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Meromorphic extension spaces


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MEROMORPHIC EXTENSION SPACES
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The extension of meromorphic maps from a spreaded domain over a Stein manifold to its envelope of holomorphy has been investigated by some authors. This problem for meromorphic functions was proved by Kajiwara and Sakai [9], for meromorphic maps with values in a compact subalgebraic space by Hirschowitz [8].

The extension of meromorphic maps with values in a compact Kahler manifold through an analytic set of codimension $\geq 2$ has been established first by P. Griffiths [6] in a particular case and by Siu [17] in general. In the present paper we shall prove the following two theorems are based on ideas of Dloussky [2].

**THEOREM 2.2** — Let $\theta : X \to Y$ be a Hartogs meromorphic extension map. Assume that $Y$ is a Hartogs meromorphic extension space. Then for every meromorphic mapping $f$ from a domain $D$ over a Stein manifold to $X$, there exists an analytic subset $A$ of codimension at least 2 in $^\wedge D$ such that $f$ extends meromorphically to $^\wedge D \setminus A$. Moreover, if $X$ is a compact Kahler manifold and $Y$ is a Hartogs meromorphic extension space and $\theta$ is a Hartogs meromorphic extension map then $X$ is a meromorphic extension space.

**THEOREM 3.1.** — Let $\theta : X \to Y$ be a finite proper surjective holomorphic map. Then $X$ is a meromorphic extension space if and only if $Y$ has the same property.

Using Theorem 3.1 we prove that every compact non-singular elliptic Kahler surface is a meromorphic extension space. Moreover using Theorem 2.2 we also prove that every complex Lie group is a meromorphic extension space.

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1. Meromorphic extension spaces.

We first recall that a meromorphic map \( f: X \to Y \) is an analytic set \( \Gamma(f) \) in \( X \times Y \) such that the canonical projection \( p(f): \Gamma(f) \to X \) is proper and there exists an open subset \( X(f) \) of \( X \) such that \( \Gamma(f) \cap (X(f) \times Y) \) is the graph of a holomorphic map from \( X(f) \) into \( Y \). \( \Gamma(f) \) is called the graph of \( f \). It is known [14] that in the case where \( X \) is normal, the indeterminacy locus of \( f \)

\[
I(f) = \{ x \in X : f \text{ is not holomorphic at } x \} = \{ x \in X : \dim p(f)^{-1}(x) > 0 \}
\]

is an analytic set of codimension \( \geq 2 \).

We now give the following

**Definition 1.1.** Let \( X \) be a complex space. We say that \( X \) is a meromorphic extension space if the two following conditions are satisfied:

\( H) \) every meromorphic map from a spreaded domain \( D \) over a Stein manifold into \( X \) can be extended meromorphically to \( \uparrow D \), the envelope of holomorphy of \( D \).

\( R) \) Every meromorphic map from \( Z \setminus S \) into \( X \), where \( Z \) is a normal complex space and \( S \) is an analytic set of codimension \( \geq 2 \) in \( Z \) can be meromorphically extended to \( Z \).

In the case where only the condition \( H) \) (resp. \( R) \)) holds, \( X \) is called a Hartogs (resp. Riemann) meromorphic extension space.

We have the following

**Proposition 1.2.** Let \( X \) be a complex space. Then the following conditions are equivalent:

(i) every meromorphic map from a Hartogs domain to \( X \) can be meromorphically extended to its envelope of holomorphy.

(ii) \( \mu_F^X \) is Stein for every Stein manifold \( R \), where \( \mu_F^X \) denotes the spread domain over \( R \) associated to the sheaf of germs of meromorphic maps on \( R \) with values in \( X \).

(iii) \( X \) is a Hartogs meromorphic extension space.
Proof. (i) \rightarrow (ii). By the Docquier-Grauert theorem [3] it suffices to show that $\mu^X_R$ is $p_r$-convex, i.e. every holomorphic embedding $\sigma : H_k(r) \rightarrow \mu^X_R$ can be holomorphically extended to $\Delta^k$, where $\Delta$ denotes the unit disc in $C$ and $H_k(r)$ is given by

$$H_k(r) = \{(z_1, z_2, \ldots, z_k) \in \Delta^k : |z_j| < r, j = 1, 2, \ldots, k-1\} \cup \{(z_1, z_2, \ldots, z_k) \in \Delta^k : |z_k| > 1-r\}, \quad 0 < r < 1,$$

$k = \dim R$.

