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Transversely affine foliations of some surface bundles over $S^1$ of pseudo-Anosov type


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TRANSVERSELY AFFINE FOLIATIONS
OF SOME SURFACE BUNDLES OVER $S^1$
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by Hiromichi NAKAYAMA

Introduction.

E. Ghys and V. Sergiescu classified codimension one foliations without compact leaves of torus bundles over $S^1$ whose monodromy matrices are hyperbolic automorphism ([2]). They cut the manifold along some fiber transverse to the foliation $\mathcal{F}$ and modified the resulting foliation $\mathcal{F}|(T^2 \times I)$ ($I = [0,1]$) so that $\mathcal{F}|(T^2 \times I)$ is tangent to each $\{*\} \times I (\ast \in T^2)$. Then $\mathcal{F}|(T^2 \times \{0\})$ is equal to $\mathcal{F}|(T^2 \times \{1\})$. However it is difficult to classify foliations without compact leaves of higher genus surface bundles over $S^1$ because it is difficult to find a fiber $S$ so that the singular foliation $\mathcal{F}|(S \times \{0\})$ coincides with $\mathcal{F}|(S \times \{1\})$ and to classify the foliation of $\Sigma \times I$. In this paper, we restrict our attention to transversely affine foliations without compact leaves of some higher genus surface bundles over $S^1$ of pseudo-Anosov type and obtain the following results:

**Main Theorem.** — Let $\Sigma$ be a closed orientable surface with genus greater than 1 and let $\pi : M \to S^1$ be an oriented $\Sigma$-bundle over $S^1$ of pseudo-Anosov type such that the real eigenvalues of its monodromy matrix are $\lambda$ and $\frac{1}{\lambda}$, and the eigenspace with respect to $\lambda$ (resp. $\frac{1}{\lambda}$) is one dimensional, where $\lambda$ (> 1) is the dilatation number of $M$. Suppose that $\mathcal{F}$ is a transversely oriented and transversely affine codimension one foliation.

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foliation of $M$ without compact leaves satisfying the Euler class equality $\chi(TF) = \pm \chi(T\pi)$ ($\in H^2(M;\mathbb{Z})$), where $TF$ and $T\pi$ denote the tangent bundles of the foliation $F$ and the bundle foliation of $\pi$ respectively. Then there is a finite covering of $F$ which is $C^0$ isotopic to a suspension foliation of a pseudo-Anosov diffeomorphism.

**Proposition.** — There is an orientable $\Sigma$-bundle over $S^1$ of pseudo-Anosov type satisfying the conditions of the main theorem. (I.e. the real eigenvalues of its monodromy matrix are $\lambda$ and $\frac{1}{\lambda}$, and the eigenspace with respect to $\lambda$ (resp. $\frac{1}{\lambda}$) is one dimensional, where $\lambda$ is the dilatation number.)

In Section 1, we give a precise definition of suspension foliations of pseudo-Anosov diffeomorphisms introduced by Meigniez [8], and prove the above proposition. For each bundle structure of pseudo-Anosov type, there exist suspension foliations of the pseudo-Anosov diffeomorphism. The hypothesis of the main theorem on the real eigenvalues of the monodromy and their eigenspaces restricts the bundle structures of $M$. S. Matsumoto showed the author examples of transversely affine foliations of $M$ which are not isotopic to the suspension foliations of pseudo-Anosov diffeomorphisms and have the same holonomy representation as the suspension foliations have ($\chi(TF) \neq \pm \chi(T\pi)$), which we also describe. In Section 2, we show the existence of a finite covering $\hat{\rho} : \hat{M} \to M$ and an embedding $\hat{g} : \Sigma \to \hat{M}$ isotopic to a fiber of the $\Sigma$-bundle $\hat{M}$ over $S^1$ such that $\hat{g}^* \hat{\rho}^* F$ is $C^0$ isotopic to a stable or unstable foliation of a pseudo-Anosov diffeomorphism which is $C^0$ isotopic to the monodromy map of $\hat{M}$ (Theorem 2). We prove the main theorem in Section 3.

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1. Pseudo-Anosov diffeomorphisms and their suspension foliations.

Let $\Sigma$ be a closed orientable surface with genus greater than 1. A *pseudo-Anosov diffeomorphism* $f : \Sigma \to \Sigma$ ([1]) is a homeomorphism with two measured foliations $(G^s, \mu^s)$ and $(G^u, \mu^u)$ such that $G^s$ and $G^u$ are mutually transverse with the same saddle singularities, $f(G^s, \mu^s) =$
(\mathcal{G}^s, \frac{1}{\lambda^u}) (\lambda > 1) \text{ and } f(\mathcal{G}^u, \mu^u) = (\mathcal{G}^u, \lambda^u \mu^u), \text{ where we adopt the definition of measured foliations written in [1] and } f \text{ is supposed to be a } C^\infty \text{ diffeomorphism except at the saddle singularities of } \mathcal{G}^s. \text{ The measured foliation } (\mathcal{G}^s, \mu^s) \text{ (resp. } (\mathcal{G}^u, \mu^u)) \text{ is called the stable (resp. unstable) foliation of } f, \text{ and } \lambda \text{ is called the dilatation number of } f.

W. Thurston showed that every diffeomorphism of } \Sigma \text{ is } C^0 \text{ isotopic to a "reducible" diffeomorphism or a periodic map or a pseudo-Anosov diffeomorphism ([1], [16]), and a pseudo-Anosov diffeomorphism is } C^0 \text{ isotopic to neither a "reducible" diffeomorphism nor a periodic map.}

Throughout this paper, we assume that } \mathcal{G}^s (a = s, u) \text{ is transversely oriented and } f \text{ preserves the transverse orientation of } \mathcal{G}^s. \text{ In particular, the number of separatrices passing through each saddle singularity is an even number.}

A surface bundle } M \text{ over } S^1 \text{ is of pseudo-Anosov type if its monodromy map is } C^0 \text{ isotopic to a pseudo-Anosov diffeomorphism. The dilatation number } \lambda \text{ of } M \text{ is defined by that of the pseudo-Anosov diffeomorphism. By the arguments of Exposé 12 of [1], } \lambda \text{ does not depend on the choice of pseudo-Anosov diffeomorphisms } C^0 \text{ isotopic to the monodromy map of } M. \text{ The monodromy matrix of } M \text{ is the linear automorphism of } H_1(\Sigma) \text{ induced by } f. \text{ Since we assume that } f \text{ preserves the transverse orientation of } \mathcal{G}^s, \lambda \text{ and } \frac{1}{\lambda} \text{ are eigenvalues of the monodromy matrix.}

Next we define suspension foliations of pseudo-Anosov diffeomorphisms. Let } M \text{ be an oriented } \Sigma \text{-bundle over } S^1 \text{ of pseudo-Anosov type and let } f \text{ be a pseudo-Anosov diffeomorphism } C^0 \text{ isotopic to the monodromy map of } M. \text{ Denote by } (\mathcal{G}^s, \mu^s) \text{ and } (\mathcal{G}^u, \mu^u) \text{ the stable and unstable foliations of } f \text{ respectively, and denote by } K \text{ the set of saddle singularities of } \mathcal{G}^s. \text{ Since } \mathcal{G}^s (a = s, u) \text{ is transversely oriented, there exists a non-singular closed 1-form } \omega^s \text{ of } \Sigma - K \text{ defining the measured foliation } (\mathcal{G}^s, \mu^s). \text{ (I.e. the kernel of } \omega^s \text{ coincides with the tangent bundle of } \mathcal{G}^s \text{ and } \int_\gamma \omega^s = \mu^s(\gamma), \text{ where } \gamma \text{ is a transverse arc of } \mathcal{G}^s \text{ oriented by the transverse orientation of } \mathcal{G}^s. \text{ Let } \mathcal{H}(\sigma, \alpha, \mathcal{G}^s, \mu^s) (\sigma = s, u, \alpha \neq 0) \text{ denote the foliation of } (\Sigma - K) \times \mathbb{R} \text{ defined by the non-singular 1-form } \lambda^\varepsilon a(t) \omega^s + \alpha dt (t \in \mathbb{R}), \text{ where } \varepsilon(s) = 1 \text{ and } \varepsilon(u) = -1. \text{ (I.e. } T\mathcal{H}(\sigma, \alpha, \mathcal{G}^s, \mu^s) = \text{ Ker}(\lambda^\varepsilon a(t) \omega^s + \alpha dt).) \text{ The completion of } \mathcal{H}(\sigma, \alpha, \mathcal{G}^s, \mu^s) \text{ in } \Sigma \times \mathbb{R} \text{ is denoted by } \hat{\mathcal{H}}(\sigma, \alpha, \mathcal{G}^s, \mu^s). \text{ For the } \mathbb{Z}\text{-action } \theta \text{ of } \Sigma \times \mathbb{R} \text{ given by } \theta_n(x, t) = (f^{-n}(x), t + n) (n \in \mathbb{Z}), \text{ the quotient space of } \Sigma \times \mathbb{R} \text{ by }
\( \theta \) is \( C^0 \) isotopic to \( M \). Since \( \theta_n^*(\lambda^{\epsilon(\sigma)t}w^\sigma + \alpha dt) = \lambda^{\epsilon(\sigma)t}w^\sigma + \alpha dt \) (here \( f^*w^\sigma = \lambda^{\epsilon(\sigma)}w^\sigma \)), \( \tilde{H}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)/\theta \) is a transversely orientable minimal \( C^0 \) foliation of \( M \) with holonomy (having a locally dense resilient leaf [4]), denoted by \( \mathcal{F}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma, f) \).

