N. Hashiguchi

PL representations of Anosov foliations


<http://www.numdam.org/item?id=AIF_1992__42_4_937_0>
PL-REPRESENTATIONS OF ANOSOV FOLIATIONS

by Norikazu HASHIGUCHI

0. Introduction.

Let $\Sigma_g$ be the closed oriented surface of genus $g(\geq 2)$ with a hyperbolic metric. The geodesic flow of the unit tangent vector bundle $T_1\Sigma_g$ of $\Sigma_g$ is an Anosov flow. To study this flow, Fried gave a good Birkhoff section for it, and Ghys showed that it is obtained from the suspension flow of some hyperbolic toral automorphism by a certain Dehn surgery (see [F] and [Gh]). In other words, the geodesic flow on $T_1\Sigma_g$ restricted to the complement of the $4g + 4$ closed orbits $\{\pm G_1, \pm G_2, \pm G_3, \ldots, \pm G_{2g+2}\}$ is topologically equivalent to the suspension of $A_g : T^2 \rightarrow T^2$ restricted to the complement of the $4g + 4$ closed orbits $\{O_l^m\}_{l=0,1,2,\ldots,2g+1, m=0,1}$ (see §1). This toral automorphism $A_g$ and the orbits $\{O_l^m\}$ are determined by the author as well as the type of the Dehn surgery (see [Ha1]).

Since the suspension of the (un)stable linear foliation of the torus by the hyperbolic toral automorphism is a transversally affine foliation, certain Dehn surgeries along leaf curves with nontrivial holonomy give rise to a transversally PL foliation (see §1). Since the (un)stable foliation of the geodesic flow a $T_1\Sigma_g$ is transverse to the fibre of the projection $T_1\Sigma_g \rightarrow \Sigma_g$, this transversally PL foliation can be seen as a PL foliated $S^1$-bundle. In other words, there exists a homomorphism

$$\Phi_g : \pi_1(\Sigma_g) \rightarrow PL_+(S^1)$$

such that the (un)stable foliation of the geodesic flow on $T_1\Sigma_g$ is topologically conjugate to the suspension of $\Phi_g$ (see [Gh]). Here $PL_+(S^1)$ is the

Key words : Anosov foliation – Total holonomy homomorphism – Godbillon-Vey invariant.
A.M.S. Classification : 57R30 – 57R32.
The purpose of this paper is to describe the homomorphism $\Phi_g$ concretely and study various properties of this homomorphism.

The organization of this paper is as follows. In §1, first we review how the Dehn surgery of transversally affine foliation gives rise to a transversally PL foliation. Secondly, we review the identification between $T_1\Sigma_g - \{ \pm G_1, \pm G_2, \pm G_3, \ldots, \pm G_{2g+2} \}$ and the suspension of $A_g$ with $\{ O^m \}$ deleted, and we see how the fibres of $T_1\Sigma_g \to \Sigma_g$ are in the suspension of $A_g$. Then we determine the holonomy homomorphism $\Phi_g$. In order to determine $\Phi_g$, it is enough to determine only one PL homeomorphism $f_g$. Since this $f_g$ satisfies

$$\left\{ f_g \circ T \left( -\frac{1}{g+1} \right) \right\}^{g+1} = \left\{ f_g \circ T \left( -\frac{1}{2(g+1)} \right) \right\}^{2(g+1)} = 1,$$

we can determine the holonomy on the generators of $\pi_1(\Sigma_g)$, and we verify that these holonomies on the generators respect the relations of $\pi_1(\Sigma_g)$. Here, $T(\theta)(\theta \in \mathbb{R}/\mathbb{Z} = S^1)$ denotes the rotation by $\theta$.

In §2, we calculate the discrete Godbillon-Vey invariant of $\Phi_g$. This invariant is defined by Ghys and Sergiescu (see [GS] and [Gh]). They gave the 2-cocycle representing the discrete Godbillon-Vey class $\overline{gv} \in H^2(PL_+(S^1); \mathbb{R})$. So we evaluate $\Phi_g^*(\overline{gv}) \in H^2(\pi_1(\Sigma_g); \mathbb{R})$ on the fundamental class of $\pi_1(\Sigma_g)$. The value is $-4(g+1)(\log \lambda_g)^2$. Let $D_+(S^1)$ denote the homomorphisms of class P of $S^1$ (see [He]). The usual Godbillon-Vey class $gv \in H^2(Diff^2_+(S^1); \mathbb{R})$ as well as $\overline{gv} \in H^2(PL_+(S^1); \mathbb{R})$ is extended to $H^2(D_+(S^1); \mathbb{R})$ (see [Gh]). As a corollary of the above calculation, we obtain again the result of Ghys which says that each $\alpha gv + \beta \overline{gv} \in H^2(D_+(S^1); \mathbb{R})$ ($\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \neq 0$) is not a topological invariant.

In §3, we give remarks related to our result. The presentation of $\pi_1(\Sigma_g)$ we used to describe $\Phi_g$ is interesting in itself. We exhibit a fundamental domain in the Poincaré disk for this presentation. Then we geometrically show that $\Phi_g$ factors through a homomorphism

$$\phi_g : \Gamma_g \to PL_+(S^1),$$

where $\Gamma_g$ is a triangle group $\Gamma(g+1, 2g+2, 2g+2)$. 
We study the deformation of \( \phi_g \) and \( \Phi_g \) in [Ha2]. We saw the foliation obtained from transversally affine foliation by Dehn surgery along leaf curves with nontrivial holonomy is transversally PL. The Godbillon-Vey invariant and the discrete Godbillon-Vey invariant of a transversally affine foliation are 0. By the result of the above calculation of the discrete Godbillon-Vey invariant, one would conjecture that each \((1, l)\)-Dehn surgery along the closed orbit \( \{O_i^m\} \) decreases \( \overline{gv} \) by \((\log \lambda_g)^2\). We will also see that this conjecture is true by using the result of Greenberg. We will show it in a future paper.

Finally, the author would like to thank Professor T. Tsuboi for helpful advice and continuous encouragement.

1. Geodesic flows on \( \Sigma_g \).

Let \( \Sigma_g \) be a closed orientable surface of genus \( g(\geq 2) \). We consider a Riemannian metric with constant negative curvature \(-1\) on it. Let \( F_t \) denote the geodesic flow on the unit tangent vector bundle \( T^1 \Sigma_g \) and \( \pi \), the projection \( T^1 \Sigma_g \to \Sigma_g \).

Fried constructed the Birkhoff section \( S \) for \( F_t \) in [F] as follows. Let \( \pm G_i \subset \Sigma_g \) be the oriented closed geodesics shown in Figure 1. Then \( G_i = \pi(\pm G_i) \subset \Sigma_g \) is a closed geodesic. \( \{G_1, G_2, G_3, \ldots, G_{2g+2}\} \) divide \( \Sigma_g \) into four \( 2g+2 \) gons \( P_1, P_2, P_3, P_4 \) where \( P_1 \) and \( P_2 \) are named so that they intersect at only \( 2g + 2 \) vertices. Let \( p_i \in \Sigma_g \) be \( G_i \cap G_{i+1} \) where \( i = 1, 2, 3, \ldots, 2g+2, G_{2g+3} = G_1 \) (see Figure 1). For \( i = 1, 2 \), we choose a family \( C_i \) of convex smooth simple closed curves which fill the interior of \( P_i \) with one singularity \( o_i \) deleted. Let \( S \) be the closure of the set of unit vectors which are tangent to the curves belonging to \( C_i \). \( \partial S = \bigcup_{i=1}^{2g+2}(+G_i) \cup \bigcup_{i=1}^{2g+2}(-G_i) \), \( \pi^{-1}(o_i) \subset S(i = 1, 2) \) and \( S \) is diffeomorphic to a torus with \( 4g + 4 \) open disks deleted. Let \( b_1 \subset S \) denote \( \pi^{-1}(o_1) \) and \( b_2 \subset S \), a component of \( S \cap \pi^{-1}(b_2) \) where \( b_2 \) is the closed geodesic through \( o_1, p_1, o_2, p_{g+2} \) and \( o_1 \). If we take \( (b_1, b_2) \) as the basis of \( S \), then the first return map of \( F_t \) about \( S \) is semi conjugate to the hyperbolic toral automorphism induced by