Let $\mathcal{O}^X_R$ denote the spread domain over $R$ associated to the sheaf of germs of holomorphic maps on $R$ with values in $X$. Obviously $\mathcal{O}^X_R$ is dense open in $\mu^X_R$. Consider the canonical map $e : \mathcal{O}^X_R \rightarrow X$ given by

$$e(g) = g(z) \quad \text{for} \quad z \in R \quad \text{and} \quad g \in (\mathcal{O}^X_R)_z.$$

It is easy to see that $e$ extends meromorphically by definition $\mu^X_R$. Hence $e : H_k(r) \rightarrow X$ is meromorphic. By hypothesis it is extended to a meromorphic map $\hat{e} : \Delta^k \rightarrow X$. Let $p : \mu^X_R \rightarrow R$ denote the locally biholomorphic canonical map and let $\hat{e} : \Delta^k \rightarrow R$ be a holomorphic extension of $p \sigma$. Since every hypersurface in $\Delta^k$ meets $H_k(r)$, it follows that $\hat{e}$ is a locally biholomorphic map. Define now a holomorphic extension $\hat{\sigma} : \Delta^k \rightarrow \mu^X_R$ by

$$\hat{\sigma}(z) = (\hat{e}(z), e(\hat{e}(z)))^{-1} \quad \text{for} \quad z \in \Delta^k$$

where $U_z$ is a neighbourhood of $z$ in $\Delta^k$ on which $\hat{e}$ is biholomorphic. Therefore (ii) is proved.

(ii) \rightarrow (iii). Given a meromorphic map $f : D \rightarrow X$, where $D$ is a spread domain over a Stein manifold. Consider $D$ as a spread domain over $\hat{\Delta}$ with the canonical map $e : D \rightarrow \hat{\Delta}$. By $D(f)$ we denote the envelope of meromorphy of $f$. Then $D(f)$ is a Stein manifold and $f$ has a canonical meromorphic extension $\hat{f}$ to $D(f)$. By the Steiness of $D(f)$ the canonical map $\beta : D \rightarrow D(f)$ can be extended to a holomorphic map $\hat{\beta} : \hat{\Delta} \rightarrow D(f)$. Therefore $\hat{f} \hat{\beta}$ is a meromorphic extension of $f$ to $\hat{\Delta}$.

(iii) \rightarrow (i) is trivial.
2. Meromorphic extension maps.

DEFINITION 2.1. — Let \( \theta : X \to Y \) be a holomorphic map between complex space. We say that \( \theta \) is a Hartogs (resp. Riemann) meromorphic extension map if for each \( y \in Y \) there exists a neighbourhood \( U \) of \( y \) such that \( \theta^{-1}(U) \) is a Hartogs (resp. Riemann) meromorphic extension space. If both conditions of Hartogs and Riemann meromorphic extension are satisfied, then \( \theta \) is called a meromorphic extension map.

THEOREM 2.2. — Let \( \theta : X \to Y \) be a Hartogs meromorphic extension map. Assume that \( Y \) is a Hartogs meromorphic extension space. Then for every meromorphic mapping \( f \) from a domain \( D \) over a Stein manifold to \( X \), there exists an analytic subset \( A \) of codimension at least 2 in \( \wedge D \) such that \( f \) extends meromorphically to \( \wedge D \setminus A \). Moreover if \( X \) is a compact Kahler manifold and \( Y \) is a Hartogs meromorphic extension space and \( \theta \) is a Hartogs meromorphic extension map then \( X \) is a meromorphic extension space.

Proof. — (i) Given \( f : D \to X \) a meromorphic map, where \( D \) is a spread domain over a Stein manifold. By hypothesis we have a following commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{f} & X \\
\downarrow{\beta} & & \downarrow{\theta} \\
D(f) & \xrightarrow{g} & Y \\
\end{array}
\]

where \( g \) is a meromorphic extension of \( \theta \cdot f \).