**Proposition.** — Let \( f \) and \( \bar{f} \) be pseudo-Anosov diffeomorphisms \( C^0 \) isotopic to the monodromy map of \( M \), and let \( (\mathcal{G}^\sigma, \mu^\sigma) \) and \( (\bar{\mathcal{G}}^\sigma, \bar{\mu}^\sigma) \) be the (un-)stable foliations of \( f \) and \( \bar{f} \) respectively \((\sigma = s, u)\). Then \( \mathcal{F}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma, f) \) is \( C^0 \) isotopic to \( \mathcal{F}(\sigma, \pm 1, \mathcal{G}^\sigma, \mu^\sigma, f) \) for any non-zero number \( \alpha \).

**Proof.** — Since \( f \) and \( \bar{f} \) are \( C^0 \) isotopic pseudo-Anosov diffeomorphisms, there is a diffeomorphism \( g \) of \( \Sigma \) isotopic to the identity map satisfying \( gf = \bar{f}g \) and \( g(\mathcal{G}^\sigma, \mu^\sigma) = (\bar{\mathcal{G}}^\sigma, k\bar{\mu}^\sigma) \) \((\sigma = s, u)\) for some \( k > 0 \) ([1], Exposé 12). Denote by \( w^\sigma \) \((\text{resp. } \bar{w}^\sigma)\) the closed 1-form defining \( (\mathcal{G}^\sigma, \mu^\sigma) \) \((\text{resp. } (\bar{\mathcal{G}}^\sigma, \bar{\mu}^\sigma))\), which is defined except at the saddle singularities of \( \mathcal{G}^\sigma \) \((\text{resp. } \bar{\mathcal{G}}^\sigma)\). Then \( g^*\bar{w}^\sigma = \pm \frac{1}{k} w^\sigma \). We define the diffeomorphism \( h: \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R} \)

by \( h(x, t) = \left( g(x), t + \frac{\epsilon(\sigma) \log(k|\alpha|)}{\log \lambda} \right) \) \((x, t) \in \Sigma \times \mathbb{R}) \). Then \( h \) satisfies that

\[
\begin{align*}
  h^*(\lambda^{\epsilon(\sigma)t}w^\sigma + \alpha dt) &= \pm |\alpha| \left( \lambda^{\epsilon(\sigma)t}w^\sigma \pm (\alpha/|\alpha|)dt \right) \\
  h\theta_n &= \bar{\theta}_n h,
\end{align*}
\]

where \( \theta_n(x, t) = (f^{-n}(x), t + n) \) and \( \bar{\theta}_n(x, t) = \left( \bar{f}^{-n}(x), t + n \right) \) \((n \in \mathbb{Z})\). This implies that \( \mathcal{F}(\sigma, \alpha, \bar{\mathcal{G}}^\sigma, \bar{\mu}^\sigma, \bar{f}) \) is \( C^0 \) isotopic to \( \mathcal{F}(\sigma, \pm 1, \mathcal{G}^\sigma, \mu^\sigma, f) \). \( \square \)

We call \( \mathcal{F}(\sigma, \pm 1, \mathcal{G}^\sigma, \mu^\sigma, f) \) \((\sigma = s, u)\) the suspension foliations of the pseudo-Anosov diffeomorphism of \( M \), denoted by \( \mathcal{F}_\pm^\sigma \). By the above proposition, the definition of the suspension foliations of the pseudo-Anosov diffeomorphism of \( M \) does not depend on the choice of pseudo-Anosov diffeomorphisms \( C^0 \) isotopic to the monodromy map of \( M \).

Next we construct a smooth model of \( \mathcal{F}_\pm^\sigma \), where \( \mathcal{F}_\pm^\sigma \) is a \( C^\infty \) foliation except at \((K \times \mathbb{R})/\theta\), denoted by \( K' \). First we choose a small closed tubular neighborhood \( V \) of \( K' \) in \( M \) such that \( \mathcal{F}_\pm^\sigma|\partial V \) is the union of \( C^\infty \) product foliations of tori whose leaves are isotopic to \( \partial V \cap \left( (\Sigma \times \{t\})/\theta \right) \) \((t \in \mathbb{R})\). By attaching the copies of the product foliation \( \{D^2 \times \{*\}; * \in S^1\} \) of \( D^2 \times S^1 \)
to \( F^\varepsilon_\pm(M - \text{int } V) \) along the leaves of \( \partial D^2 \times S^1 \) and \( \partial V \), we obtain a \( C^\infty \) foliation of \( M \), denoted by \( \tilde{F}^\varepsilon_\pm \). The foliation \( \tilde{F}^\varepsilon_\pm \) is \( C^0 \) isotopic to \( F^\varepsilon_\pm \).

The transverse orientation of \( \tilde{F}^\varepsilon_\pm \) (resp. \( \tilde{F}^-_\varepsilon \)) is given by the positive orientation of \( \lambda^\varepsilon(\sigma)^t\omega^\sigma + dt \) (resp. \( \lambda^\varepsilon(\sigma)^t\omega^\sigma - dt \)). Then the Euler class \( \chi(T\tilde{F}^\varepsilon_\pm) \) (resp. \( \chi(T\tilde{F}^-_\varepsilon) \)) is equal to \( \chi(T\pi) \) (resp. \( -\chi(T\pi) \)). By using this fact and Seke’s theorem ([12]), Meigniez ([8]) showed that \( \tilde{F}^\varepsilon_\pm \) is not isotopic to \( \tilde{F}^-_\varepsilon \).

We say that a transversely orientable codimension one foliation \( F \) is \textit{transversely affine} if there exists a system of transition functions consisting of elements of \( \text{Aff}^+\mathbb{R} = \{x \mapsto ax + b; a > 0\} \). By Seke’s theorem ([12]), transversely affine structures are characterized by the pairs \((\omega, \omega_1)\) of 1-forms of \( M \) such that

1) \( \omega \) defines the foliation \( F \),

(i.e. the tangent bundle of \( F \) coincides with \( \ker \omega \).)

2) \( d\omega = \omega \wedge \omega_1 \),

3) \( d\omega_1 = 0 \),

modulo the identifications \((\omega, \omega_1) \sim (g\omega, \omega_1 - \frac{dg}{g}) \) where \( g \) is a non-zero function of \( M \).

For example, \( \tilde{F}^\varepsilon_\pm \) is a transversely affine foliation. In fact, \( \tilde{F}^\varepsilon_\pm|(M - \text{int } V) \) has the transversely affine structure \((\lambda^\varepsilon(\sigma)^t\omega^\sigma \pm dt, -\varepsilon(\sigma) \log \lambda \cdot dt)\), and this transversely affine structure extends to \( M \).

Next we define the holonomy representation of a transversely affine foliation \( F \). Let \( x_0 \) denote the base point of \( M \) and let \( \tilde{p} : (\tilde{M}, \tilde{x}_0) \to (M, x_0) \) be a universal covering of \( M \) with the base point \( \tilde{x}_0 \) \( (p(\tilde{x}_0) = x_0) \). Then there exist two functions \( k : (\tilde{M}, \tilde{x}_0) \to (\mathbb{R}, 0) \) and \( h : (\tilde{M}, \tilde{x}_0) \to (\mathbb{R}_+^*, 1) \) \((\mathbb{R}_+^* = \{t > 0\}) \) satisfying \( \tilde{p}^*(\omega, \omega_1) = \left( \frac{dk}{h}, \frac{dh}{h} \right) \) ([12]). For each element \( \gamma \in \pi_1(M, x_0) \), there is an element \((a, b) \in R_+^* \times \mathbb{R} \) such that \( k \cdot \gamma = ak + b \) and \( h \cdot \gamma = ah \). We define the \textit{holonomy representation} \( \text{hol}_F : \pi_1(M, x_0) \to \text{Aff}^+\mathbb{R} \) of \( F \) by \( \text{hol}_F(\gamma) = (x \mapsto ax + b) \). The holonomy representation is uniquely determined up to an inner automorphism of \( \text{Aff}\mathbb{R} = \{x \mapsto ax + b; a \neq 0\} \).

For example, the holonomy representation of \( \tilde{F}^\varepsilon_\pm \) is as follows (up to an inner automorphism of \( \text{Aff}\mathbb{R} \)). Let \( \beta \) be a section of \( \pi : M \to S^1 \) passing through the base point \( x_0 \) and oriented by the positive orientation of \( S^1 \).
Then $\text{hol}_{\varphi_{\pm}}([\beta])$ is equal to $(x \mapsto \lambda^{-\varepsilon(\sigma)}x)$. Let $\iota : \Sigma \to M$ denote the inclusion map of the fiber passing through $x_0$ and let $y_0 = \iota^{-1}(x_0)$. Then $\text{hol}_{\varphi_{\pm}}(\iota_* \pi_1(\Sigma, y_0))$ is contained in the group of translations $\{x \mapsto x + b\}$, identified with $\mathbb{R}$, and $[\text{hol}_{\varphi_{\pm}} \cdot \iota_*] \in H^1(\Sigma; \mathbb{R})$ is cohomologous to $[\text{Per}_\mu \sigma]$, where $\text{Per}_\mu \sigma : \pi_1(\Sigma, y_0) \to \mathbb{R}$ is defined by $\text{Per}_\mu \sigma(\gamma) = \int_\gamma \omega^\sigma$.

S. Matsumoto constructed examples of transversely affine foliations of $M$ which are not isotopic to the suspension foliations of the pseudo-Anosov diffeomorphisms.