\[
A_g = \begin{pmatrix} 2g^2 - 1 & 2g(g - 1) \\ 2g(g + 1) & 2g^2 - 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).
\]
Similarly, we can construct the Birkhoff section $S'$ for $F_t$ over $P_3 \cup P_4$ from families $C_3$ and $C_4$ which are mapped on $C_2$ and $C_1$ by the reflection of a plane $V$, respectively. Here, $V$ divides $\Sigma_g$ into $P_1 \cup P_3$ and $P_2 \cup P_4$. $F_t$ can be constructed from the matrix $A_g$ as follows (see [Ha1]). The matrix $A_g$ acts on $T^2$ as a diffeomorphism $\overline{A_g}$ and let $M$ be the torus bundle over $S'$ with monodromy $\overline{A_g}$, and $\phi_t$, the suspension flow of $\overline{A_g}$. More explicitly, let

\[ \tilde{\phi}_t : \mathbb{R}^3 \to \mathbb{R}^3 \]

be the flow defined by $\tilde{\phi}_t(x, y, z) = (x, y, z + t)$. Consider the equivalence relation $\sim$ on $\mathbb{R}^3$ generated by $(x+1, y, z) \sim (x, y, z), (x, y+1, z) \sim (x, y, z)$ and $(x, y, z + 1) \sim \left( t \begin{pmatrix} A_g(x) \\ y \end{pmatrix}, z \right)$. Then $M$ is the quotient space $\mathbb{R}^3/\sim$, and we obtain the induced Anosov flow

\[ \phi_t : M \to M. \]

Let

\[ q : \mathbb{R}^3 \to M \]
be the quotient map. We construct an Anosov flow
\[ \varphi_t : T_1 \Sigma_g \to T_1 \Sigma_g \]
from the suspension flow \( \phi_t \) by \((1, 1)\)-Dehn surgeries along \( 4g + 4 \) closed
orbits \( \{O_l^m\} \) of period 1. Here,
\[ O_l^m = q \left( \left\{ \left( \frac{l}{2(g + 1)}, \frac{m}{2}, t \right) \in \mathbb{R}^3; t \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\} \right) \]
\( (l = 0, 1, \ldots, 2g + 1, m = 0, 1) \).

These closed orbits correspond to \( 4g + 4 \) oriented closed geodesics
\( \{ \pm G_1, \pm G_2, \ldots, \pm G_{2g+2} \} \)
shown in Figure 1. Then in [Ha1], we showed \( F_t \) is topologically equivalent
to \( \varphi_t \).

We will show that the unstable foliation \( \mathcal{F}_u \) of \( \varphi_t \) is transversally
piecewise linear. It is easy to see that the unstable foliation \( \mathcal{F}_u \) of \( \phi_t \) is
induced from a linear foliation on a torus and transversally affine. More
precisely, the holonomy pseudogroup of \( \mathcal{F}_u \) is generated by the affine maps
\[ y = (\lambda_g)^{\sigma} x + \tau (\sigma \in \mathbb{Z}, \tau \in \mathbb{R}) \]
where \( \lambda_g = 2g^2 - 1 + 2g\sqrt{g^2 - 1} \) is the larger eigenvalue of \( A_g \). The
leaves of \( \mathcal{F}_u \) are diffeomorphic to \( \mathbb{R}^2 \) or \( S^1 \times \mathbb{R} \). Each \( O_l^m \) is a leaf
curve in a cylindrical leaf \( L_l^m \) and the holonomy of \( O_l^m \) is an affine map
\( y = \lambda_g^{-1} x \). There is a neighborhood \( N \) of \( O_l^m \) such that it is homeomorphic
to \( [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \times S^1, \{0\} \times \{0\} \times S^1 = O_l^m \) and \( \{0\} \times [-\varepsilon, \varepsilon] \times S^1 \)
is a component of \( N \cap L_l^m \) where \( \varepsilon \) is a very small positive number and
\( S^1 = \mathbb{R}/\mathbb{Z} \). The holonomy of \( L_l^m \) along \( O_l^m \) is \( y = \lambda_g^{-1} x \) and the holonomy
of \( L_l^m \) along the leaf curve \( \{(0, t, 0) \in N; -\varepsilon \leq t \leq \varepsilon\} \) is the identity.

The \((1, 1)\)-Dehn surgery along \( O_l^m \) used in [F] is as follows. Let a torus
\( T_l^m = \{(\theta, u) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\} \) be the blowing-up of \( O_l^m \) with a flow \( \phi_t^{(l,m)} \)
on \( T_l^m \) induced from \( \phi_t \). We divide \( T_l^m \) into the circles
\[ S^1_r = \left\{ (\theta, r) \in T_l^m; 0 \leq \theta \leq \frac{1}{2} \right\} \cup \left\{ (\theta, 2\theta + r - 1) \in T_l^m; \frac{1}{2} \leq \theta \leq 1 \right\} \]
\( (0 \leq r \leq 1) \).
We can perturb the latter part of $S^l$ in order that $S^l_r$ is a smooth circle and transverse to $\phi_t^{(l,m)}$. Contracting each $S^l_r$ to a point, we obtain a closed orbit $O^m_l$ in a cylindrical leaf $L^m_l$. There is also a neighborhood $N'$ of $O^m_l$ such that it is induced from $N$ and it has the same properties as $N$. The holonomy of $L^m_l$ along $O^m_l$ is $y = \lambda^{-1}_g x$. The holonomy of $L^m_l$ along $\{(0,t,0) \in N'; -\varepsilon \leq t \leq \varepsilon\}$ is piecewise linear as follows. By the choice of $S^l_0$, the rectangle $\{(s,t,0) \in N'; 0 \leq s \leq \varepsilon, -\varepsilon \leq t \leq \varepsilon\}$ is induced from the rectangle $\{(s,t,0) \in N; 0 \leq s \leq \varepsilon, -\varepsilon \leq t \leq \varepsilon\}$. And the rectangle $\{(s,t,0) \in N'; -\varepsilon \leq s \leq 0, -\varepsilon \leq t \leq \varepsilon\}$ is induced from

\[\{(s,t,0) \in N; -\varepsilon \leq s \leq 0, -\varepsilon \leq t \leq \varepsilon\} \cup \{(s,0,\theta) \in N; -\varepsilon \leq s \leq 0, -1 \leq \theta \leq 0\} \cup \{(s,t-1) \in N; -\varepsilon \leq s \leq 0, 0 \leq t \leq \varepsilon\} \]

So the holonomy along $\{(0,t,0) \in N'; -\varepsilon \leq t \leq \varepsilon\}$ is

\[y = \begin{cases} x & \text{for } x \geq 0 \\ \lambda_g x & \text{for } x \leq 0. \end{cases}\]

Hence, $F^u_l$ is transversally piecewise linear in the neighborhood of $O^m_l$. It implies that $F^u_l$ is transversally piecewise linear.

Similarly in general, operating $(1,n)$-Dehn surgeries along finite leaf curves of a transversally affine foliation, we obtain a transversally piecewise linear foliation.

Since the unstable foliation of $F_t$ is transverse to the fibre of the projection $\pi : T_1 \Sigma_g \to \Sigma_g$, the transversally piecewise linear foliation $F^u_l$ can be seen as a PL-foliated $S^1$ bundle. So the total holonomy of $F^u_l$

\[\Phi_g : \pi_1(\Sigma_g) \to PL_+(S^1)\]

is determined. Now we consider the spaces

\[L^3 = [0, 1] \times [0, 1] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \subset \mathbb{R}^3\]

and

\[L^3_\phi = L^3 - \left( \bigcup_{l=0}^{2g+1} q^{-1}(O^0_l) \right) \cup \left( \bigcup_{l=0}^{2g+1} q^{-1}(O^1_l) \right).\]
By the above construction, \( T^1 \Sigma_g^* = T^1 \Sigma_g - \{ \pm G_1, \pm G_2, \ldots, \pm G_{2g+2} \} \) is obtained from \( \text{L}^3_* \) by identifying opposite faces of \( \text{L}^3_* \) by the equivalence relation \( \sim \). We note that, in particular, for each \( i \in \{ -g, -(g-1), \ldots, 3g + 1, 3g + 2 \} \), two segments in \( \partial \text{L}^3_* \)

\[
\left\{ \left( x, \frac{g+1}{g} \left( x - \frac{i}{2(g+1)} \right) + \frac{1}{2}, -\frac{1}{2} \right) ; x \in \mathbb{R} \right\} \cap \text{L}^3_*
\]

and

\[
\left\{ \left( x, -\frac{g+1}{g} \left( x - \frac{i}{2(g+1)} \right) + \frac{1}{2}, \frac{1}{2} \right) ; x \in \mathbb{R} \right\} \cap \text{L}^3_*
\]

are identified.