We show that \( \gamma = \gamma|_{D(f) \setminus \gamma^{-1}(I(g))} : D(f) \setminus \gamma^{-1}(I(g)) \to \wedge D \setminus I(g) \) is locally pseudoconvex. Let \( z \in \wedge D \setminus I(g) \). Take a neighbourhood \( V \) of \( g(z) \) in \( Y \) and a Stein neighbourhood \( U \) of \( z \) in \( \wedge D \setminus I(g) \) such that \( g(U) \subset V \). As in Proposition 1.2 ((i)\( \to \) (ii)), it follows that \( \gamma^{-1}(U) \) is \( p_r \)-convex. Therefore \( \gamma^{-1}(U) \) is Stein [3], and the local pseudoconvexity of \( D(f) \setminus \gamma^{-1}(I(g)) \) over \( \wedge D \setminus I(g) \) is proved.
We now write \( I(g) = \cap Z(h_\alpha) \), where \( h_\alpha \) is holomorphic on \(^\wedge D\) and vanishes on \( I(g) \) and \( Z(h_\alpha) \) denotes the zero-set of \( h_\alpha \). Since \( \gamma_\alpha : D(f)\gamma^{-1}(I(g)) \rightarrow ^\wedge D\cap I(g) \) is locally pseudoconvex and \(^\wedge D\cap Z(h_\alpha) \) is Stein, \( \gamma_\alpha^{-1}(^\wedge D\cap Z(h_\alpha)) \) also is Stein for every \( \alpha \). For each \( \alpha \) consider the holomorphic map \( \beta_\alpha = \beta|_{D\cap Z(h_\alpha)} : D\cap Z(h_\alpha) \rightarrow \gamma_\alpha^{-1}(^\wedge D\cap Z(h_\alpha)) \). Then \( \beta_\alpha \) can be extended to a holomorphic map \(^\wedge \beta_\alpha : (^\wedge D\cap Z(h_\alpha)) = ^\wedge D\cap Z(h_\alpha) \rightarrow D(f) \). By uniqueness the maps \(^\wedge \beta_\alpha \) define a holomorphic map \(^\wedge \beta : ^\wedge D\cap I(g) \rightarrow D(f) \) such that \(^\wedge \beta \circ e = \beta \) on \( D\cap e^{-1}(I(g)) \). The map \(^\wedge f = \bar{f} \cdot ^\wedge \beta : ^\wedge D\cap I(g) \rightarrow X \) is meromorphic and is a meromorphic extension of \( f \).

(ii) Now assume that \( X \) is a compact Kahler manifold and \( Y \) is a Hartogs meromorphic extension space and \( \theta \) is a Hartogs meromorphic extension map. By [17] and since (i) it implies that \( X \) is a meromorphic extension space.

3. Finite proper holomorphic surjections and meromorphic extension spaces.

The aim of this section is to prove Theorem 3.1 on invariance of meromorphic extendibility under finite proper holomorphic surjections.

**Theorem 3.1.** — Let \( \theta : X \rightarrow Y \) be a finite proper holomorphic surjective map. Then \( X \) is a meromorphic extension space if and only if \( Y \) has the same property.

For the proof of the theorem we need following four lemmas.

**Lemma 3.2.** — Let \( \varphi : W \rightarrow Z\setminus S \) be unbranched finite covering map, where \( W, Z \) are complex manifolds and \( S \) is an analytic set in \( Z \) of codimension \( \geq 1 \). Then there exists a following commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{e} & \tilde{W} \\
\varphi \downarrow & & \tilde{\varphi} \\
Z\setminus S & \rightarrow & Z
\end{array}
\]

where \( (\tilde{W}, \tilde{\varphi}, Z) \) is a branched covering map and \( e \) is an open embedding.

**Proof.** — See [5] and [18].
LEMMA 3.3. — Let \( \varphi : G \to D \) be a branched covering map, where \( G \) is a normal complex space and \( D \) is a spread domain over a Stein manifold such that points of \( D \) are separated by holomorphic functions on \( D \). Assume that \( H \) is the branch locus of \( \varphi \) and \( D_0 = D \setminus H \), \( G_0 = G \setminus \varphi^{-1}(H) \), \( \varphi_0 = \varphi|_{G_0} \).

Then there exists an analytic set \( H' \) in \( D \) contained in \( H \) such that \( ^{(D \setminus H')} = ^D \) and a commutative diagram of normal complex spaces

\[
\begin{array}{ccc}
G \setminus \varphi^{-1}(H') & \xrightarrow{\beta} & W \\
\downarrow \varphi_0 & & \downarrow 4 \\
D \setminus H' & \xrightarrow{\varphi_0} & ^D \\
\end{array}
\]

where \( \varphi_0 \), \( 4 \), \( \beta : G \setminus \varphi^{-1}(H') \to \text{Im} \beta \) are branched covering maps, \( \alpha \) is an open embedding and \( \beta^{-1}(\alpha(e(G_0))) = G_0 \).