**Theorem (S. Matsumoto).** — Let $\Sigma$ be a closed orientable surface with genus greater than 1 and let $\pi : M \to S^1$ be an orientable $\Sigma$-bundle over $S^1$ of pseudo-Anosov type such that the saddle singularities of the (un-)stable foliation $\mathcal{G}^\sigma (\sigma = s, u)$ of the pseudo-Anosov diffeomorphism $f$ isotopic to the monodromy map of $M$ are the fixed points of $f$ and have 4 separatrices (4-saddle singularities). Then, for each $k \in \mathbb{Z}$ satisfying $|k| \leq -\chi(\Sigma)/2$, there exists a transversely affine foliation $\mathcal{F}_k^\sigma$ of $M$ satisfying the following conditions:

1) $\langle \chi(T\mathcal{F}_k^\sigma), [\Sigma] \rangle = 2k$ where $[\Sigma] \in H_2(M; \mathbb{Z})$ denotes the homology class represented by the fiber of $\pi$.

2) $\text{hol}_{\mathcal{F}_k^\sigma}$ is equal to $\text{hol}_{\varphi_{\pm}}$ up to an inner automorphism of $\text{Aff}\mathbb{R}$.

3) $\mathcal{F}_k^\sigma$ has no compact leaves.

**Proof.** — Let $K = \{s_1, s_2, s_3, \ldots, s_n\}$ denote the set of the saddle singularities of the (un-)stable foliation $\mathcal{G}^\sigma (\sigma = s, u)$ of $f$. The foliation of $(\Sigma - K) \times \mathbb{R}$ defined by the non-singular 1-form $\lambda^\varepsilon(\sigma) \omega^\sigma$ is denoted by $\mathcal{H}^\sigma_v$. Since $\mathcal{H}^\sigma_v$ is invariant under the $\mathbb{Z}$-action $\theta_n(x, t) = (f^{-n}(x), t + n)$, $n \in \mathbb{Z}$, $\mathcal{H}^\sigma_v/\theta$ is the foliation of $M - K'$ ($K' = (K \times \mathbb{R})/\theta$), denoted by $\mathcal{F}_v^\sigma$. The transverse orientation of $\mathcal{F}_v^\sigma$ is given by the positive orientation of $\lambda^\varepsilon(\sigma) \omega^\sigma$.

Denote by $\sigma^i_j$ ($j = 1, 2, 3, 4$) the separatrices of $\mathcal{G}^\sigma$ passing through the saddle singularity $s_i$ ($1 \leq i \leq n$). To simplify the explanation, we assume that $f(\sigma^i_j) = \sigma^i_{j+1}$ ($1 \leq j \leq n, 1 \leq i \leq 4$).

The leaf $(\sigma^i_j \times \mathbb{R})/\theta$ of $\mathcal{F}_v^\sigma$ is diffeomorphic to $S^1 \times \mathbb{R}$ and has holonomy. Hence there exists a small closed tubular neighborhood $V_i$ of $(\{s_i\} \times \mathbb{R})/\theta$ in $M$ such that $\partial V_i$ is transverse to $\mathcal{F}_v^\sigma$ and $\mathcal{F}_v^\sigma | \partial V_i$ consists of four 2-dimensional Reeb components (Fig. 1).
Next we construct two transversely oriented foliations \( \mathcal{K}_+ \) and \( \mathcal{K}_- \) of \( S^1 \times D^2 \) satisfying the following conditions (Fig. 2):

1) \( \mathcal{K}_\pm | (S^1 \times \partial D^2) \) is isotopic to \( \mathcal{F}_\nu | \partial V_i \) with the same transverse orientation.

2) \( \mathcal{K}_\pm \) has two annular leaves tangent to \( S^1 \times \{ * \} \) (\( * \in D^2 \)), and the other leaves of \( \mathcal{K}_\pm \) are transverse to \( S^1 \times \{ * \} \) (any \( * \in D^2 \)).

3) The transverse orientation of \( S^1 \times \{ 0 \} \) (\( 0 \in D^2 \)) induced by the transverse orientation of \( \mathcal{K}_+ \) (resp. \( \mathcal{K}_- \)) coincides with the positive (resp. negative) orientation of \( S^1 \).

(\( \mathcal{K}_\pm \) consists of two plus half Reeb components [14] and one dead-end component of \( D^1 \times S^1 \times S^1 \).)

By attaching \( \mathcal{F}_\nu | (M - \bigcup_{i=1}^{n} \text{int} V_i) \) with \( k - \frac{\chi(S^2)}{2} \) copies of \( \mathcal{K}_+ \) and \(-k - \frac{\chi(S^2)}{2} \) copies of \( \mathcal{K}_- \) along the leaves of \( \mathcal{F}_\nu | (\bigcup_{i=1}^{n} \partial V_i), \partial \mathcal{K}_+ \) and \( \partial \mathcal{K}_- \), we obtain a transversely orientable \( C^\infty \) foliation of \( M \), denoted by \( \mathcal{F}_k \).

By Thurston’s proposition of [15], \( \langle \chi(T \mathcal{F}_k), [\Sigma] \rangle = 2k \). Furthermore, \( \mathcal{F}_k \) has no compact leaves, because all the leaves of \( \mathcal{F}_\nu | (M - \bigcup_{i=1}^{n} \text{int} V_i) \) are non-compact.
The transversely affine structure of $\mathcal{F}^\sigma_k$ is given as follows. First we define the transversely affine structure of $\mathcal{F}^\sigma_v(M - \bigcup_{i=1}^n \text{int} V_i)$ by 

\[(\lambda^t \omega, -\varepsilon(\sigma) \log \lambda \cdot dt).\]

The foliation $\mathcal{K}_\pm$ also has a transversely affine structure. By Seki's theorem ([12]), which shows the uniqueness of the transversely affine structure of a foliation with holonomy, the transversely affine structures of $\mathcal{F}^\sigma_v(M - \bigcup_{i=1}^n \partial V_i)$ and $\partial \mathcal{K}_\pm$ are unique. Therefore the transversely affine structure of $\mathcal{K}_\pm$ can be attached to that of $\mathcal{F}^\sigma_v(M - \bigcup_{i=1}^n \text{int} V_i)$. For this transversely affine structure of $\mathcal{F}^\sigma_k$, the holonomy representation is equal to $\text{hol}_{\mathcal{F}^\sigma_k}$ up to an inner automorphism of $\text{Aff}\mathbb{R}$.

\[\Box\]

Remark. — If $2k \neq \pm \chi(\Sigma)$, then $\mathcal{F}^\sigma_k$ is not homotopic to $\tilde{\mathcal{F}}^\sigma_k$. Therefore $\mathcal{F}^\sigma_k$ is not isotopic to $\tilde{\mathcal{F}}^\sigma_k$.

In the end of this section, we prove the proposition in the introduction.

Proof of Proposition. — Let $f$ denote the hyperbolic automorphism of the torus $T^2$ given by the $2 \times 2$ matrix $\begin{pmatrix} 5 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2$. Then the fixed points of $f$ are $[(0,0)]$, $\left[\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right]$, $\left[\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right]$, $\left[\begin{pmatrix} 3 \\ 5 \end{pmatrix}\right]$ and $\left[\begin{pmatrix} 4 \\ 5 \end{pmatrix}\right]$, where $T^2$ is identified with the quotient of $\mathbb{R}^2$ by the integer
lattice and the element of $T^2$ represented by $z \in \mathbb{R}^2$ is denoted by $[z]$. Let $K$ denote the set $\left\{ \left[ \left( \frac{1}{5}, \frac{2}{5} \right) \right], \left[ \left( \frac{4}{5}, \frac{3}{5} \right) \right] \right\}$ and let $\alpha$, $\beta$ and $\varepsilon$ denote the generators of $\pi_1(T^2 - K)$ where $\alpha$, $\beta$ and $\varepsilon$ are represented by $(\{0,1\} \times \{0\})/\sim, (\{0\} \times \{0,1\})/\sim$ and a loop winding around $\left[ \left( \frac{1}{5}, \frac{2}{5} \right) \right]$, respectively.

Let $S_1$ and $S_2$ denote two copies of $T^2 - \left\{ \left( t, 2t \right) \right\}; -\frac{1}{5} \leq t \leq \frac{1}{5}$}. By attaching $S_1$ to $S_2$ along $\left\{ \left( t, 2t \right); -\frac{1}{5} < t < \frac{1}{5} \right\}$ alternatively, we obtain a double covering $p : \hat{\Sigma}_2 \to T^2 - K$, where $\hat{\Sigma}_2$ is a 2-punctured surface with genus 2. Let $\eta : \pi_1(T^2 - K) \to \mathbb{Z}/2\mathbb{Z}$ denote the homomorphism satisfying $\eta(\alpha) = \eta(\beta) = \eta(\varepsilon) = 1$. Then $p_*\pi_1(\hat{\Sigma}_2) = \text{Ker} \eta$. Since $\eta f_*([\alpha]) = \eta f_*([\beta]) = \eta f_*([\varepsilon]) = 1$, there is a lift $f'$ of $f$.