In order to determine \( \Phi_g \), we need to study the “bundle” structure \( \text{L}^3_*/\sim \rightarrow \Sigma_g \) corresponding to \( \pi|_{T\Sigma_g} : T\Sigma_g \rightarrow \Sigma_g \). So we see how the fibre of \( \pi|_{T\Sigma_g} \) are in \( \text{L}^3_* \). This correspondence is only determined up to parallel transformations in x-direction. From the above construction, \( \text{Int}(S) \), which is a \( 4g + 4 \) punctured torus, corresponds to \([0,1] \times [0,1] \times \{0\} \cap \text{L}^3_* \). More precisely, \( \text{Int}(S) \cap \pi^{-1}(P_i)(i = 1,2) \) corresponds to

\[
R_i = [0,1] \times \left[ \frac{i-1}{2}, \frac{i}{2} \right] \times \{0\} \cap \text{L}^3_*.
\]

\( \text{Int}(S') \cap \pi^{-1}(P_3) \) corresponds to

\[
R_3 = \left\{ \left( t \left( B_g \left( \begin{array}{c} x \\ y \end{array} \right) \right), \frac{1}{2} \right) \in \mathbb{R}; (x,y) \in [0,1] \times \left[ \frac{1}{2}, 1 \right] \right\},
\]

or

\[
R'_3 = \left\{ \left( t \left( B_g^{-1} \left( \begin{array}{c} x \\ y \end{array} \right) \right), -\frac{1}{2} \right) \in \mathbb{R}; (x,y) \in [0,1] \times \left[ \frac{1}{2}, 1 \right] \right\},
\]

and \( \text{Int}(S') \cap \pi^{-1}(P_4) \) corresponds to

\[
R_4 = \left\{ \left( t \left( B_g \left( \begin{array}{c} x \\ y \end{array} \right) \right), \frac{1}{2} \right) \in \mathbb{R}; (x,y) \in [0,1] \times \left[ 0, \frac{1}{2} \right] \right\},
\]

or

\[
R'_4 = \left\{ \left( t \left( B_g^{-1} \left( \begin{array}{c} x \\ y \end{array} \right) \right), -\frac{1}{2} \right) \in \mathbb{R}; (x,y) \in [0,1] \times \left[ 0, \frac{1}{2} \right] \right\},
\]
where
\[
B_g = \begin{pmatrix}
-g & -(g-1) \\
-(g+1) & -g
\end{pmatrix}
\]
satisfying that \((B_g)^2 = A_g\). Since \(b_1 = \pi^{-1}(o_1)\) is the base of \(S\), it corresponds to the center line of \(R_1\), i.e.,
\[
L_1 = \left\{ \left( x, \frac{1}{4} \right), 0 \in L^3, x \in [0,1] \right\}
\]
(see Figures 1 and 2).

Similarly, \(\pi^{-1}(o_2)\) corresponds to the center line of \(R_2\)
\[
L_2 = \left\{ \left( x, \frac{3}{4} \right), 0 \in L^3, x \in [0,1] \right\}
\]
Since \(b_2\) is another base of \(S\), it corresponds to \(\left\{ \frac{1}{4(g+1)} \right\} \times [0,1] \times \{0\} \cap L^3_\ast\).

The correspondence between closed orbits \(\{ \pm G_i \}\) and \(\{ O_i^n \}\) is as follows. From the definition of \(b_2\), \(\partial(\text{Int}(S) \cap \pi^{-1}(P_1)) \cap b_2 \cap \pi^{-1}(p_1)\) is between \(q^+_2 = \partial(\text{Int}(S) \cap \pi^{-1}(P_1)) \cap (+G_2)\) and \(q^+_1 = \partial(\text{Int}(S) \cap \pi^{-1}(P_1)) \cap (+G_1)\). So, operating \((1,1)\)-Dehn surgery along \(O_0^0\) (resp. \(O_0^1\)), we obtain \(+G_2\) (resp. \(+G_1\)). Similarly, since \(\partial(\text{Int}(S) \cap \pi^{-1}(P_1)) \cap b_2 \cap \pi^{-1}(p_{g+2})\) is between \(q^+_{g+2} = \partial(\text{Int}(S) \cap \pi^{-1}(P_1)) \cap (-G_{g+2})\) and \(q^+_{g+3} = \partial(\text{Int}(S) \cap \pi^{-1}(P_1)) \cap (-G_{g+3})\), \(O_0^0\) (resp. \(O_1^1\)) corresponds to \(-G_{g+3}\) (resp. \(-G_{g+2}\)). \(\partial(\text{Int}(S) \cap \pi^{-1}(P_1))\) intersects \(+G_2 + G_1, +G_{2g+2}, \ldots, +G_3\) (resp. \(-G_{g+3}, -G_{g+2}, \ldots, -G_2, -G_1, -G_{2g+2}, \ldots, -G_{g+4}\)) in this order. Hence,
\[
O_0^0, O_0^1, O_{2g+1}^0, O_2^1, O_{3}^1, \ldots, O_{2g+1}^1
\]
correspond to
\[
+G_{2g+2}, +G_{2g+1}, \ldots, +G_3, -G_{g+1}, -G_g, \ldots, -G_1, -G_{2g+2}, \ldots, -G_{g+4},
\]
respectively. \(\pi^{-1}(p_i) \cap \partial(\text{Int}(S) \cap \pi^{-1}(P_1))\) is two open intervals \((q^+_i, q^-_i)\) and \((q^+_i, q^-_i) \subset \partial(\text{Int}(S) \cap \pi^{-1}(P_1))\) where \(q^-_i = \partial(\text{Int}(S) \cap \pi^{-1}(P_1)) \cap (-G_i)(i = 1,2)\). The other part of \(\pi^{-1}(p_1)\) is \(\pi^{-1}(p_1) \cap \partial(\text{Int}(S') \cap \pi^{-1}(P_3))\) which are two open intervals. Because of the correspondence between \(\text{Int}(S) \cap \pi^{-1}(P_1)\) (resp. \(\text{Int}(S') \cap \pi^{-1}(P_3)\)) and \(R_1\) (resp. \(R_3\) \subset
\[
\{ (x, y, \frac{1}{2}) \mid x, y \in \mathbb{R} \}, \pi^{-1}(p_1) \text{ can be considered to be corresponding to the union of four segments}
\]

\[
S_1 = \left\{ (x, 0, 0) \in \mathbb{L}_*^3 ; x \in \left( 0, \frac{1}{2(g+1)} \right) \right\}
\cup \left\{ \left( x, \frac{g+1}{g} \left( x - \frac{1}{2} \right) + \frac{1}{2}, -\frac{1}{2} \right) \in \mathbb{L}_*^3 ; x \in \left( \frac{1}{2(g+1)}, \frac{1}{2} \right) \right\}
\cup \left\{ \left( x, \frac{1}{2}, 0 \right) \in \mathbb{L}_*^3 ; x \in \left( \frac{1}{2}, \frac{1}{2} + \frac{1}{2(g+1)} \right) \right\}
\cup \left\{ \left( x, -\frac{g+1}{g} \left( x - \frac{1}{2} - \frac{1}{2(g+1)} \right) + \frac{1}{2}, \frac{1}{2} \right) \in \mathbb{L}_*^3 ; x \in \left( \frac{1}{2} + \frac{1}{2(g+1)}, 1 \right) \right\},
\]

or

\[
S'_1 = \left\{ (x, 1, 0) \in \mathbb{L}_*^3 ; x \in \left( 0, \frac{1}{2(g+1)} \right) \right\}
\cup \left\{ \left( x, -\frac{g+1}{g} \left( x - \frac{1}{2} \right) + \frac{1}{2}, -\frac{1}{2} \right) \in \mathbb{L}_*^3 ; x \in \left( \frac{1}{2(g+1)}, \frac{1}{2} \right) \right\}
\cup \left\{ \left( x, \frac{1}{2}, 0 \right) \in \mathbb{L}_*^3 ; x \in \left( \frac{1}{2}, \frac{1}{2} + \frac{1}{2(g+1)} \right) \right\}
\cup \left\{ \left( x, \frac{g+1}{g} \left( x - \frac{1}{2} - \frac{1}{2(g+1)} \right) + \frac{1}{2}, \frac{1}{2} \right) \in \mathbb{L}_*^3 ; x \in \left( \frac{1}{2} + \frac{1}{2(g+1)}, 1 \right) \right\},
\]