Proof. — Since \( D \) and \( G \) are normal, it follows that either \( H \) is a hypersurface in \( D \) or \( H = \emptyset \). The case where \( H = \emptyset \) is trivial. Therefore we can assume that \( H \) is a hypersurface. Then there exists an analytic set \( ^H \) in \( ^D \) such that

\( ^D_0 = ^D \setminus ^H \) [2].

Observe that \( ^H \cap D \subseteq H \). We write \( H = (^H \cap D) \cup H' \), where \( H' \) is an analytic set in \( D \) such that \( ^{(D \setminus H')} = ^D \). By [11] the map \( ^\varphi_0 : ^G_0 \to ^D_0 \) is an unbranched covering map and using Lemma 3.2 to \( ^\varphi_0 \), we can construct a commutative diagram

\[
\begin{array}{ccc}
G' & \xrightarrow{\beta} & W \\
\downarrow \varphi' & & \downarrow 4 \\
D' & \xrightarrow{\varphi_0} & ^D \\
\end{array}
\]

where \( D' = D \setminus H' \), \( G' = G \setminus \varphi^{-1}(H') \) and \( \varphi' = \varphi|_{G'} \), in which \( 4 \) is a branched covering map of the normal complex space \( W \) onto \( ^D \) and...
\(\alpha\) is an open embedding. Put \(^\wedge \alpha = \alpha e\). We shall prove \(^\wedge \alpha\) can be extended to a holomorphic map \(\beta\) from \(G'\) to \(W\). Since the Steiness is invariant under finite proper holomorphic surjections [13], \(W\) is Stein. Thus by the normality of \(G'\) it suffices to show that \(^\wedge \alpha\) is locally compact on \(G'\), i.e., for every \(z \in G'\) there exists a neighbourhood \(U\) of \(z\) such that \(^\wedge \alpha(U \cap G_0)\) is relatively compact in \(W\). Assume that \(z_0 \in \varphi'^{-1}(H')\) and \(\{z_n\} \subset G_0\) converging to \(z_0\).

Then
\[
\lim_n 4^\wedge \alpha(z_n) = \lim \varphi'(z_n) = \varphi_o(z_0) \in D' \hookrightarrow ^\wedge D.
\]

Thus from property of \(4\), it follows that \(\{^\wedge \alpha(z_n)\}\) is relatively compact in \(W\). This yields the local compactness of \(^\wedge \alpha\).

Let \(\beta : G' \to W\) be a holomorphic extension of \(^\wedge \alpha\). Since \(\varphi'\) and \(4\) are finite proper maps and \(D'\) is contained in \(^\wedge D\) as an open subset, it is easy to see that \(\beta : G' \to \beta(G')\) is finite proper. Hence by the normality of \(W\) and by the equality \(\dim G' = \dim W\), it follows that \(\beta(G')\) is open in \(W\) and \(\beta : G' \to \beta(G')\) is a branched covering map. Finally, if \(^\wedge \alpha(z_0) = \beta(z_1)\), where \(z_0 \in G_0\) and \(z_1 \in G'\), then
\[
\varphi'(z_1) = 4\beta(z_1) = 4^\wedge \alpha(z_0) = \varphi_o(z_0)\]

This implies \(z_1 \in G_0\). Hence \(\beta^{-1}(^\wedge \alpha(G_0)) = G_0\).

The lemma is proved.

**Lemma 3.4.** — Let \(X\) be a meromorphic extension space and \(Z\) a normal Stein space. Assume that \(H\) is a hypersurface of \(Z\) and \(G\) is an open subset of \(Z\) meeting every irreducible branch of \(H\). Then every meromorphic map \(f : (D \setminus H) \cup G \to X\) can be meromorphically extended to \(Z\).

**Proof.** — Since \(Z\) is normal, \(\text{codim } S(Z) \geq 2\) [4], where \(S(Z)\) denotes the singular locus of \(Z\). We write by the Steiness of \(Z^S(Z)\) in the form
\[
\begin{align*}
S(Z) &= \cap \{Z(h) : h \text{ is holomorphic on } Z, h|_{S(Z)} = 0 \text{ and } \\
& \quad h \neq 0 \text{ on every irreducible branch of } H\}.
\end{align*}
\]

From hypothesis, it suffices to show that for every such \(h\) the map \(f_h = f|_{Z(h)\setminus H}\), where \(Z_h = Z \setminus Z(h)\), can be meromorphically extended on \(Z_h\). Put \(G_h = G \setminus Z(h)\) and \(H_h = H \cap Z_h\). Then \(G_h\) also meets every
irreducible branch of $H_h$. Consider the meromorphic map $f_h|_{(Z_h \setminus H_h) \cup G_h}$. Since $\mathcal{H}(Z_h \setminus H_h) \cup G_h = Z_h$ [2] it follows that $f_h|_{(Z_h \setminus H_h) \cup G_h}$ can be extended to a meromorphic map $f_h^*$ to $Z_h$.