By collapsing two holes of $\hat{\Sigma}_2$, $f'$ extends to a homeomorphism $f''$ of the closed orientable surface $\Sigma_2$ with genus 2, which is a pseudo-Anosov diffeomorphism ([1], Exposé 13). We take two lifts of $\left\{ \left[ \left( \frac{1}{2}, t \right) \right]; 0 \leq t \leq 1 \right\}$ and $\left\{ \left[ \left( \frac{1}{2}, \frac{3}{2} + t \right) \right]; 0 \leq t \leq 1 \right\}$ as the generators of $H_1(\Sigma_2)$. Since $f$ maps $\left\{ \left[ \left( \frac{1}{2}, t \right) \right]; 0 \leq t \leq 1 \right\}$ (resp. $\left\{ \left[ \left( \frac{1}{2}, t \right) \right]; 0 \leq t \leq 1 \right\}$) on $\left\{ \left[ \left( \frac{5}{2} + 3t, 3t + 1 \right) \right]; 0 \leq t \leq 1 \right\}$ which intersects $\left\{ \left[ \left( t, 2t \right); -\frac{1}{5} < t < \frac{1}{5} \right\}$ two times, the isomorphism of $H_1(\Sigma_2; \mathbb{Z})$ induced by $f''$ is represented by the $4 \times 4$ matrix \[ \begin{pmatrix} 2 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 \end{pmatrix} \], whose eigenvalues are $\frac{7 \pm 3\sqrt{5}}{2}$ and $\frac{-1 \pm \sqrt{-3}}{2}$. Therefore the $\Sigma_2$-bundle over $S^1$ whose monodromy map is $C^0$ isotopic to $f''$ satisfies the conditions of the main theorem.
2. An embedded surface with the (un-)stable foliation.

The purpose of this section is to prove the existence of a finite covering of $\mathcal{F}$ whose restriction to a fiber is $C^0$ isotopic to an (un-)stable foliation of a pseudo-Anosov diffeomorphism (Theorem 2). First we show the following theorem.

**Theorem 1.** — Let $\pi : M \to S^1$ be as in the main theorem. If $\mathcal{F}$ is a transversely oriented and transversely affine foliation of $M$ without compact leaves, then the holonomy representation of $\mathcal{F}$ is equal to $\text{hol}_{\mathcal{F}_s}$ or $\text{hol}_{\mathcal{F}_u}$ up to an inner automorphism of $\text{Aff}_R$, where $\text{hol}_{\mathcal{F}_{\sigma}}$ ($\sigma = s, u$) is the holonomy representation of the suspension foliation of the pseudo-Anosov diffeomorphism defined in Section 1.

*Proof.* — We define homomorphisms $u : R \to \text{Aff}^+_R$ by $u(b) = (x \mapsto x + b)$ and $v : \text{Aff}^+_R \to R^*_+$ by $v(x \mapsto ax + b) = a$. Then the sequence $0 \to R \to \text{Aff}^+_R \to R^*_+ \to 1$ is an exact sequence ([8]).

Let $\iota : \Sigma \to M$ be the inclusion map of a fiber, and let $f : \Sigma \to \Sigma$ be a monodromy map of $M$ according to $\iota$. (I.e. there is a diffeomorphism $\phi : (\Sigma \times I)/((x, 1) \sim (f(x), 0)) \to M (I = [0, 1])$ such that $\phi|((\Sigma \times \{0\}) = \iota$).

Choose a fixed point $y_0$ of $f$, and the base point of $M$ is given by $\iota(y_0)$. Let $\ell$ denote the loop $\phi(y_0 \times I)$ of $M$ oriented by the positive orientation of $\{y_0\} \times I$, let $\beta$ denote the element of $\pi_1(M, \iota(y_0))$ represented by $\ell$. Then $\iota_+ f_+ \gamma = \beta^{-1}(\iota_+ \gamma) \beta$ for any $\gamma \in \pi_1(\Sigma, y_0)$.

For the homomorphism $\log \cdot v \cdot \text{hol}_\mathcal{F} \cdot \iota_+ : \pi_1(\Sigma, y_0) \to R$, the following equation holds for any $\gamma \in \pi_1(\Sigma, y_0)$:

\[
\log \cdot v \cdot \text{hol}_\mathcal{F} \cdot \iota_+ (f_+ \gamma) = \log \cdot v \cdot \text{hol}_\mathcal{F}(\beta^{-1}(\iota_+ \gamma) \beta)
\]

\[
= \log \cdot v \cdot \text{hol}_\mathcal{F}(\beta) + \log \cdot v \cdot \text{hol}_\mathcal{F}(\iota_+ \gamma) + \log \cdot v \cdot \text{hol}_\mathcal{F}(\beta^{-1})
\]

\[
= \log \cdot v \cdot \text{hol}_\mathcal{F} \cdot \iota_+ (\gamma).
\]

This shows that the cohomology class $[\log \cdot v \cdot \text{hol}_\mathcal{F} \cdot \iota_+] (\in H^1(\Sigma; R))$ is a fixed point of $f^# : H^1(\Sigma; R) \to H^1(\Sigma; R)$. Since $f^# : H_1(\Sigma; Z) \to H_1(\Sigma; Z)$ has no eigenvalue equal to 1, $[\log \cdot v \cdot \text{hol}_\mathcal{F} \cdot \iota_+] = 0$ in $H^1(\Sigma; R)$, and $v \cdot \text{hol}_\mathcal{F} \cdot \iota_+(\pi_1(\Sigma, y_0)) = \{1\}$. Thus the following commutative diagram
exists:
\[
\begin{array}{cccccc}
1 & \to & \pi_1(\Sigma, y_0) & \to & \pi_1(M, t(y_0)) & \to & \pi_1(S^1) & \to & 1 \\
\downarrow & & \downarrow H_N & & \downarrow \text{hol}_\mathcal{F} & & \downarrow H_L \\
1 & \to & \mathbb{R} & \to & \text{Aff}^+(\mathbb{R}) & \to & \mathbb{R}^*_+ & \to & 1 \\
\end{array}
\]

where the upper sequence is the homotopy exact sequence of the fibration $\pi$.

For the cohomology class $[H_N]$ represented by $H_N$, the following equation holds for any element $\gamma \in \pi_1(\Sigma, y_0)$:

\[
f^\# [H_N](\gamma) = u^{-1} \text{hol}_\mathcal{F} (t_*(f_\gamma)) = u^{-1} \text{hol}_\mathcal{F} (\beta^{-1}(t_*(\gamma))\beta) = u^{-1}(x \mapsto x + ce)
\]

where $\text{hol}_\mathcal{F}(\beta) = (x \mapsto \frac{1}{c} x + d)$ and $\text{hol}_\mathcal{F}(t_\gamma) = (x \mapsto x + e)$

\[
= cu^{-1}(\text{hol}_\mathcal{F}(t_\gamma)) = c[H_N](\gamma).
\]

First assume that $[H_N] \neq 0$ in $H^1(\Sigma; \mathbb{R})$. Then $c$ is an eigenvalue of $f^\#$ and $[H_N]$ is an eigenvector with respect to $c$. By the conditions of the monodromy matrix, $c$ is equal to $\lambda$ or $\frac{1}{\lambda}$. Since the cohomology class $[\text{Per}_\mu s]$ (resp. $[\text{Per}_\mu u]$) is also an eigenvector of $f^\#$ with respect to $\lambda$ (resp. $\frac{1}{\lambda}$), there is a non-zero number $c'$ such that $[H_N] = c'[\text{Per}_\mu s]$ (resp. $[H_N] = c'[\text{Per}_\mu u]$) if $c = \lambda$ (resp. $c = \frac{1}{\lambda}$). Therefore $\text{hol}_\mathcal{F}$ is equal to $\text{hol}_{\mathcal{F}_\pm}$ or $\text{hol}_{\mathcal{F}_\pm}^*$ up to an inner automorphism of $\text{Aff}\mathbb{R}$.

If $[H_N] = 0$, then $\text{hol}_\mathcal{F}\pi_1(M, t(y_0))$ is an abelian subgroup. Such transversely affine foliations were studied in [12], [17]. Since $\mathcal{F}$ has no compact leaves, $\mathcal{F}$ has no holonomy and $\mathcal{F}$ is defined by a non-singular closed 1-form ([12], Theorem 7, 8). The cohomology class of this closed 1-form is $\pi^*(c''[dt])$ for some non-zero number $c''$ where $[dt]$ is the generator of $H^1(S^1; \mathbb{Z})$. By the theorem ([6]) of Laudenbach-Blank in a weak form, $\mathcal{F}$ is isotopic to a bundle foliation (the referee showed the author the existence of direct proofs). This contradicts the non-existence of compact leaves of $\mathcal{F}$. \qed
THEOREM 2. — Let \( \pi : M \to S^1 \) be an oriented \( \Sigma \)-bundle over \( S^1 \) of pseudo-Anosov type. If \( \mathcal{F} \) is a transversely oriented and transversely affine foliation of \( M \) without compact leaves such that \( \chi(T\mathcal{F}) = \pm \chi(T\pi) \) and the holonomy representation of \( \mathcal{F} \) is equal to \( \text{hol}_{\mathcal{F}}^\pm \) (resp. \( \text{hol}_{\mathcal{F}}^\pm \)) up to an inner automorphism of \( \text{Aff}\mathbb{R} \), then there exists a finite covering \( \hat{\mathcal{F}} : \hat{M} \to M \) and an embedding \( \hat{g} : \Sigma \to \hat{M} \) isotopic to a fiber of the \( \Sigma \)-bundle \( \hat{M} \) over \( S^1 \) such that \( \hat{g}^*\hat{\mathcal{F}} \) is \( C^0 \) isotopic to the stable (resp. unstable) foliation of a pseudo-Anosov diffeomorphism which is \( C^0 \) isotopic to the monodromy map of \( M \).