(see Figure 2).

In the same way as the case of \( \pi^{-1}(p_1) \), we can see that each \( \pi^{-1}(p_i) (i = 2, 3, 4, \ldots, 2g + 2) \) corresponds to

\[
\left\{ \left( x - \frac{i-1}{2g+2}, y, z \right) \mid (x, y, z) \in S_1 \right\} \equiv S_i \subset \mathbb{L}_*^3,
\]
Figure 2
or
\begin{equation}
\left\{ \left( x - \frac{i - 1}{2g + 2}, y, z \right) \right\}; (x, y, z) \in S'_i \right\} \equiv S'_i \subset L^3
\end{equation}
(mod \( \mathbb{Z} \) in \( x \) - coordinate).

Let \( l_1 \subset \Sigma_g \) be a geodesic arc between \( o_1 \) and \( p_1 \). \( \pi^{-1}(l_1) \) is an annulus and it corresponds to a helicoid \( H_1 \subset L^3 \) such that its center line is \( L_1 \) and its edge is \( S_1 \) which is a spiral around \( L_1 \). Similarly, let \( l_i \) (resp. \( l'_i \)) \( \subset \Sigma_g \) be the geodesic arc between \( o_1 \) (resp. \( o_2 \)) and \( p_i \) \((i = 1, 2, 3, \ldots, 2g + 2)\), then \( \pi^{-1}(l_i) \) (resp. \( \pi^{-1}(l'_i) \)) corresponds to a helicoid \( H_i \) (resp. \( H'_i \)) \( \subset L^3 \) such that its center line is \( L_1 \) (resp. \( L_2 \)) and its edge is \( S_i \) (resp. \( S'_i \)). We only need \( H_i \) and \( H'_i \) to calculate \( \Phi_g \). Other fibres correspond to some curves in \( L^3 \) as follows (see Figure 2). For \( p \in \text{Int}(G_i \cap \partial P_j) \) \((i = 1, 2, 3, \ldots, 2g + 2, j = 1, 2)\), \( \pi^{-1}(p) \) corresponds to the union of two curves and for \( p' \in \text{Int}P_j \) \((j = 1, 2)\), \( \pi^{-1}(p') \) corresponds to a compatibly oriented spiral around \( L_j \). Curves corresponding to fibres over \( \text{Int}(P_3) \) (resp. \( \text{Int}(P_4) \)) are the spirals around the center lines of \( R_3 \) or \( R'_3 \) (resp. \( R_4 \) or \( R'_4 \)).

To sum up, we obtain a continuous map

\[ \pi' = (\pi|_{L^3/\sim}) \circ (q|_{L^3}) : L^3 \to \Sigma_g \]

such that \((\pi')^{-1}(l_i) = H_i \) \((\pi')^{-1}(l'_i) = H'_i \) \((i = 1, 2, 3, \ldots, 2g + 2)\) and \((\pi')^{-1}(P_j) \) \((j = 1, 2)\) is \((2g + 2 \text{ gon}) \times [0, 1]\) which twists around \( L_j \). \((\pi')^{-1}(P_3) \) (resp. \((\pi')^{-1}(P_4) \)) is \((2g + 2 \text{ gon}) \times [0, 1]\) which twists around the center line of \( R_3 \) or \( R'_3 \) (resp. \( R_4 \) or \( R'_4 \)).

To calculate the piecewise linear total holonomy of a foliated \( S^1 \) bundle over \( \Sigma_g \), a presentation of the fundamental group \( \pi_1(\Sigma_g) \) is adopted here. That is,

\[ \pi_1(\Sigma_g) = \langle a_1, a_2, a_3, \ldots, a_{2g+2} \rangle; \quad a_1a_2a_3 \ldots a_{2g+2} = a_1a_3 \ldots a_{2g+1} = a_2a_4 \ldots a_{2g+2} = 1 \]

(see Figure 1 and Proposition 2 in §3).

Let \( \alpha_i \) be the loop in \( \Sigma_g \) which starts \( o_1 \) and passes \( p_i, o_2, p_{i+1} \) and reaches \( o_1 \) such that \( \alpha_i \) represents \( a_i \) in \( \pi_1(\Sigma_g) \). It is easy to see that

\[ (\pi')^{-1}(\alpha_i) = H_i \cup H'_i \cup H'_{i+1} \cup H_{i+1}. \]

Now we calculate the PL total holonomy \( \phi_g \) of the unstable foliation \( F^u \) of \( \varphi_t \). The total holonomy of the stable foliation is conjugate to \( \Phi_g \).
by $T(1/2)$. An eigenvector of $A_g$ corresponding to $\lambda_g$ is \((1, \sqrt{\frac{g+1}{g-1}})\). Let $\Pi_s(s \in \mathbb{R})$ be the plane
\[
\left\{ \left( x, \sqrt{\frac{g+1}{g-1}} x + s, z \right) \in \mathbb{R}^3; x, z \in \mathbb{R} \right\}.
\]
The leaves of $\mathcal{F}^u|_{L_2/\sim}$ are made of
\[
\left\{ \Pi_s \cap L_3^s; s \in \left( -\sqrt{\frac{g+1}{g-1}}, 1 \right) \right\}.
\]
Now, for $p \in (\pi')^{-1}(\alpha_1) = L_1$, there exists $s \in \left( -\sqrt{\frac{g+1}{g-1}}, 1 \right)$ such that $p \in \Pi_s$ and we move $p$ on
\[
\Pi_s \cap (H_1 \cup H_1' \cup H_2' \cup H_2)
\]
along $\alpha_1$ in $L_3^s$ following the next rule: if a point reaches a face of the cube $L_3^s$, then the point is moved in the opposite face by the equivalence relation $\sim$ and starts from the face into $\text{Int}(L_3^s)$. During this move, $p$ passes $S_1, L_2, S_2$ one by one. Finally, $p$ returns to $L_1$. But any points which pass some $q^{-1}(O_i^m)$ cannot return to $L_1$. Hence we obtain a return map defined on $L_1/\sim - \{12 \text{ points}\}$. This return map can be extended to the homeomorphism
\[
f_g^{-1} : L_1/\sim = \mathbb{R}/\mathbb{Z} = S^1 \to L_1/\sim.
\]
Then $f_g$ is a PL homeomorphism whose left (right) differential coefficients are $(\lambda_g)^\sigma (\sigma = -1, 0, 1)$ as it is described below. (Here we parametrize $L_1$ by the $x$-coordinate.) $f_g$ has four non-differentiable points and these points are caused by four of twelve points deleted from $L_1/\sim$.