The lemma is proved.

**Lemma 3.5.** — Let $\pi : Z \to W$ be a branched covering map and $f : Z \to X$ a meromorphic map which can be factorized through $\pi|_{\pi^{-1}(V)}$ for some non-empty open subset $V$ of $W$. Then $f$ can be factorized through $\pi$.

**Proof.** — Let $H$ denote the branch locus of $\pi$. It is easy to check that there exists a holomorphic map $g$ from $W \setminus (H \setminus \pi(I(f)))$ to $X$ such that $g\pi = f$ on $\pi^{-1}(W \setminus (H \cup \pi(I(f))))$. Since $\pi \times \text{id} : Z \times X \to W \times X$ is proper,

$$\Gamma(g) = (\pi \times \text{id})\Gamma(f)$$

is an analytic set in $W \times X$. Hence from property of $\pi$ and $p(f)$, it follows that $\Gamma(g)$ defines a meromorphic map $g_1$ on $W$ such that $g_1 \cdot \pi = f$.

The lemma is proved.

We now can prove Theorem 3.1.

a) First prove sufficiency of the theorem.

(i) Given $f : D \to X$ a meromorphic map, where $D$ is a spread domain over a Stein manifold. From hypothesis we have a following commutative diagram

$$
\begin{array}{ccc}
D & \longrightarrow & X \\
\downarrow \beta & & \downarrow \gamma \\
D(f) & \longrightarrow & Y \\
\end{array}
$$

where $g$ is a meromorphic extension of $\theta \cdot f$.

As in Theorem 2.2, $\beta|_{D,\pi^{-1}(I(g))}$ can be extended to a holomorphic map $\mathcal{H}^\beta : D \setminus I(g) \to D(f)$. Put $A = (\text{id} \times \theta)^{-1}(p(g)^{-1}(I(g)))$. Then $\Gamma(\mathcal{H}^\beta) \subset (D \setminus I(g)) \times X \subset (D(f) \times X) \setminus A$, where $\mathcal{H}^\beta = \mathcal{H}^\beta |_{D(f)}$, and is closed in $(\gamma(D(f)) \times X) \setminus A$. Indeed, let $\{(x_n, z_n)\} \subset \Gamma(\mathcal{H}^\beta)$ converge to
(\(x_0, z_0\)) \in \gamma(D(f)) \times X \setminus A. Since (x_0, z_0) \in A, (id \times \theta)(x_0, z_0) = (x_0, \theta z_0) \in p(g)^{-1}(I(g)). If x_0 \in I(g), then (x_0, z_0) \in (\wedge D \setminus I(g)) \times X. Hence (x_0, z_0) \in \Gamma(\wedge f). In the case where x_0 \in I(g), we have (x_0, \theta z_0) \in I(g). This is impossible, because of the relation \(\Gamma(\wedge f) \supset \{(x_n, z_n)\} \to (x_0, \theta z_0) \in I(g). Therefore \Gamma(\wedge f) is closed in \(\gamma(\wedge D(f)) \times X \setminus A. Since \dim \Gamma(\wedge f) = \dim \wedge D > \dim A, by the Remmert-Stein theorem [7] \Gamma(\wedge f) is an analytic set in \wedge D \times X. Since \theta is proper, it follows that \Gamma(\wedge f) defines a meromorphic extension of \(f to \wedge D.

(ii) Let now \(f: Z \setminus S \to X, where Z is a normal complex space and S is an analytic set in Z of codimension \(\geq 2. From the Riemann meromorphic extendibility of \(Y we have a following commutative diagram

\[
\begin{array}{ccc}
Z \setminus S & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
Z & \xrightarrow{g} & Y
\end{array}
\]

Similarly as in (i), where \(\wedge D \setminus I(g) and \wedge f\) are replaced by \(Z, Z \setminus (I(g) \cup S)\) and \(f\) respectively we obtain a meromorphic extension \(\wedge f\) of \(f to Z.\)

b) We now prove necessity of the theorem.