The holonomy representation \( \text{hol}_{\mathcal{F}} \) satisfies that either \( \nu \cdot \text{hol}_{\mathcal{F}}(\beta) = \frac{1}{\lambda} \) and \( [H_N] = c[\text{Per}_\mu, s] \) (\( c \neq 0 \)) or \( \nu \cdot \text{hol}_{\mathcal{F}}(\beta) = \lambda \) and \( [H_N] = c[\text{Per}_\mu, u] \). To simplify the following proof of Theorem 2, we assume that \( \nu \cdot \text{hol}_{\mathcal{F}}(\beta) = \lambda \) and \( [H_N] = c[\text{Per}_\mu, u] \).

By the Roussarie's lemma ([11], [9]), there exists an embedding \( g : \Sigma \to M \) isotopic to a fiber of \( M \) such that \( g^*\mathcal{F} \) is a singular foliation with 4-saddle singularities, which are saddle singularities with four separatrices.

Let \( f : \Sigma \to \Sigma \) be a monodromy map of \( M \) with respect to \( g(\Sigma) \). (I.e. there exists a diffeomorphism \( \phi : (\Sigma \times I)/(\{x, 1\} \sim (f(x), 0)) \to M \) satisfying \( \phi((\Sigma \times \{0\})) = g \).) We define the infinite cyclic covering \( q : N \to M \) \((N = \Sigma \times \mathbb{R})\) by \( q(x, t) = \phi(f^i(x), t-i) \) \((i \leq t \leq i+1, i \in \mathbb{Z})\). In the following, we give the base point \( \bar{x}_0 \) of \( N \) by \((y_0, 0)\) where \( y_0 \) is a fixed point of \( f \), and the base point \( x_0 \) of \( M \) by \( g(y_0) \). The holonomy representation does not depend on the choice of the base points up to inner automorphisms.

Let \( r : (\hat{M}, \bar{x}_0) \to (N, x_0) \) be a universal covering of \( N \) with the base point and let \( p = q \cdot r \). For the transversely affine structure \((\omega, \omega_1)\) of \( \mathcal{F} \), there are two functions \( h : (\hat{M}, \bar{x}_0) \to (\mathbb{R}^+, 1) \) and \( k : (\hat{M}, \bar{x}_0) \to (\mathbb{R}, 0) \) such that \( p^*(\omega, \omega_1) = \left( \frac{dk}{h}, \frac{dh}{h} \right) \).

In order to prove Theorem 2, we need the following lemmas.

**Lemma 1.** — \( q^*\mathcal{F} \) is defined by a non-singular closed 1-form. Especially \( q^*\mathcal{F} = (q|\Sigma \times \{0\})^*\mathcal{F} \) is defined by a closed 1-form.

**Proof.** — For each element \( \gamma \in \pi_1(N, \bar{x}_0) \), \( q_*\gamma \in \pi_1(M, x_0) \) is homotopic to an element of \( g_*\pi_1(\Sigma, y_0) \). Hence \( \text{hol}_{\mathcal{F}}(q_*\gamma) \) is a translation, and \( h \cdot q_*\gamma(x) = h(x) \) \((x \in \hat{M})\) by the definition of the holonomy
representation. For any elements $z_1$ and $z_2$ ($\in \widetilde{M}$), $h(z_1) = h(z_2)$ if $r(z_1) = r(z_2)$.

We define $s : (N, \pi_0) \to (\mathbb{R}^*, 1)$ by $s = h \cdot r^{-1}$. Since $r^*(q^*\omega_1 - \frac{ds}{s}) = p^*\omega_1 - \frac{d(s \cdot r)}{s} = 0$, $q^*\omega_1$ is equal to $\frac{ds}{s}$. Hence $d(sq^*\omega) = ds \wedge q^*\omega + sdq^*\omega = 0$. Therefore $q^*\mathcal{F}$ is defined by the non-singular closed 1-form $sq^*\omega$. 

In the following, the non-singular closed 1-form $sq^*\omega$ is denoted by $\Omega$, which defines $q^*\mathcal{F}$.

**Lemma 2.** There exists a non-singular vector field $X$ of $M$ transverse to both $\mathcal{F}$ and $g(\Sigma)$.

**Proof.** Let $s_i (1 \leq i \leq n)$ denote the saddle singularities of $\mathcal{F}|g(\Sigma)$. Then there exists a non-singular vector field $X$ of $M$ and pairwise disjoint small neighborhoods $U_i$ of $s_i$ contained in $g(\Sigma)$ such that $X$ is transverse to $\mathcal{F}$ and tangent to $g(\Sigma) - \bigcup_{i=1}^n U_i$.

The saddle singularity $s_i$ is called positive (resp. negative) if the orientation of $X$ at $s_i$ is equal to the positive (resp. negative) orientation of the base space $S^1$. Let $I_+$ (resp. $I_-$) denote the number of positive (resp. negative) saddle singularities. By Thurston’s lemma ([15]), the following equations hold:

1) $-I_+ + I_- = \langle \chi(T\mathcal{F}), [g(\Sigma)] \rangle$,
2) $-I_+ - I_- = \chi(\Sigma)$,

where $\chi(T\mathcal{F}) \in H^2(M; \mathbb{Z})$ denotes the Euler class of the tangent bundle of $\mathcal{F}$, and $[g(\Sigma)]$ denotes the element of $H_2(M; \mathbb{Z})$ represented by $g(\Sigma)$. Since $\chi(T\mathcal{F}) = \pm \chi(T\pi)$, either $I_+$ or $I_-$ is equal to 0. Hence the saddle singularities of $\mathcal{F}|g(\Sigma)$ are all negative or all positive. If all the saddle singularities of $\mathcal{F}|g(\Sigma)$ are positive (resp. negative), then we can perturb $X$ toward the positive (resp. negative) direction of the base space $S^1$ in a neighborhood of $g(\Sigma)$ so that $X$ is transverse to both $\mathcal{F}$ and $g(\Sigma)$.

**Lemma 3.** There exists an embedding $\Gamma : \Sigma \times \mathbb{R}_+ \to N$ such that $\Gamma(\Sigma \times \{0\}) = \Sigma \times \{0\}$, $\Gamma(\Sigma \times \mathbb{R}_+) \subset \Sigma \times \mathbb{R}_+$ and $\Gamma^*\Omega = \iota_{t*}\Omega \pm dt$, where the inclusion map $\iota_t : \Sigma \to N$ ($t \in \mathbb{R}$) is defined by $\iota_t(x) = (x, t)$.

**Proof.** Let $\check{X}$ denote the lift of $X$ with respect to $q$. Then there is a non-singular vector field $Y$ of $N$ such that $\Omega(Y) = \pm 1$, $Y = u\check{X}$ for some
non-zero function \( u \) of \( N \), and the orientation of \( Y \) at \( \Sigma \times \{0\} \) coincides with the positive orientation of \( \{\ast\} \times \mathbb{R} \) \((\ast \in \Sigma)\). The integral manifolds of \( Y \) are called the leaves of \( Y \), which are to be oriented by \( Y \).

Let \( z \) be an element of \( N \). Denote by \( L \) the leaf of \( Y \) passing through \( z \). The point \( w \) of \( L \) satisfying \( \int_{z}^{w} \Omega|L = \Omega(Y)t \) \((t \in \mathbb{R})\) is denoted by \( \psi(z,t) \). Then \( \psi \) is the flow of \( Y \) because \( \Omega \left( \frac{\partial \psi}{\partial t} \right) = \frac{d}{dt} \left( \int_{0}^{t} \Omega \left( \frac{\partial \psi}{\partial t} \right) dt \right) = \frac{d}{dt} (\Omega(Y)t) = \Omega(Y) \). Note that \( \psi \) is not always defined in the whole \( N \times \mathbb{R} \). However \( \psi \) is defined on \( (\Sigma \times \{0\}) \times \mathbb{R}_{+} \), which will be shown in the following.

Let \( L(x) \) denote the leaf of \( Y \) passing through \((x,0) \in \Sigma \times \{0\} \subset N \), and let \( L_{i}(x) = L(x) \cap (\Sigma \times [i,i+1]) \) and \( L_{+}(x) = L(x) \cap (\Sigma \times [0,\infty)) \).

When \( L_{+}(x) \) is contained in \( \Sigma \times [0,n_{0}) \) for some integer \( n_{0} (> 0) \), \( \psi \) is defined on \((x,0) \times \mathbb{R}_{+} \) because \( \psi|[\Sigma \times [0,n_{0}]) \) is the flow of the compact manifold \( \Sigma \times [0,n_{0}] \) transverse to the boundary.

Suppose that \( L_{+}(x) \) is not contained in a compact region. Then \( L_{i}(x) \) is not empty for every \( i \geq 0 \) \((i \in \mathbb{Z})\). Let \( \ell \) denote \( \min_{y \in \Sigma} \Omega(Y) \left( \int_{L_{0}(y)} \right) > 0 \). \( \ell \) is the shortest time to reach \( \Sigma \times \{1\} \) from \( \Sigma \times \{0\} \) by the flow \( \psi \). We define the covering transformation \( \theta : \Sigma \times \mathbb{R} \rightarrow \Sigma \times \mathbb{R} \) of \( q \) by the flow \( \psi \). We define \( \theta^{*} = \theta^{*}(s \cdot \theta)(q \theta)^{*} \omega = \lambda s q^{*} \omega = \lambda \Omega \), \( \theta^{*} = \lambda \Omega \). Thus the following inequality holds:

\[
\Omega(Y) \int_{L_{i}(x)} \Omega = \Omega(Y) \int_{\theta^{-i}L_{i}(x)} (\theta^{i})^{*} \Omega = \Omega(Y) \int_{\theta^{-i}(x,i)} \lambda^{i} \Omega \geq \lambda^{i} \ell,
\]

where \( \{x_{i}\} = L(x) \cap (\Sigma \times \{i\}) \). Hence \( \Omega(Y) \int_{L_{+}(x)} \Omega = \infty \) and \( \psi \) is defined on \((x,0) \times \mathbb{R}_{+} \). Therefore \( \psi \) is defined on \( (\Sigma \times \{0\}) \times \mathbb{R}_{+} \).