Now we prepare some notations. In order to describe a PL homeomorphism $h$ of $S^1 = \mathbb{R}/\mathbb{Z}$, the lift homeomorphism of $h, \tilde{h} : \mathbb{R} \to \mathbb{R}$, is described by using non-differentiable points of $\tilde{h}$. If $a, b \in \mathbb{R} (a < b)$ are non-differentiable points of $\tilde{h}$ and $\tilde{h}|_{[a, b]}$ is
\[
y = \lambda x + \nu(\lambda, \nu \in \mathbb{R}),
\]
then \( \tilde{h}_{[a,b]} \) is denoted by
\[
  a \mapsto c = \lambda a + \nu
\]
\[
  [\lambda]
\]
\[
  b \mapsto d = \lambda b + \nu.
\]
For example, \( T(\theta)(\theta \in S^1 = \mathbb{R}/\mathbb{Z}) \) is denoted by
\[
  T(\theta) : 0 \mapsto \theta
\]
\[
  [1]
\]
\[
  1 \mapsto \theta + 1.
\]
Then \( f_g \) or its lift \( \tilde{f}_g \) is described as follows:
\[
  \frac{4g^2 - 2 + (1 - 4g) \sqrt{g^2 - 1}}{4(g + 1)} \mapsto \frac{2 + \sqrt{g^2 - 1}}{4(g + 1)}
\]
\[
  [(\lambda_g)^{-1}]
\]
\[
  \frac{2g - \sqrt{g^2 - 1}}{4(g + 1)} \mapsto \frac{(4g - 1) \sqrt{g^2 - 1} - 4g^2 + 2g + 4}{4(g + 1)}
\]
\[
  \frac{4g + 2 + \sqrt{g^2 - 1}}{4(g + 1)} \mapsto \frac{4g + 2 - 3\sqrt{g^2 - 1}}{4(g + 1)}
\]
\[
  \frac{2g + 4 + 3\sqrt{g^2 - 1}}{4(g + 1)} \mapsto \frac{2g + 4 - \sqrt{g^2 - 1}}{4(g + 1)}
\]
\[
  \frac{4g^2 - 2 + (1 - 4g) \sqrt{g^2 - 1}}{4(g + 1)} + 1 \mapsto \frac{2 + \sqrt{g^2 - 1}}{4(g + 1)} + 1.
\]
Similarly, with respect to \( \alpha_i(i = 2, 3, \ldots, 2g + 2) \), we obtain a PL-homeomorphism
\[
  T \left( \frac{i - 1}{2(g + 1)} \right) \circ f_g^{-1} \circ T \left( \frac{i - 1}{2(g + 1)} \right).
\]
The next lemma is proved by the induction.
Lemma 1. — Let \( f_g^{(i)}(i = 1, 2) \) be \( f_g \circ T \left( \frac{1}{i(g+1)} \right) \). \( \mathcal{L}_g \) and \( \mathcal{M}_g \) denote \( 2\lambda_g(g+1) \) and \( \sqrt{\lambda_g(g+1)} \), respectively.

(1) For \( m = 1, 2, \ldots, g - 1, g \), \( \left\{ f_g^{(1)} \right\}^m \):

\[
\frac{4g^2 + 2 + (1 - 4g)\sqrt{g^2 - 1}}{4(g+1)} \rightarrow \frac{m - 1}{\mathcal{M}_g} \rightarrow \frac{2 + \sqrt{g^2 - 1}}{4(g+1)}
\]

[1]

\[
\frac{2g + 4 - \sqrt{g^2 - 1}}{4(g+1)} \rightarrow \frac{(4g - 1)\sqrt{g^2 - 1} - 4g^2 + 2g + 4}{4(g+1)} + \frac{m - 1}{\mathcal{M}_g}
\]

\( [\lambda_g]^{-1} \)

\[
\frac{4g + 6 + \sqrt{g^2 - 1}}{4(g+1)} \rightarrow \frac{4g + 2 - 3\sqrt{g^2 - 1}}{4(g+1)} + \frac{m - 1}{\mathcal{M}_g}
\]

[1]

\[
\frac{2g + 8 + 3\sqrt{g^2 - 1}}{4(g+1)} \rightarrow \frac{2g + 4 - \sqrt{g^2 - 1}}{4(g+1)}
\]

\( [\lambda_g] \)

\[
\frac{4g^2 + 4g + 6 + (1 - 4g)\sqrt{g^2 - 1}}{4(g+1)} \rightarrow \frac{m - 1}{\mathcal{M}_g} \rightarrow \frac{4g + 6 + \sqrt{g^2 - 1}}{4(g+1)}
\]

(2) For \( m = 3, 4, \ldots, g, g + 1 \), \( \left\{ f_g^{(2)} \right\}^m \):

\[
\frac{(8g^2 - 4g - 1)\sqrt{g^2 - 1} - 8g^3 + 4g^2 + 6g}{4(g+1)} + \frac{m - 3}{\mathcal{L}_g} \rightarrow \frac{2g + 2 - \sqrt{g^2 - 1}}{4(g+1)}
\]

\( [\lambda_g] \)

\[
\frac{(8g^2 - 8g - 1)\sqrt{g^2 - 1} - 8g^3 + 8g^2 + 6g - 2}{4(g+1)} + \frac{m - 3}{\mathcal{L}_g} \rightarrow \frac{2g + 4 - \sqrt{g^2 - 1}}{4(g+1)}
\]

\( [\lambda_g] \)
PL-REPRESENTATION OF ANOSOV FOLIATIONS

\[
\frac{12g^2 - 4 + (1 - 12g)\sqrt{g^2 - 1}}{4(g + 1)} + \frac{m - 3}{\mathcal{L}_g} \mapsto \frac{4g + 6 + \sqrt{g^2 - 1}}{4(g + 1)}
\]

[1]

\[
\frac{2g + 2 - \sqrt{g^2 - 1}}{4(g + 1)} \mapsto \frac{(12g - 1)\sqrt{g^2 - 1} - 12g^2 + 6g + 12 - m - 3}{4(g + 1)}
\]

\[(\lambda_g)^{-1}\]

\[
\frac{2g + 4 - \sqrt{g^2 - 1}}{4(g + 1)} \mapsto \frac{(8g - 1)\sqrt{g^2 - 1} - 8g^2 + 6g + 10 - m - 3}{4(g + 1)}
\]

\[(\lambda_g)^{-2}\]

\[
\frac{4g + 4 + \sqrt{g^2 - 1}}{4(g + 1)} \mapsto \frac{8g^3 - 8g^2 + 10 - (8g^2 - 8g - 1)\sqrt{g^2 - 1} - m - 3}{4(g + 1)}
\]

\[(\lambda_g)^{-1}\]

\[
\frac{4g + 6 + \sqrt{g^2 - 1}}{5(g + 1)} \mapsto \frac{8g^3 - 4g^2 + 8 - (8g^2 - 4g - 1)\sqrt{g^2 - 1} - m - 3}{4(g + 1)}
\]

[1]

\[
\frac{(8g^2 - 4g - 1)\sqrt{g^2 - 1} - 8g^3 + 4g^2 + 10g + 4 - m - 3}{4(g + 1)} \mapsto \frac{6g + 6 - \sqrt{g^2 - 1}}{4(g + 1)}.
\]

**Lemma 2.** The map

\[\Phi_g : \pi_1(\Sigma_g) \mapsto PL_+(S^1)\]

defined by

\[\Phi_g(a_i) = T\left(-\frac{i - 1}{2(g + 1)}\right) \circ f_g \circ T\left(\frac{i - 1}{2(g + 1)}\right) (i = 1, 2, \ldots, 2(g + 1))\]

is a group homomorphism.

**Proof.** By Lemma 1,

\[\left\{f_g^{(1)}\right\}^{g + 1} = 1\]

and

\[\left\{f_g^{(2)}\right\}^{2(g + 1)} = \left[\left\{f_g^{(2)}\right\}^{g + 1}\right]^2 = 1.\]
Hence

\[ \Phi_g(a_1a_3 \ldots a_{2g+1}) = \left\{ f_g^{(1)} \right\}^g \circ f_g \circ T \left( \frac{2g}{2(g+1)} \right) = \left\{ f_g^{(1)} \right\}^{g+1} = 1, \]

\[ \Phi_g(a_2a_4 \ldots a_{2(g+1)}) = T \left( -\frac{1}{g+1} \right) \circ \Phi_g(a_1a_3 \ldots a_{2g+1}) \circ T \left( \frac{1}{g+1} \right) = 1, \]

and

\[ \Phi_g(a_1a_2 \ldots a_{2(g+1)}) = \left\{ f_g^{(2)} \right\}^{2g+1} \circ f_g \circ T \left( \frac{2g+1}{2(g+1)} \right) = \left\{ f_g^{(2)} \right\}^{2(g+1)} = 1. \]

These verify that \( \Phi_g \) is a group homomorphism. \( \square \)

To sum up, we obtain the following theorem.