(i) Let \(f\) be a meromorphic map from a spread domain \(D over a Stein manifold to X. By Proposition 1.2 we can assume that \(D is a Hartogs domain. Consider the commutative diagram

\[
\begin{array}{ccc}
G_1 & \xrightarrow{g_1} & X \\
\downarrow \varphi_1 & & \downarrow \theta \\
D_1 & \xrightarrow{f_1} & Y
\end{array}
\]

where \(D_1 = D \setminus I(g), G_1 = (D_1 x_1 x)_{red} is the fiber product, f_1 = f|_{D_1} and \varphi_1, g_1 are canonical projections.

Without loss of generality we may assume that \(G_1 is normal. Observe that \(\varphi_1 is a branched covering map. Let \(H_1 denote the branched locus of \(\varphi_1. Since \dim H_1 > \dim I(f), it follows that \(\bar{H_1} is an analytic
set in \( D \). Using Lemma 3.2 to the unbranched covering map \( \varphi_1: G_1 \setminus \varphi_1^{-1}(H_1 \cup I(f)) \to D_1 \setminus (\overline{H_1} \cup I(f)) \) we have a following commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
G & \xrightarrow{g} & X \\
\downarrow{\varphi} & & \downarrow{0} \\
D & \xrightarrow{f} & Y \\
\end{array}
\end{array}
\]

in which \( \varphi \) is a branched covering map and \( G \) is normal. Since \( \dim G_1 = \dim (\varphi \times \theta)^{-1}\Gamma(f) \geq \dim (\varphi \times \theta)^{-1}p(f)^{-1}(I(f)) \), by the Remmert-Stein theorem [7], \( \Gamma(g_1) \) is an analytic set in \( G \times X \). Hence by property of \( \theta \), it defines a meromorphic extension of \( g \) on \( G \) such that \( \theta g = f \). In notations of Lemma 3.3 we have a following commutative diagram of normal complexe spaces

\[
\begin{array}{c}
\begin{array}{ccc}
G' & \xrightarrow{\beta} & W \\
\downarrow{\varphi'} & & \downarrow{4} \\
G_0 & \xrightarrow{\alpha} & D' \\
\downarrow{\varphi_0} & & \downarrow{\hat{\varphi}_0} \\
D_0 & \xrightarrow{\hat{\varphi}_0} & \hat{\varphi}_0^{-1}(\hat{D}_0) \\
\end{array}
\end{array}
\]

in which \( \varphi_0 \) and \( \hat{\varphi}_0 \) are unbranched covering maps, \( \varphi' \), \( 4 \), \( \beta : G' \to \beta(G') \) are branched covering maps. Moreover \( G_0 = \beta^{-1}(\text{ze}(G_0)) \). Thus \( g|_G \) can be meromorphically factorized through \( \beta : G' \to \beta(G') \). Let \( \hat{g}_0 \) be a meromorphic extension of \( g|_{G_0} \) on \( \hat{\varphi}_0^{-1}(\hat{G}_0) \) and \( \hat{g} \) a meromorphic map on \( \beta(G') \) such that \( \hat{g}\beta = g|_{G'} \). Define a meromorphic map \( g_2 \) from \( \hat{\varphi_0}^{-1}(\hat{G}_0) \cup \beta(G') \) into \( X \) by

\[
g_2 = \hat{g}_0 \text{ on } \hat{\varphi}_0^{-1}(\hat{G}_0) \quad \text{and} \quad g_2 = \hat{g} \text{ on } \beta(G').
\]

Since \( 4 \) is finite proper and every irreducible branch of \( \hat{H} \) meets \( D' \), it follows that this holds for \( 4^{-1}(\hat{H}) \) and \( \beta(G') \). Thus by Lemma 3.4 we have a meromorphic extension \( g_3 \) of \( g_2 \) on \( W \). From Lemma 3.5, \( g_3 \) can be meromorphically factorized through \( 4 \). Hence \( f \) is extended to a meromorphic map to \( \hat{D} \).
(ii) Finally we show that $Y$ has the Riemann meromorphic extension property. Given $f: Z \setminus S \to Y$ a meromorphic map, where $Z$ is a normal complex space and $S$ is an analytic set in $Z$ of codimension $\geq 2$ which can be assumed to contain the singular locus of $Z$.

As in (i) we can construct a following commutative diagram of normal complex spaces

\[
\begin{array}{c}
G_1 \\
\downarrow g_1 \\
G_0 \\
\downarrow \phi_1 \\
Z \setminus S \\
\downarrow g_0 \\
\downarrow f \\
(Z \setminus S) \setminus I(f) \\
\downarrow f_0 \\
Y
\end{array}
\]

where $\phi_0$, $\phi_1$ are branched covering maps and $g_0$, $g_1$ are meromorphic maps. The problem is local without loss of generality we may assume that there exists a branched covering map $\gamma: Z \to \Delta^n$, $n = \dim Z$. Let $H$ denote the branch locus of $\phi_1$.