We define an embedding \( \Gamma : \Sigma \times \mathbb{R}_{+} \rightarrow N \) by \( \Gamma(x,t) = \psi((x,0),t) \). Then

\[
\Gamma^{*}\Omega(v,a) \quad (v \in T_{x} \Sigma, a \in T_{t} \mathbb{R}_{+} = \mathbb{R})
= \Gamma^{*}\Omega \left( (\iota_{t})_{*}v + a \left( \frac{\partial}{\partial t} \right) \right)
= \iota_{t}^{*} \Gamma^{*}\Omega(v) + a \Omega \left( \frac{\partial}{\partial t} \right)
\]
\[ (\Gamma \cdot t_0)^* \Omega(v) + a\Omega(Y) \]
\[ = (\psi_t \cdot t_0)^* \Omega(v) \pm a \quad (\psi_t(z) = \psi(z, t), \ z \in N, \ t \in \mathbb{R}) \]
\[ = \iota_0^* \psi_t^* \Omega(v) \pm a \]
\[ = \iota_0^* \Omega(v) \pm a \quad \text{(See [3], Chapter VIII, Lemma 1.1.2)} \]
\[ = (p_1^* \iota_0^* \Omega \pm dt)((t_t)_* v + a(\frac{\partial}{\partial t})) \quad (p_1(x, t) = x) \]
\[ = (t_0^* \Omega \pm dt)(v, a). \]

Therefore \( \Gamma^* \Omega = t_0^* \Omega \pm dt. \)

\[ \square \]

**Lemma 4.** — There exists a non-zero number \( c \) such that \( \int_\gamma \iota_0^* \Omega = c[\text{Per}_\mu u](\gamma) \) for any \( \gamma \in \pi_1(\Sigma, y_0) \).

**Proof.** — For any \( \gamma \in \pi_1(\Sigma, y_0) \), \( \text{hol}_\mathcal{F}(g_* \gamma) = (x \mapsto x + \int_{(t_0)_* \gamma} \Omega) \). In fact,
\[
k \cdot g_* \gamma(\tilde{x}_0) - k(\tilde{x}_0)
= \int_{\tilde{g}^* \gamma} dk \quad \text{where } \tilde{g}^* \gamma \text{ is the lift of } g_* \gamma \text{ with respect to } p \quad \text{whose starting point is } \tilde{x}_0,
\]
\[ = \int_{\tilde{g}^* \gamma} h \rho^* \omega \]
\[ = \int_{\tilde{g}^* \gamma} r^*(sq^* \omega) \]
\[ = \int_{r_* \tilde{g}^* \gamma} \Omega \]
\[ = \int_{(t_0)_* \gamma} \Omega. \]

Since \( \text{hol}_\mathcal{F}(g_* \gamma) \) is also equal to \( (x \mapsto x + c[\text{Per}_\mu u](\gamma)) \) for some non-zero number \( c \), \( \int_{(t_0)_* \gamma} \Omega = c[\text{Per}_\mu u](\gamma). \) \[ \square \]

By changing the differentiable structure of \( \Sigma \), there exists a closed 1-form \( \tilde{\omega}^\sigma \) (\( \sigma = s, u \)) of \( \Sigma \) such that \( \tilde{\omega}^\sigma \) defines \( (\mathcal{G}^\sigma, \mu^\sigma) \) and \( \tilde{\omega}^\sigma = 0 \) at the saddle singularities of \( \mathcal{G}^\sigma \). (i.e. there is a homeomorphism \( \rho \) of \( \Sigma \) isotopic to the identity map such that \( \rho^*(\mathcal{G}^\sigma, \mu^\sigma) \) is the measured foliation defined by \( \tilde{\omega}^\sigma \).) By Lemma 4, \( \int_\gamma \iota_0^* \Omega = c \int_\gamma \tilde{\omega}^u \) for any \( \gamma \in \pi_1(\Sigma, y_0) \).
LEMMA 5. — There exist embeddings \( \eta_+ : \Sigma \rightarrow \Sigma \times \mathbb{R}_+ \) satisfying the following conditions:

1) \( c\tilde{\omega}^u = \eta_+^*(\iota_0\Omega + dt) = \eta_-^*(\iota_0\Omega - dt) \).

2) \( \eta_\pm(\Sigma) \) is transverse to \( \{\ast\} \times \mathbb{R}_+ \) for each \( \ast \in \Sigma \), and \( \eta_\pm \) is isotopic to \( \Sigma \times \{0\} \).

Proof. — By the above argument, \([\iota_0\Omega]\) and \([c\tilde{\omega}^u]\) are cohomologous in \( H^1(\Sigma; \mathbb{R}) \). Hence there is a function \( \xi : \Sigma \rightarrow \mathbb{R} \) such that \( \iota_0\Omega - c\tilde{\omega}^u = d\xi \). We define \( \eta_+ : \Sigma \rightarrow \Sigma \times \mathbb{R}_+ \) by \( \eta_+(x) = (x, \text{Max}(\xi) - \xi(x)) \) and \( \eta_- : \Sigma \rightarrow \Sigma \times \mathbb{R}_+ \) by \( \eta_-(x) = (x, \xi(x) - \text{Min}(\xi)) \). Then

\[
\eta_\pm^*(p_1^+ \iota_0\Omega \pm p_2^+ dt) = (p_1(x, t) = x, \quad p_2(x, t) = t) = (p_1 \eta_\pm^* \iota_0\Omega \pm (p_2 \eta_\pm^*) dt \quad = \iota_0\Omega - d\xi \quad = c\tilde{\omega}^u. \]

Proof of Theorem 2. — There exists a sufficiently large integer \( m (> 0) \) such that \( \Gamma_{\eta_+}(\Sigma) \) and \( \Gamma_{\eta_-}(\Sigma) \) are contained in \( \Sigma \times [0, m] \). Let \( q' : N \rightarrow \widehat{M} \) denote the quotient map of \( N \) by \( \theta^m \). Denote by \( \widehat{p} : \widehat{M} \rightarrow M \) the finite covering satisfying \( q = \widehat{p} \cdot q' \). If \( \Gamma^*\Omega = \iota_0\Omega + dt \) (resp. \( \Gamma^*\Omega = \iota_0\Omega - dt \)), then we define \( \widehat{g} : \Sigma \rightarrow \widehat{M} \) by \( q'\Gamma_{\eta_+} \) (resp. \( q'\Gamma_{\eta_-} \)). Then \( \widehat{g} : \Sigma \rightarrow \widehat{M} \) is an embedding isotopic to the fiber of \( \widehat{M} \). Since \( \widehat{g}^*\widehat{p}^*\mathcal{F} \) is defined by \( (\Gamma_{\eta_\pm}^*) \Omega = \eta_\pm^*(\iota_0\Omega \pm dt) = c\tilde{\omega}^u \), \( \widehat{g}^*\widehat{p}^*\mathcal{F} \) is \( C^0 \) isotopic to \( \mathcal{G}^u \), which is an unstable foliation of a pseudo-Anosov diffeomorphism which is \( C^0 \) isotopic to the monodromy map \( f^m \) of \( \widehat{M} \).

Remark. — The foliation \( \mathcal{H} \) obtained by cutting \( \widehat{p}^*\mathcal{F} \) along \( \widehat{g}(\Sigma) \) is a \( C^0 \) foliation of \( \Sigma \times I \) with a transverse invariant measure with full support such that \( \mathcal{H}|(\Sigma \times \{0\}) \) is the (un-)stable foliation of a pseudo-Anosov diffeomorphism which is \( C^0 \) isotopic to \( f^m \). If we choose the pseudo-Anosov diffeomorphism as the monodromy map of \( \widehat{M} \), then \( \mathcal{H}|(\Sigma \times \{0\}) \) is equal to \( \mathcal{H}|(\Sigma \times \{1\}) \). (Here \( \mathcal{H} \) is not a foliation at the saddle singularities of \( \mathcal{H}|(\Sigma \times \partial I) \) by the ordinary definition of foliations. Such foliations are called pseudo-foliations in [9]. However, in this paper, we call them also foliations.)
3. Foliations of $\Sigma \times I$ with transverse invariant measures.

By Theorems 1 and 2 (see also Remark of Section 2), the main theorem obviously follows from the following Theorem 3.

**Theorem 3.** — Let $\Sigma$ be a closed orientable surface with genus greater than 1. Let $f$ be a pseudo-Anosov diffeomorphism with an (un-)stable foliation $(\mathcal{G}^\sigma, \mu^\sigma)$ ($\sigma = s, u$). Suppose that $\mathcal{H}$ is a transversely orientable $C^0$ foliation of $\Sigma \times I$ ($I = [0,1]$) satisfying the following conditions:

1) $\mathcal{H}$ has a transverse invariant measure $\nu$ with full support.

2) $\mathcal{H}|(\Sigma \times \{0\}) = \mathcal{H}|(\Sigma \times \{1\}) = \mathcal{G}^\sigma$.