\textbf{Theorem.} — Let \( \Sigma_g \) be the orientable closed surface of genus \( g \geq 2 \), consider a Riemannian metric on \( \Sigma_g \) of constant negative curvature \( -1 \), and let

\[ \Psi_g : \pi_1(\Sigma_g) \to PSL(2,\mathbb{R}) \]

denote the total holonomy of the unstable foliation of the geodesic flow \( F_t \) on the unit tangent vector bundle. Then \( \Psi_g \) is topologically conjugate to the above homomorphism \( \Phi_g \). That is to say, there exists a homeomorphism \( h : S^1 \to S^1 \) such that

\[ \Psi_g(\gamma)(\theta) = (h \circ \Phi_g(\gamma) \circ h^{-1})(\theta) \text{ for all } \gamma \in \pi_1(\Sigma_g) \text{ and } \theta \in S^1. \]

\textbf{Proof.} — In [Gh] and [Ha1], it is shown that the unstable foliation of \( F_t \) is topologically equivalent to the unstable foliation of \( \varphi_t \), that is, their total holonomies are topologically conjugate each other. On the other hand, \( \Phi_g \) is the total holonomy of \( \varphi_t \) (cf. [CN], Chapter V). So \( \Psi_g \) is topologically conjugate to \( \Phi_g \). \( \square \)
Remark. — $\Phi_g$ is independent of the choice of the metric of constant negative curvature $-1$ but dependent on the choice of the basis of the Birkhoff section.

2. The discrete Godbillon-Vey invariant of $\Phi_g$.

The discrete Godbillon-Vey invariant $\overline{GV}$ (see [Gr], [GS], [Gh] and [T]) is the $\mathbb{R} \otimes \mathbb{Z} \mathbb{R}$ valued 2-cocycle of $PL_+(S^1)$ defined by

$$\overline{GV}(h_1, h_2) = \frac{1}{2} \sum_{x \in S^1} C(h_2, h_1 \circ h_2)(x)(h_1, h_2 \in PL_+(S^1)),$$

where $C(k_1, k_2)(x) = \log k'_1(x + 0) \otimes \Delta(\log k'_2)(x) - \log k'_2(x + 0) \otimes \Delta(\log k'_1)(x)(k_1, k_2 \in PL_+(S^1), x \in S^1)$ and for a map $k : S^1 \to \mathbb{R}, \Delta k(x) = k(x + 0) - k(x - 0)$ if $k$ has

$$k(x \pm 0) \lim_{\varepsilon \to 0} k(x \pm \varepsilon)(\text{for } x \in S^1).$$

From this definition, we have the next lemma.

**Lemma 3.** — For $\theta \in S^1$,

1. $\overline{GV}(T(\theta) \circ h_1, h_2) = \overline{GV}(h_1, h_2)$,
2. $\overline{GV}(h_1 \circ T(\theta), h_2) = \overline{GV}(h_1, T(\theta) \circ h_2)$,
3. $\overline{GV}(h_1, h_2 \circ T(\theta)) = \overline{GV}(h_1, h_2)$.

Let $\Sigma_g \in H_2(\pi_1(S^g); \mathbb{Z}) = \mathbb{Z}$ be the fundamental class. According to [EM], $\Sigma_g$ is represented by the 2-cycle

$$\Sigma_g = (a_1, a_3) + (a_1a_3, a_5) + \ldots + (a_1a_3a_5 \ldots a_{2g-3}, a_{2g-1})$$
$$+ (a_2, a_4) + (a_2a_4, a_6) + \ldots + (a_2a_4a_6 \ldots a_{2g-2}, a_{2g})$$
$$- \{(a_1, a_2) + (a_1a_2, a_3) + \ldots + (a_1a_2a_3 \ldots a_{2g-1}, a_{2g})$$
$$- (a_{2g+2}, a_{2g+1})\}$$
$$= (a_1, a_3) + (a_1a_3, a_5) + \ldots + (a_1a_3a_5 \ldots a_{2g-3}, a_{2g-1})$$
$$+ (a_2, a_4) + (a_2a_4, a_6) + \ldots + (a_2a_4a_6 \ldots a_{2g-2}, a_{2g})$$
$$- \{(a_1, a_2) + (a_1a_2, a_3) + \ldots + (a_1a_2a_3 \ldots a_{2g-1}, a_{2g})$$
$$+ (a_{2g+2}, a_{2g+1})\}$$
$$= (a_2, a_4) + (a_2a_4, a_6) + \ldots + (a_2a_4a_6 \ldots a_{2g-2}, a_{2g})$$
$$- \{(a_1, a_2) + (a_1a_2, a_3) + \ldots + (a_1a_2a_3 \ldots a_{2g-1}, a_{2g})$$
$$+ (a_{2g+2}, a_{2g+1})\}$$
$$= (a_2, a_4) + (a_2a_4, a_6) + \ldots + (a_2a_4a_6 \ldots a_{2g-2}, a_{2g})$$
$$- \{(a_1, a_2) + (a_1a_2, a_3) + \ldots + (a_1a_2a_3 \ldots a_{2g-1}, a_{2g})$$
$$+ (a_{2g+2}, a_{2g+1})\}$$
The next lemma is proved by Lemma 3 and the fact \( T \left( -\frac{2g+1}{2(g+2)} \right) = T \left( \frac{1}{2(g+1)} \right) \).

**Lemma 4.**

(1) \( \overline{GV}(\Phi_{g*}(a_1a_3 \ldots a_{2m-3}, a_{2m-1})) = \overline{GV} \left( (f_g^{(1)})^{m-1}, f_g^{(1)} \right) \) \((m = 2, 3, \ldots, g)\).

(2) \( \overline{GV}(\Phi_{g*}(a_2a_4 \ldots a_{2m-2}, a_{2m})) = \overline{GV} \left( (f_g^{(1)})^{m-1}, f_g^{(1)} \right) \) \((m = 2, 3, \ldots, g)\).

(3) \( \overline{GV}(\Phi_{g*}(a_1a_2 \ldots a_{m-1}, a_m)) = \overline{GV} \left( (f_g^{(2)})^{m-1}, f_g^{(2)} \right) \) \((m = 2, 3, \ldots, g+1)\).

(4) \( \overline{GV}(\Phi_{g*}(a_{2g+2}a_{2g+1} \ldots a_{2g-m}, a_{2g-m})) = \overline{GV} \left( (f_g^{(2)})^{-m-3}, f_g^{(2)} \right) \) \((m = 0, 1, \ldots, g - 2)\).

(5) \( \overline{GV}(\Phi_{g*}(a_{2g+2}a_{2g+1})) = \overline{GV} \left( (f_g^{(2)})^{-1}, (f_g^{(2)})^{-1} \right) \).