Then $\tilde{H}$ is an analytic set in $Z$ because of the inequality $\text{codim } I(f) \geq 2$. Take a hypersurface $\tilde{H}$ in $\Delta^n$ containing the branch locus of $\gamma$ such that $\gamma(S \cup H) \subset \tilde{H}$. Using Lemma 3.3 we give a following commutative diagram

\[
\begin{array}{c}
G_1 \\
\downarrow \beta \\
W \\
\downarrow \alpha \\
\Delta^n \setminus \gamma(S) \\
\downarrow 4 \\
\Delta^n \setminus \tilde{H}
\end{array}
\]

where $\eta = \gamma \phi_1$, $4$, $\beta: G_1 \to \beta(G_1)$ are branched covering maps.

Obviously $\beta^{-1}(\alpha(G_1 \setminus \eta^{-1}(\tilde{H}))) = G_1 \setminus \eta^{-1}(\tilde{H})$. Thus $g_1$ can be meromorphically factorized through $\beta: G_1 \to \beta(G_1)$. Hence $g_0$ and $g_1$ induce a meromorphic map $g_2$ on $G_1 \setminus \eta^{-1}(\tilde{H}) \cup \beta(G_1)$ with values in $X$. Since
every irreducible branch of $\tilde{H}$ meets $\Delta^0 \gamma(S)$, it follows that this holds for $4^{-1}(\tilde{H})$ and $\beta(G_1)$. By Lemma 3.4, $g_2$ can be extended to a meromorphic map $g_3$ on $W$. Thus from Lemma 3.5 we give a meromorphic extension of $f$ to $\Delta^n$.

Theorem 3.1 is completely proved.

4. Some applications.

We first recall that an elliptic surface is a compact regular surface $V$ equipped with a holomorphic map $\theta$ from $V$ onto a non-singular curve $C$ such that $\theta^{-1}(x)$ is an elliptic curve outside a finite set in $C$.

Using now Theorem 3.1 we prove the following.

**Theorem 4.1.** Let $V$ be an elliptic Kahler surface. Then $V$ is a meromorphic extension space.

**Proof.** From a result of Siu [17], $V$ is a Riemann meromorphic extension space. Thus it remains to prove that $V$ has the Hartogs meromorphic extension property.

(i) In [12] Kodaira constructed for $V$ a branched covering map $\alpha$ from an elliptic surface $\tilde{V}$ on $V$ such that for each $x \in C$ there exists a sufficiently small disc $U_x$ containing $x$ for which $(\theta \alpha)^{-1}(U_x)$ is biholomorphic to a locally pseudoconvex open subset of a projective surface $P_x$. Put $\eta = \theta \cdot \alpha$. Given $f: D \to \eta^{-1}(U_x)$ a meromorphic map, where $D$ is a spread domain over a Stein manifold.

Let $^\wedge f : ^\wedge D \to P_x$ be a meromorphic extension of $f|_{D \cap U_x}$. Put

$$G = ^\wedge f_0^{-1}(\eta^{-1}(U_x)), \text{ where } ^\wedge f_0 = ^\wedge f|_{^\wedge D \cap I(\wedge f)}.$$  

We may suppose that $D$ is a Hartogs domain. Since $D \setminus I(f) \subset G$ we have $^\wedge G = ^\wedge D$. Let now $G \neq ^\wedge D \setminus I(\wedge f)$. Then we can find a point $z_0 \in \partial G$ in $^\wedge D \setminus I(\wedge f)$ and two Stein neighbourhoods of $z_0$ and $^\wedge f_0(z_0)$ in $^\wedge D \setminus I(\wedge f)$ and $P_x$ respectively such that $^\wedge f_0(U) \subset W$ and $z_0 \in ^\wedge (U \cap G)$. Since $W \cap \eta^{-1}(U_x)$ is Stein and $^\wedge f_0(U \cap G) \subset W \cap \eta^{-1}(U_x)$, it follows that $^\wedge f_0(z_0) \in W \cap \eta^{-1}(U_x)$. This yields $z_0 \in G$. Hence $G = ^\wedge D \setminus I(\wedge f)$. On the other hand, since $\alpha^\wedge f_0$ and $\eta^\wedge f_0$ are extended to meromorphic maps $g : ^\wedge D \to V$ and $h : ^\wedge D \to U_x$ respectively. We have $\theta g = h$.

