0}^{\infty} K_{jd}$ and the support of K_{jd} is contained in a box of dimension

$2^{j+2}s^{-2} \times \cdots \times 2^{j+2}s^{-2} \times 2^{d-k}s^{-1}$. We can apply Lemma 3 to K_{jd} where the grid has the dimensions we have just mentioned. Call $\{R_\nu^{jd}\}$ the boxes of that grid. Therefore, if $0 \le j \le [\lg s] + 1 - k$, the left hand side of inequality (19) is bounded by $\|K_{jd}\|_{L^\infty}\left(\sup_\nu \int_{R^\nu_{jd}} V^p\right)$. But (18) gives

$$\|K_{jd}\|_{L^\infty} \le c_m 2^{-md} s^{2n-3} 2^{k-j(\frac{n-2}{2})}.$$

On the other hand, decomposing each R_ν^{jd} in cubes with height equal to $2^{j+2}s^{-2}$

$$\begin{aligned}\sup_\nu \int_{R_\nu^{jd}} V^p &\le \frac{2^d(2^k s)^{-1}}{2^{j+2}s^{-2}}\left(\sup_\nu \int_{Q_\nu^{jd}} V^p\right)\\ &\le c s^{-2n+1+2\alpha p} 2^{j(n-\alpha p-1)} 2^{-k} |||V|||^p_{\alpha,p}\end{aligned}$$

where $\{Q_\nu^{jd}\}_\nu$ is a grid made by cubes of volume $2^{n(j+2)}s^{-2n}$. Putting both estimates together and summing up in d we obtain the desired result in this case.

Now consider $j \ge [\lg s] + 1 - k$. Then by (18)

$$\|K_{jd}\|_{L^\infty} \le c_m s^{2n-3+3m} 2^{k-j(\frac{n-2}{2})} 2^{-m(k+j)} 2^{-dm}$$

and

$$\sup_\nu \int_{R_\nu^{jd}} V^p \le \sup_\nu \int_{Q^\nu_{jd}} V^p \quad \text{if } 2^{d-k}s^{-1} \le 2^{j+2}s^{-2}$$

$$\sup_\nu \int_{R_\nu^{jd}} V^p \le \frac{2^d(2^k s)^{-1}}{2^{j+2}s^{-2}}\left(\sup_\nu \int_{R^\nu_{jd}} V^p\right) \quad \text{if } 2^{d-k}s^{-1} \ge 2^{j+2}s^{-2}.$$

Using both estimates in Lemma 3 and summing up in d we obtain the desired bound also in this case.

Finally we need the following lemma

LEMMA 6. — *For $j = 0, 1, \ldots$, define K_j as in (17). Then*

$$\|\int K_j(x_n, x-y) f(y)\,dy\|_{L^2(\mathbf{R}^{n-1}\times(-1/2,1/2))} \le c\tau_j' \|f\|_{L^2(\mathbf{R}^n)} \tag{20}$$

where

$$\tau_j' = \begin{cases} 2^j s^{-2} & \text{if } j \le [\lg s] + 1 - k \\ c2^{-k}s^{-1} & \text{if } j \ge [\lg s] + 1 - k. \end{cases}$$

Before proving the lemma notice that a standard interpolation theorem together with the estimates (19) and (20) give us

$$\|\int K_j(x_n, x-y) f(y)\,dy\|_{L^2(V)} \le c\tau_j'' \|f\|_{L^2(V^{-1})}$$

where

$$\tau_j''=\begin{cases} cs^{2(\alpha-1)}2^{j((1-\alpha)+(n-2)/2p)}|||V|||_{\alpha,p} & \text{if } 0\le j\le [\lg s]+1-k \\ c_m s^{2\alpha-1-\frac{1}{p}}2^{j(\frac{n}{2p}-\alpha)}2^{-k(1-\frac{1}{p'})}(2^{k+s}s^{-1})^{-\frac{m}{p}} & \text{if } j\ge [\lg s]+1-k. \end{cases}$$

Then summing up in j we obtain (take $m>\dfrac{n}{2}-\alpha$)

$$\|\int K(x_n,x-y)f(y)dy\|_{L^2(V)}\le cs^{2\beta}2^{-k(1-\alpha+\frac{n-2}{2p})}|||V|||_{\alpha,p}\|f\|_{L^2(V^{-1})}$$

where K is given in (14) and $2\beta=\alpha-1+\dfrac{n-2}{2p}$. Therefore, the proof of Theorem 2 is finished as long as we prove Lemma 6.