If \( h_1, h_2 \in PL_+(S^1) \), then it is the order of the non-differentiable points of \( h_2 \) and \( h_1 \circ h_2 \) that determines the value of \( \overline{GV}(h_1, h_2) \). Let

\[ d_g^{(i)}(m; \sigma, \tau) \in S^1 (i = 1, 2; m \in \mathbb{Z}; \sigma, \tau \in \{-2, -1, 0, 1, 2\}) \]

be the non-differentiable points of \( \left( f_g^{(i)} \right)^m \) such that

\[ \left\{ \left( f_g^{(i)} \right)^m \right\}' (d_g^{(i)}(m; \sigma, \tau) - 0) = (\lambda_g)^\sigma \]

and

\[ \left\{ \left( f_g^{(i)} \right)^m \right\}' (d_g^{(i)}(m; \sigma, \tau) + 0) = (\lambda_g)^\tau. \]

\( \left( f_g^{(i)} \right)^m (d_g^{(i)}(m; \sigma, \tau)) \) is denoted by \( r_g^{(i)}(m; -\sigma, -\tau) \) which is the non-differentiable points of \( \left( f_g^{(i)} \right)^{-m} \) such that

\[ \left\{ \left( f_g^{(i)} \right)^{-m} \right\}' (r_g^{(i)}(m; -\sigma, -\tau) - 0) = (\lambda_g)^{-\sigma} \]

and

\[ \left\{ \left( f_g^{(i)} \right)^{-m} \right\}' (r_g^{(i)}(m; -\sigma, -\tau) + 0) = (\lambda_g)^{-\tau}. \]

We know the values of \( d_g^{(i)}(m; \sigma, \tau) \) and \( r_g^{(i)}(m; \sigma, \tau) \) by Lemma 2 except \( d_g^{(2)}(m; \sigma, \tau) \) and \( r_g^{(2)}(m; \sigma, \tau) \)(\(m = 1, 2\)), but it is easy to calculate them.
$S^1 = \mathbb{R}/\mathbb{Z}$ has the cyclic order $<$ determined by the orientation of $S^1$. The orders of non-differentiable points which are used to calculate \( GV(\Phi_{g*}(\Sigma_g)) \) are as follows.

**Lemma 5.**

1. $d_g^{(1)}(m; 1, 0) < d_g^{(1)}(1; 0, 1) < d_g^{(1)}(1; 1, 0)$
   
   $< d_g^{(1)}(1; 0, -1) = d_g^{(1)}(m; 0, -1) < d_g^{(1)}(1; -1, 0) = d_g^{(1)}(m; -1, 0)$
   
   $< d_g^{(1)}(m; 0, 1) < d_g^{(1)}(m; 1, 0)$

   \( (m = 2, 3, \ldots, g) \).

2. $d_g^{(2)}(2; 0, 1) < d_g^{(2)}(2; 1, 2) < d_g^{(2)}(1; 1, 0) = d_g^{(2)}(2; 2, 1)$
   
   $< d_g^{(2)}(2; 1, 0) < d_g^{(2)}(1; 0, -1) = d_g^{(2)}(2; 0, -1) < d_g^{(2)}(2; -1, -2) $
   
   $< d_g^{(2)}(1; -1, 0) = d_g^{(2)}(2; -2, -1) < d_g^{(2)}(1; 0, 1) = d_g^{(2)}(2; -1, 0)$
   
   $< d_g^{(2)}(2; 0, 1)$.

3. $d_g^{(2)}(3; 0, 1) < d_g^{(2)}(1; 1, 0) < d_g^{(2)}(3; 1, 2) < d_g^{(2)}(3; 2, 1)$
   
   $< d_g^{(2)}(3; 1, 0) < d_g^{(2)}(1; 0, -1) = d_g^{(2)}(3; 0, -1) < d_g^{(2)}(3; -1, -2) $
   
   $< d_g^{(2)}(1; -1, 0) = d_g^{(2)}(3; -2, -1) < d_g^{(2)}(1; 0, 1)$
   
   $< d_g^{(2)}(3; -1, 0) < d_g^{(2)}(3; 0, 1)$.

4. $d_g^{(2)}(m; 0, 1) < d_g^{(2)}(m; 1, 2) < d_g^{(2)}(m; 2, 1) < d_g^{(2)}(m; 1, 0)$
   
   $< d_g^{(2)}(1; 0, -1) = d_g^{(2)}(m; 0, -1) < d_g^{(2)}(m; -1, -2)$
   
   $< d_g^{(2)}(1; -1, 0) = d_g^{(2)}(m; -2, -1) < d_g^{(2)}(1; 0, 1) < d_g^{(2)}(m; -1, 0)$
   
   $< d_g^{(2)}(1; 1, 0) < d_g^{(2)}(m; 0, 1)$

   \( (m = 4, 5, \ldots, g + 1) \).

5. $r_g^{(2)}(2; 0, -1) < r_g^{(2)}(2; -1, -2) < d_g^{(2)}(1; -1, 0) = r_g^{(2)}(2; -2, -1)$
   
   $< r_g^{(2)}(1; 0, 1) < r_g^{(2)}(2; -1, 0) < d_g^{(2)}(1; 1, 0) < r_g^{(2)}(2; 0, 1)$
   
   $< r_g^{(2)}(2; 1, 2) < r_g^{(2)}(2; 2, 1) < d_g^{(2)}(1; 0, -1) = r_g^{(2)}(2; 1, 0)$
   
   $< r_g^{(2)}(2; 0, -1)$.

6. $d_g^{(2)}(1; 0, -1) = r_g^{(2)}(m; 0, -1) < r_g^{(2)}(m; -1, -2)$
   
   $< d_g^{(2)}(1; -1, 0) = r_g^{(2)}(m; -2, -1) < d_g^{(2)}(1; 0, 1) < r_g^{(2)}(m; -1, 0)$
\[ \begin{align*}
&< d^{(2)}_g(1; 1, 0) < r^{(2)}_g(m; 0, 1) < r^{(2)}_g(m; 1, 2) < r^{(2)}_g(m; 2, 1) \\
&< r^{(2)}_g(m; 1, 0) < d^{(2)}_g(1; 0, -1) = r^{(2)}_g(m; 0, -1) \\
&\quad (m = 3, 4, \ldots, g).
\end{align*} \]

(7) \[ \begin{align*}
&< r^{(2)}_g(1; 1, 0) = r^{(2)}_g(2; 0, -1) < r^{(2)}_g(1; 0, -1) = r^{(2)}_g(2; -1, -2) \\
&< r^{(2)}_g(2; -2, -1) < r^{(2)}_g(1; -1, 0) = r^{(2)}_g(2; -1, 0) < r^{(2)}_g(2; 0, 1) \\
&< r^{(2)}_g(2; 1, 2) < r^{(2)}_g(2; 2, 1) < r^{(2)}_g(2; 1, 0) \\
&< r^{(2)}_g(1; 1, 0) = r^{(2)}_g(2; 0, -1).
\end{align*} \]

Consequently,

**Proposition 1.**

\[ \Phi^*_{g}(GV)(\Sigma_g) = GV(\Phi^*_{g}(\Sigma_g)) = -4(g + 1)\log \lambda_g \otimes \log \lambda_g. \]

**Proof.** — For \( m = 2, 3, \ldots, g \), Lemma 4 (1), (2) and Lemma 5 (1) imply that

\[ \begin{align*}
&\quad GV(\Phi^*_{g_*}(a_1 a_3 \ldots a_{2m-3}, a_{2m-1})) = GV(\Phi^*_{g_*}(a_2 a_4 \ldots a_{2m-2}, a_{2m})) \\
&\quad = GV\left((f^{(1)}_g)^{m-1}, f^{(1)}_g\right) = \frac{1}{2} \sum_{x \in S^1} C \left(f^{(1)}_g, (f^{(1)}_g)^m\right)(x) \\
&\quad = \frac{1}{2} \left\{ C \left(f^{(1)}_g, (f^{(1)}_g)^m\right)(d^{(1)}_g(m; 1, 0)) + C \left(f^{(1)}_g, (f^{(1)}_g)^m\right)(d^{(1)}_g(1; 0, 1)) \\
&\quad + C \left(f^{(1)}_g, (f^{(1)}_g)^m\right)(d^{(1)}_g(1; 1, 0)) + C \left(f^{(1)}_g, (f^{(1)}_g)^m\right)(d^{(1)}_g(1; 0, -1)) \\
&\quad + C \left(f^{(1)}_g, (f^{(1)}_g)^m\right)(d^{(1)}_g(1; -1, 0)) + C \left(f^{(1)}_g, (f^{(1)}_g)^m\right)(d^{(1)}_g(m; 0, 1)) \right\} \\
&\quad (\text{where } d^{(1)}_g(1; 0, -1) = d^{(1)}_g(m; 0, -1) \\
&\quad \text{and } d^{(1)}_g(1; -1, 0) = d^{(1)}_g(m; -1, 0) \\
&\quad = \frac{1}{2} \left\{ (0 \otimes (-\log \lambda_g) - 0 \otimes 0) + (\log \lambda_g \otimes 0 - 0 \otimes \log \lambda_g) \\
&\quad + (0 \otimes 0 - 0 \otimes (-\log \lambda_g)) \\
&\quad + (\log(\lambda_g)^{-1} \otimes \log(\lambda_g)^{-1} - \log(\lambda_g)^{-1} \otimes \log(\lambda_g)^{-1}) \\
&\quad + (0 \otimes (\log(\lambda_g)^{-1} - 0 \otimes (\log(\lambda_g)^{-1}))) \\
&\quad + (0 \otimes \log \lambda_g - \log \lambda_g \otimes 0) \right\} = 0.
\end{align*} \]
Similarly, 
\[
\overline{GV}(\Phi_g^*(a_1, a_2)) = \overline{GV}\left(f_g^{(2)}, f_g^{(2)}\right) \\
= \frac{1}{2} \sum_{x \in \mu} C (f_g^{(2)}, (f_g^{(2)})^2) (x) = 3 \log \lambda_g \otimes \log \lambda_g,
\]