It is easy to see that $\Gamma(\wedge f_0)$ is contained and closed in $(\text{id} \times \alpha)^{-1}\Gamma(g) \times (\text{id} \times \alpha)^{-1}\rho(g)^{-1}(I(g))$, by the Remmert-Stein theorem, $\Gamma(\wedge f_0)$ defines a meromorphic extension $\tilde{f}$ of $f$. [528] LE MAU HAI AND NGUYEN VAN KHUE
From the relation $\eta f = f$, it follows that $\tilde{f}$ induces a meromorphic extension of $f$ with values in $\eta^{-1}(U_\lambda)$.

(ii) Let now $f$ be a meromorphic map from a spread domain $D$ over a Stein manifold into $\tilde{V}$. Consider the following commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{f} & \tilde{V} \\
\downarrow{\beta} & & \downarrow{\eta} \\
\hat{D} & \xrightarrow{\gamma} & C
\end{array}
\]

By (i) as in Theorem 2.1 we can find a holomorphic extension $^\wedge \beta$ of $\beta|_{D \setminus I(g)}$ on $^\wedge D \setminus I(g)$. Let $^\wedge f_1 : ^\wedge D \to V$ be a meromorphic extension of $f_1 = \alpha f^\wedge \beta : D \setminus I(g) \to V$. Then as in Theorem 2.1, it follows that $\Gamma(f^\wedge \beta)$ defines a meromorphic extension of $f$. Hence $\tilde{V}$ is a Hartogs meromorphic extension space.

(iii) Given a meromorphic map $f$ from $Z \setminus S$ into $\tilde{V}$, where $Z$ is a normal complex space and $S$ is an analytic set in $Z$ of codimension $\geq 2$. Let $g : Z \to V$ be a meromorphic extension of $\alpha f$. Then as in (i) we infer that $\Gamma(f)$ defines a meromorphic extension of $f$.

(iv) From (ii) and (iii), $\tilde{V}$ is a meromorphic extension space. Hence by Theorem 3.1, $V$ is a meromorphic extension space.

The theorem is proved.

**Theorem 4.2.** — Every complex Lie group is a meromorphic extension space.

**Proof.** — Let $G$ be a complex Lie group.

(i) Given $f : D \to G$ a meromorphic map, where $D$ is a spread domain over a Stein manifold. Since $\text{codim} I(f) \geq 2$, $f|_{D \setminus I(f)}$ can be holomorphically extended to $^\wedge D [1]$. Thus $G$ is a Hartogs meromorphic extension space.
(ii) Given now \( f \) a meromorphic map from \( Z \setminus S \rightarrow G \), where \( Z \) is a normal complex space and \( S \) is an analytic set in \( Z \) of codimension \( \geq 2 \). Let \( \varphi \) be a plurisubharmonic exhaustion function \([10]\) on \( G \). Since codim \( I(f) \geq 2 \) and codim \( S \geq 2 \), \( \varphi f \) is plurisubharmonic on \( Z \) \([8]\). By \([19]\) there exists a holomorphic bundle map \( \theta \) from \( G \) onto a complex torus \( T \) such that the fibers of \( \theta \) are Stein manifolds. Consider the holomorphic map \( \varphi f|_{(Z,S)\setminus R} \). Then, by the Kahlerness of the torus \( T \), \( \theta f \) is meromorphic on \( Z \) \([17]\). Let \( \gamma : Z \rightarrow Z \) be the Hironaka singular resolution of \( Z \). By (i), \( h = \theta f \gamma \) is holomorphic on \( Z \). For each \( z_0 \in \gamma^{-1}(S) \) take the two neighbourhoods \( U \) and \( V \) of \( z_0 \) and \( h(z_0) \) respectively such that \( h(U) \leq V \) and \( \theta^{-1}(V) \) is a Stein manifold. Then we have \( f \gamma(U \setminus \gamma^{-1}(S)) \leq \theta^{-1}(V) \). By the upper semi-continuity of \( \varphi f \gamma \) on \( Z \) and since \( \varphi \) is an exhaustion function on \( G \) it follows that \( f \gamma|_{U \setminus h^{-1}(S)} \) can be extended holomorphically at \( z_0 \). Since \( z_0 \) is arbitrary \( f \gamma \) is extended holomorphically to \( Z \). Then \( (f \gamma) \gamma^{-1} \) is a meromorphic extension of \( f \).

The theorem is proved.

**BIBLIOGRAPHY**


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