Proof of Lemma 6. — Using the definitions of K_j and K in (17) and (14) respectively we have

$$\int K_j(x_n,x-y)f(y)dy=\int\psi_j(s^2(x'-y'))f(y)\int p_{1s}^k(x_n,\xi)e^{i(x-y)\xi}\,d\xi$$
$$=\int\widehat{f}(\eta)q(x_n,\eta)e^{ix\eta}\,d\eta$$

where

$$q_j(x_n,\eta',\eta_n)=\int p_{1s}^k(x_n,\xi',\eta_n)s^{-2(n-1)}\overset{\wedge'}{\psi}_j\left(\frac{\xi'-\eta'}{s^2}\right)d\xi'. \tag{21}$$

Notice that the pseudodifferential operator defined by q_j is of variable coefficients just with respect to x_n. Then using Plancherel's theorem we have

$$\left\|\int\widehat{f}(\eta)q(x_n,\eta)e^{ix\cdot\eta}\,d\eta\right\|^2_{L^2(\mathbf{R}^{n-1}\times(-1/2,1/2))}$$
$$=\int_{-1/2}^{1/2}\left\|\int\left(\int\widehat{f}(\eta',\eta_n)q(x_n,\eta',\eta_n)e^{ux_n\eta_n}\,d\eta_n\right)e^{ix'\cdot\eta'}\,d\eta'\right\|^2_{L^2(\mathbf{R}^{n-1})}dx_n$$
$$=\int_{\mathbf{R}^{n-1}}\int_{-1/2}^{1/2}\left|\int\widehat{f}(\eta',\eta_n)q(x_n,\eta',\eta_n)e^{ix_n\eta_n}\,d\eta_n\right|^2dx_nd\eta'.$$

Therefore it is enough to prove ([Jo], p. 68)

$$\left|\frac{\partial^\beta}{\partial x_n^\beta}\frac{\partial^\gamma}{\partial\xi_n^\gamma}q_j\right|\le c\tau_j'(1+|\xi_n|)^{-\gamma}\quad\gamma=0,1,2. \tag{22}$$

Consider first $\beta=\gamma=0$. From the definition of ψ_j we have

(23)

$$s^{-2(n-1)}\overset{\wedge'}{\psi}_j\left(\frac{\xi'-\eta'}{s^2}\right)$$
$$=(2^js^{-2})^{n-1}\overset{\wedge'}{\psi}_0\left(\frac{\xi'-\eta'}{2^{-j}s^2}\right)-(2^{j-1}s^{-2})^{n-1}\overset{\wedge'}{\psi}_0\left(\frac{\xi'-\eta'}{2^{-j}s^2}\right).$$

On the other hand, recall that the support of $p^k_{1s}(x_n, \xi', \eta_n)$ is contained in the cylinder $\big||\xi'| - s^2(1+x_n)\big| \le 2^{k+1}s$. Therefore if $2^{-j}s^2 \le 2^k s$ (i.e., $j \ge [\lg s] + 1 - k$) by Young's inequality (11), (7) and (22) we have

$$\|q_j\|_{L^\infty} \le \|p^k_{1s}\|_{L^\infty} \|s^{-2(n-1)} \overset{\wedge'}{\psi}_0 (\tfrac{\cdot}{s^2})\|_{L^1} \le c2^{-k}s^{-1}.$$

If $2^{-j}s^2 \ge 2^k s$ write $\overset{\wedge'}{\psi}_0 = \sum_{m=1}^{\infty} \overset{\wedge'}{\psi}_{0m}$ where

$$\overset{\wedge'}{\psi}_{0m} (\xi' = \overset{\wedge'}{\psi}_0 (\xi') \left(\chi_{\{|\xi'| \le 2^m\}}(\xi') - \chi_{\{|\xi'| \le 2^{m-1}\}}(\xi')\right).$$

We shall prove (22) for $m = 0$ being the general case a consequence of the exponential decay of $\overset{\wedge'}{\psi}_0$. Thus

$$\begin{aligned}\|q_j\|_{L^\infty} &\le \|p^k_{1s}\|_{L^\infty} (2^j s^{-2})^{n-1} \| \overset{\wedge'}{\psi}_{00} \|_{L^\infty} \\ &\cdot \{\mathrm{vol}(\mathrm{supp}\, p^k_{1s}(x_n, \cdot, \eta_n) \cap \{|\xi'| \le 2^{-j}s^2\}) \le c2^j s^{-2}.\end{aligned}$$

If $\beta \ne 0$, or $\gamma \ne 0$ we use the above procedure together with (7). Notice that we only have to compute derivatives with respect to ξ_n where the right decay is obtained.

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