Then $\mathcal{H}$ is $C^0$ isotopic to $\mathcal{H}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)(\Sigma \times I)$ with the boundary fixed for some non-zero number $\alpha$, where $\mathcal{H}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)$ is the foliation of $\Sigma \times \mathbb{R}$ defined in Section 1.

In order to prove Theorem 3, we need some consideration.

First we consider some properties of singular foliations of $\Sigma$. Let $\mathcal{G}$ be a singular foliation of $\Sigma$ (all the singularities of $\mathcal{G}$ are saddle ones). A leaf $L$ of $\mathcal{G}$ is called ordinary if $L$ is neither a saddle singularity nor a separatrix, and $\mathcal{G}$ is called minimal if all the leaves except for the saddle singularities are dense in $\Sigma$. The next lemma is the generalization of Levitt’s pantalon decomposition theorem ([7]) to singular foliations having saddle singularities with many separatrices.

**Lemma 6.** — Let $\mathcal{G}$ be a transversely orientable minimal singular foliation of $\Sigma$. Then there exist disjoint simple closed curves $\gamma_i$ ($1 \leq i \leq n$) satisfying the following conditions:

1) $\gamma_i$ ($1 \leq i \leq n$) is transverse to $\mathcal{G}$. Denote by $S_j$ ($1 \leq j \leq m$) the connected components obtained by cutting $\Sigma$ along $\bigcup_{i=1}^{n} \gamma_i$. Then,

2) $\mathcal{G}|S_j$ ($1 \leq j \leq m$) is a singular foliation transverse to $\partial S_j$ with a unique saddle singularity whose separatrices reach $\partial S_j$.

3) All the ordinary leaves of $\mathcal{G}|S_j$ are properly embedded arcs which connect different boundaries of $S_j$, and there are ordinary leaves $\beta^j_1, \beta^j_2, \ldots, \beta^j_{\rho_j}$ which cut $S_j$ into a 2-disk.
Proof. — Suppose that disjoint submanifolds $S_j$ $(1 \leq j \leq q \leq m)$ satisfying the conditions 2) and 3) of Lemma 6 are constructed. Denote by $N$ the closure of $\Sigma - \bigcup_{j=1}^{q} S_j$.

If $\mathcal{G}|N$ has no saddle singularities, then $N$ is the disjoint union of annuli, say $A_i$ $(1 \leq i \leq n)$, and each $\mathcal{G}|A_i$ is the product foliation $\left\{ D^1 \times \{*\}; * \in S^1 \right\}$. Denote by $\gamma_i$ one of the boundaries of $A_i$. Then $\gamma_i$'s $(1 \leq i \leq n)$ satisfy the conditions of Lemma 6.

Next suppose that $\mathcal{G}|N$ has a saddle singularity $s$. Denote by $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_r$ the separatrices of $s$ in the clockwise order. Since the singular foliation $\mathcal{G}$ is minimal, $\sigma_{2k}$ $(k = 1, 2, 3, \ldots, r)$ intersects $\partial N$. Hence there exist pairwise disjoint closed transversals $\rho_k$ $(k = 1, 2, 3, \ldots, r)$ contained in the interior of $N$ and intersecting $\sigma_{2k} - \{s\}$. Let $z_k$ denote the point of $\sigma_{2k} \cap \left( \bigcup_{\ell=1}^{r} \rho_{\ell} \right)$ nearest to $s$ along $\sigma_{2k}$. The closed transversal $\rho_{\ell}$ containing $z_k$ is denoted by $\rho'_{\ell}$ and the restriction of $\sigma_{2k}$ to $[s, z_k]$ is denoted by $w_k$. Then there exists a sufficiently small closed neighborhood $S_{q+1}$ $(\subset \text{int } N)$ of $\bigcup_{k=1}^{r} (w_k \cup \rho'_{k})$ whose boundary is transverse to $\mathcal{G}$. The singular foliation $\mathcal{G}|S_{q+1}$ satisfies the conditions 2) and 3) of Lemma 6. By induction on the number of the saddle singularities of $\mathcal{G}|\bigcup_{j=1}^{q} S_j$, Lemma 6 holds. □

Next we prove the following lemmas about foliations obtained by cutting $\mathcal{H}$ along $\bigcup_{i=1}^{n} (\gamma_i \times I)$.

Let $S$ be an orientable surface with boundary. A transversely orientable $C^0$ foliation $\mathcal{U}$ of $S \times I$ having a transverse invariant measure $\nu$ with full support is called a unit foliation if it satisfies the following conditions:

1) $(\mathcal{U}, \nu)|(S \times \{0\})$ is a measured foliation of $S$ transverse to $\partial S$ satisfying the conditions 2) and 3) of Lemma 6.

2) $(\mathcal{U}, \nu)|(S \times \{1\}) = (\mathcal{U}, \nu)|(S \times \{0\})$.

3) $\mathcal{U}$ is transverse to $\partial S \times I$. 
LEMMA 7. — Let \((U, \nu)\) be a unit foliation. Then \(U|(\partial S \times I)\) has no vertical leaves, where a leaf of \(U|(\partial S \times I)\) is called vertical if it is isotopic to \(* \times I\) with \(* \times \partial I\) fixed.

Proof. — If \(U|(\partial S \times I)\) has a vertical leaf, then all the leaves of the component of \(U|(\partial S \times I)\) containing the vertical leaf are vertical because \(U\) has the transverse invariant measure \(\nu\).

Let \(\ell\) be a vertical leaf of \(U|(\partial S \times I)\) such that \(\partial \ell\) is not contained in any separatrix of \(U|(S \times \partial I)\). Let \(x_0\) (resp. \(x_1\)) denote the endpoint of \(\ell\) contained in \(\partial S \times \{0\}\) (resp. \(\partial S \times \{1\}\)). Denote by \(\beta_{x_0}\) (resp. \(\beta_{x_1}\)) the ordinary leaf of \(U|(S \times \partial I)\) containing \(x_0\) (resp. \(x_1\)), and denote by \(y_0\) (resp. \(y_1\)) the other endpoint of \(\beta_{x_0}\) (resp. \(\beta_{x_1}\)). Since \(U|(\partial S \times I)\) has no holonomy, \(U|(\partial S \times I)\) contains no interior compact leaves. Hence there exists a properly embedded arc \(\alpha \subset (\partial S \times I)\) connecting \(y_0\) and \(y_1\) and isotopic to \(* \times I\) \((* \in \partial S)\) with \(* \times \partial I\) fixed such that \(\alpha\) is either transverse or tangent to \(U|(\partial S \times I)\).

If \(\alpha\) is transverse to \(U|(\partial S \times I)\), then there exists a null-homotopic closed transversal near \(\ell \cup \beta_{x_0} \cup \alpha \cup \beta_{x_1}\). Since this contradicts the existence of the transverse invariant measure \(\nu\) with full support, \(\alpha\) is tangent to \(U|(\partial S \times I)\).

By Roussarie's theorem ([11], see also [9] for foliations with saddle singularities in the boundary), a null-homotopic simple closed curve \(\ell \cup \beta_{x_0} \cup \alpha \cup \beta_{x_1}\) bounds a leaf of \(U\) homeomorphic to the 2-disk \(D^2\). By Reeb's global stability theorem, there exists an immersion \(\psi : D^2 \times [-1,1] \to S \times I\) satisfying the following conditions 1), 2) and 3):

1) \(\psi(D^2 \times \{t\})(t \in (-1,1))\) is a leaf of \(U\).

2) \(\psi(D^2 \times (-1,1))\) is an embedding.

3) Both \(\psi(\partial D^2 \times \{1\})\) and \(\psi(\partial D^2 \times \{-1\})\) contain two saddle singularities of \(U|(S \times \partial I)\).

By considering the transverse orientation of \(U|(S \times \{0\})\) in the neighborhood of the saddle singularity of \(U|(S \times \{0\})\), there exists a number \(t_0 \in (-1,1)\) sufficiently near 1 or \(-1\) such that \(\psi(D^2 \times \{t_0\})\) contains a properly embedded short arc crossing the saddle singularity of \(U|(S \times \{0\})\) (Fig. 3). However this contradicts the non-existence of saddle connections of \(U|(S \times \{0\})\).

Thus \(U|(\partial S \times I)\) has no vertical leaves. \(\square\)
Remark. — The original proof of Roussarie’s theorem demands that the foliations are of class $C^r$ ($r \geq 2$). However it has already been known that his theorem is true for $C^0$ foliations (see [3], [5], [13]).

A unit foliation $(\mathcal{U}, \nu)$ is called normalized if $\mathcal{U}|(\partial S \times I)$ is transverse to $\{x\} \times I$ for any $x \in \partial S$.

**Lemma 8.** — Let $(\mathcal{U}, \nu)$ be a normalized unit foliation. For any $x, y \in \partial S$, $\nu(\{x\} \times I) = \nu(\{y\} \times I)$ and the orientation of $\{x\} \times I$ induced by the transverse orientation of $\mathcal{U}$ coincides with that of $\{y\} \times I$.

**Proof.** — If $x$ and $y$ are contained in the same connected component of $\partial S$, then $\nu(\{x\} \times I) = \nu(\{y\} \times I)$ and the orientation of $\{x\} \times I$ induced by the transverse orientation of $\mathcal{U}$ coincides with that of $\{y\} \times I$.

Let $\mathcal{G}$ denote $\mathcal{U}|(S \times \{0\})$. Suppose that an ordinary leaf $\beta$ of $\mathcal{G}$ connects $x$ and $y$ ($x, y \in \partial S$). Since $\{x\} \times I$ is homotopic to $(\beta \times \{1\}) \cup (\{y\} \times I) \cup (\beta \times \{0\})$, $\nu(\{x\} \times I)$ is equal to $\nu(\{y\} \times I)$. If the orientation of $\{x\} \times I$ induced by the transverse orientation of $\mathcal{U}$ is opposite to that of $\{y\} \times I$, then there is a null-homotopic closed transversal, which contradicts the existence of the transverse invariant measure $\nu$ with full support.

Let $\gamma$ and $\gamma'$ be connected components of $\partial S$. Denote by $\sigma$ and $\sigma'$ the separatrices of $\mathcal{G}$ intersecting $\gamma$ and $\gamma'$, respectively. Then there exists a series of separatrices $\sigma = \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_k = \sigma'$ where $\sigma_i$ is adjacent to $\sigma_{i+1}$ for each $i$. Since there is an ordinary leaf of $\mathcal{G}$ near $\sigma_i \cup \sigma_{i+1}$ for each
LEMMA 9. — Let \((U_1, \nu_1)\) and \((U_2, \nu_2)\) be normalized unit foliations of \(S \times I\) satisfying \((U_1, \nu_1)|\partial(S \times I) = (U_2, \nu_2)|\partial(S \times I)\), then there exists a homeomorphism \(h : S \times I \to S \times I\) such that \(h|\partial(S \times I) = \text{id}\) and \(h(U_1, \nu_1) = (U_2, \nu_2)\).

Proof. — Let \(G\) denote \(U_1|(S \times \{0\})\), and let \(\beta_j (1 \leq j \leq p)\) be the ordinary leaves of \(G\) which cut \(S\) into a 2-disk. By Roussarie's theorem ([11]), there are pairwise disjoint properly embedded disks \(D_j\) (resp. \(D'_j\)) transverse to \(U_1\) (resp. \(U_2\)) and bounded by \(\partial(\beta_j \times I)\). Since \(U_1|D_j\) and \(U_2|D'_j\) are foliations whose leaves are properly embedded arcs, there is a homeomorphism \(h : \partial(S \times I) \cup \left( \bigcup_{j=1}^{p} D_j \right) \to \partial(S \times I) \cup \left( \bigcup_{j=1}^{p} D'_j \right)\) such that \(h(U_1, \nu_1) = (U_2, \nu_2)\).

Let \(\hat{U}_1\) (resp. \(\hat{U}_2\)) denote the foliation of \(D^3\) obtained by cutting \(\hat{U}_1\) (resp. \(\hat{U}_2\)) along \(\bigcup_{j=1}^{p} D_j\) (resp. \(\bigcup_{j=1}^{p} D'_j\)) (Fig. 4). \(\hat{U}_i\) \((i = 1, 2)\) has \(2p\) collapsing leaves homeomorphic to \(I\) and two saddle singularities in the boundary. The leaves of \(\hat{U}_i\) near the collapsing leaves are all homeomorphic to \(D^2\). By Poincaré-Bendixson's theorem, the ordinary leaves of \(\partial\hat{U}_i\) are all homeomorphic to \(S^1\) and the union of the leaves of \(\partial\hat{U}_i\) containing a saddle singularity is a bouquet. Hence the leaves of \(\hat{U}_i\), containing no saddle singularities of \(\partial\hat{U}_i\) are homeomorphic to the 2-disks, and the union of the leaves of \(\hat{U}_i\) containing the saddle singularity is the union of 2-disks whose intersection point is the saddle singularity. Therefore \(h\) extends to a homeomorphism of \(S \times I\) which satisfies the conditions of Lemma 9.

Proof of Theorem 3. — Let \(\gamma_i (1 \leq i \leq n)\) denote the disjoint simple closed curves transverse to \(G^\sigma\) constructed by Lemma 6, and let \(S_j (1 \leq j \leq m)\) denote the connected components obtained by cutting \(\Sigma\) along \(\bigcup_{i=1}^{n} \gamma_i\). Since \(H\) has the transverse invariant measure \(\nu\) with full support, \(H\) has no interior compact leaves. By Roussarie's theorem ([11]), \(\gamma_i \times I\) can be taken by an isotopy of \(\Sigma \times I\) with \(\Sigma \times \partial I\) fixed so that \(\gamma_i \times I\) is transverse to \(H\). Since all the leaves of \(H|_{(\gamma_i \times I)}\) are properly embedded arcs, \(\nu(\gamma_i \times \{0\})\) is equal to \(\nu(\gamma_i \times \{1\})\). By the unique ergodicity of the (un-)stable foliation.
of the pseudo-Anosov diffeomorphism ([1]), \(\nu((\Sigma \times \{0\}) = \nu((\Sigma \times \{1\})\). Therefore \((\mathcal{H}|(S_j \times I), \nu|(S_j \times I))\) is a unit foliation.

By Lemma 7, \(\mathcal{H}|(S_j \times I)\) has no vertical leaves. We change \(\Sigma \times I\) again by an isotopy with \(\Sigma \times \partial I\) fixed so that \(\{\ast\} \times I\) is transverse to \(\mathcal{H}\) for any \(\ast \in \bigcup_{i=1}^{n} \gamma_i\). Then \((\mathcal{H}|(S_j \times I), \nu|(S_j \times I))\) is a normalized unit foliation.

We take the transverse orientation of \(\mathcal{H}\) so that the transverse orientation of \(\mathcal{H}|(\Sigma \times \{0\})\) coincides with that of \(\mathcal{G}^\sigma\). Since all the leaves of \(\mathcal{H}|(\gamma_i \times I)\) are properly embedded arcs, the transverse orientation of \(\mathcal{H}|(\Sigma \times \{1\})\) also coincides with that of \(\mathcal{G}^\sigma\).

By Lemma 8, the orientations of \(\{\ast\} \times I\ (\ast \in \partial S_j)\) induced by the transverse orientation of \(\mathcal{H}\) are either all positive or all negative. For each \(\gamma_i\) and \(\gamma_j\), there is an arc in a leaf of \(\mathcal{G}^\sigma\) connecting \(\gamma_i\) with \(\gamma_j\) by the minimality of \(\mathcal{G}^\sigma\). Thus the orientations of \(\{\ast\} \times I\ (\ast \in \bigcup_{i=1}^{n} \gamma_i)\) are either all
positive or all negative. If they are positive (resp. negative), then we put \( \delta(\mathcal{H}) = 1 \) (resp. \( \delta(\mathcal{H}) = -1 \)).

Denote by \( c \) the positive number satisfying \( c \nu|/(\Sigma \times \partial I) = \mu^\sigma \). In the following, the transverse invariant measure of \( \mathcal{H} \) is given by \( c \nu \).

Let \( \alpha \) denote the positive number satisfying \( c \nu((\ast) \times I) = \alpha \int_0^1 \lambda^{-\varepsilon(\sigma)t} \, dt \) \((\ast \in \gamma_i)\). The foliation \( \mathcal{H}(\sigma, \alpha \delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma) \) of \( \Sigma \times \mathbb{R} \) (defined by \( \lambda^{\varepsilon(\sigma)t} \omega^\sigma + \alpha \delta(\mathcal{H}) dt \) in \( (\Sigma - K) \times \mathbb{R} \)) has a transverse invariant measure \( \widetilde{\nu} = \int (\omega^\sigma + \alpha \delta(\mathcal{H}) \lambda^{-\varepsilon(\sigma)t} \, dt) \). The transverse orientation of \( \mathcal{H}(\sigma, \alpha \delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma) \) is given by the positive orientation of \( \lambda^{\varepsilon(\sigma)t} \omega^\sigma + \alpha \delta(\mathcal{H}) dt \).

In the following, we construct a homeomorphism \( h'' : \Sigma \times I \to \Sigma \times I \) satisfying \( h''(\mathcal{H}, c \nu) = \mathcal{H}(\sigma, \alpha \delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)|(\Sigma \times I), \widetilde{\nu}|/(\Sigma \times I) \).

First we define the homeomorphism \( h : \Sigma \times \partial I \to \Sigma \times \partial I \) by the identity map. The transversely oriented measured foliations of \( S^1 \times I \) transverse to both \( S^1 \times \partial I \) and \( \{\ast\} \times I \) (for any \( \ast \in S^1 \)), are determined by the lengths of \( S^1 \times \{0\} \) and \( \{\ast\} \times I \), and the orientations of \( S^1 \times \partial I \) and \( \{\ast\} \times I \) \((\ast \in S^1)\) ([1]). Hence \( h \) extends to \( h' : (\Sigma \times \partial I) \cup (\bigcup_{i=1}^n \gamma_i \times I) \to (\Sigma \times \partial I) \cup (\bigcup_{i=1}^n \gamma_i \times I) \) such that \( h'(\mathcal{H}, c \nu) = \mathcal{H}(\sigma, \alpha \delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma), \widetilde{\nu} \) and \( h'((\ast) \times I) = (\ast) \times I \) for any \( \ast \in \bigcup_{i=1}^n \gamma_i \). By Lemma 9, \( h' \) extends to \( h'' : \Sigma \times I \to \Sigma \times I \) which brings \( \mathcal{H} \) to \( \mathcal{H}(\sigma, \alpha \delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)|(\Sigma \times I) \). Therefore \( \mathcal{H} \) is \( C^0 \) isotopic to \( \mathcal{H}(\sigma, \alpha \delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)|(\Sigma \times I) \) with the boundary fixed.

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