\[
\overline{GV}(\Phi_g^*(a_1 a_2, a_3)) = \overline{GV}\left(f_g^{(2)}, f_g^{(2)}\right) \\
= \frac{1}{2} \sum_{x \in \mu} C (f_g^{(2)}, (f_g^{(2)})^3) (x) = 3 \log \lambda_g \otimes \log \lambda_g,
\]

\[
\overline{GV}(\Phi_g^*(a_1 a_2 \ldots a_{m-1}, a_m)) = \overline{GV}\left((f_g^{(2)})^{m-1}, f_g^{(2)}\right) \\
= \frac{1}{2} \sum_{x \in \mu} C (f_g^{(2)}, (f_g^{(2)})^m) (x) = 2 \log \lambda_g \otimes \log \lambda_g
\]

\[
(m = 4, 5, \ldots, g + 1),
\]

\[
\overline{GV}(\Phi_g^*(a_{2g+2}^{-1} a_{2g+1}^{-1} \ldots a_{2g-m}^{-1}, a_{2g-m})) = \overline{GV}\left((f_g^{(2)})^{-m-3}, f_g^{(2)}\right) \\
= \frac{1}{2} \sum_{x \in \mu} C (f_g^{(2)}, (f_g^{(2)})^{-m-2}) (x) = 2 \log \lambda_g \otimes \log \lambda_g
\]

\[
(m = 1, 2, \ldots, g - 2),
\]

\[
\overline{GV}(\Phi_g^*(a_{2g+2}^{-1} a_{2g+1}^{-1} a_{2g}^{-1}, a_{2g})) = \overline{GV}\left((f_g^{(2)})^{-3}, f_g^{(2)}\right) \\
= \frac{1}{2} \sum_{x \in \mu} C (f_g^{(2)}, (f_g^{(2)})^{-2}) (x) = 3 \log \lambda_g \otimes \log \lambda_g,
\]

\[
\overline{GV}(\Phi_g^*(a_{2g+2}, a_{2g+1})) = \overline{GV}\left((f_g^{(2)})^{-1}, (f_g^{(2)})^{-1}\right) \\
= \frac{1}{2} \sum_{x \in \mu} C (f_g^{(2)}), (f_g^{(2)})^{-2}) (x) = -3 \log \lambda_g \otimes \log \lambda_g.
\]

Therefore,
\[
\overline{GV}(\Phi_g^*(\Sigma_g)) \\
= \{0 \times 2(g - 1) - (3 + 3 + 2 \times 2(g - 2) + 3)\} \log \lambda_g \otimes \log \lambda_g \\
= -4(g + 1) \log \lambda_g \otimes \log \lambda_g.
\]

Let \(\text{Homeo}^{+}(S^1)\) be the group of orientation preserving homeomorphism \(f\) of \(\mathbb{R}\) satisfying that
\[
f(x + 1) = f(x) + 1, \text{ for all } x \in \mathbb{R}.
\]
DEFINITION ([He]). — For $f \in \text{Homeo}_+^{-}(S^1)$, we say that $f$ is of class $P$ if $f$ is differentiable except at most countably many points of $\mathbb{R}$ and there exists a function $h: \mathbb{R} \to \mathbb{R}$ satisfying that

(i) $h(x + 1) = h(x)$ for all $x \in \mathbb{R}$,
(ii) $h(x) > a > 0$ for all $x \in \mathbb{R}$,
(iii) $h|_{[0,1]}$ is of bounded variation,
(iv) $f'$ coincides with $h$ except at most countably many points of $\mathbb{R}$.

$\tilde{D}_+(S^1)$ denotes the homeomorphisms of class $P$ of $\mathbb{R}$. In [He], some remarks about $\tilde{D}_+(S^1)$ are stated.

Remark.

(1) If $f \in \tilde{D}_+(S^1)$ then both of $f, f^{-1}$ are absolutely continuous and Lipschitz continuous.

(2) If $f \in \tilde{D}_+(S^1)$ then so is $f^{-1}$ and if $f, g \in \tilde{D}_+(S^1)$ then so is $f \circ g$. Hence $\tilde{D}_+(S^1)$ is a group.

(3) In the above definition, $\log h$ is of bounded variation.

(4) $f$ is of class $P$ if and only if for all $x \in \mathbb{R}, f$ has the right derivative $f'(x + 0)$ and $\log f'(- + 0)$ has bounded variation on $[0,1]$.

$D_+(S^1)$ denotes the orientation preserving homeomorphisms of $S^1$ whose lifts belong to $\tilde{D}_+(S^1)$. So $D_+(S^1)$ is a group and $PL_+(S^1) \cup PSL(2,\mathbb{R}) \subset D_+(S^1) \subset \text{Homeo}_+(S^1)$. Let $\rho : \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ be the homomorphism defined by $\rho(a \otimes b) = a \times b(a, b \in \mathbb{R})$. $\overline{g\nu}$ denotes the composition $\rho \circ G\nu$ which can define the $\mathbb{R}$-valued 2-cocycle of $D_+(S^1)$. On the other hand, for $h_1, h_2 \in \text{Diff}_+^2(S^1)$, the Godbillon-Vey cocycle is defined by

$$g\nu(h_1, h_2) = \frac{1}{2} \int_{S^1} \left| \begin{array}{cc} \log h_2'(x) & \log(h_1 \circ h_2)'(x) \\ (\log h_2')'(x) & (\log(h_1 \circ h_2)')'(x) \end{array} \right| dx.$$ 

From Remark (3) and (4), this integral has a finite value for $h_1, h_2 \in D_+(S^1)$. So the Godbillon-Vey cocycle can be defined for $D_+(S^1)$ by the same formula.

Now we prove that each of non-trivial linear combinations $g\nu$ and $\overline{g\nu}$ is not a topological invariant in $D_+(S^1)$. 

Corollary ([Gh], Théorème 1). — Each
\[ \alpha g v + \beta \overrightarrow{v} (\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \neq 0) \]
is not a topological invariant in \( D_+(S^1) \).

Proof. — The above proposition implies that
\[ (\Phi^*_{\overrightarrow{g}}(\overrightarrow{v})) (\Sigma_g) = -4(g+1)(\log \lambda) \]
It is well known that
\[ (\Psi^*_{\overrightarrow{g}}(\overrightarrow{v})) (\Sigma_g) = -2\pi \cdot \text{volume of } \{ \text{PSL}(2, \mathbb{R})/\Psi_g(\pi_1(\Sigma_g)) \} = (2\pi)^2 2(1-g) = -8(g-1)\pi^2. \]
By definitions,
\[ (\Psi^*_{\overrightarrow{g}}(\overrightarrow{v})) (\Sigma_g) = (\Phi^*_{\overrightarrow{g}}(\overrightarrow{v})) (\Sigma_g) = 0. \]
Hence,
\[ (\Psi^*_{\overrightarrow{g}}(\alpha g v + \beta \overrightarrow{v})) (\Sigma_g) = -8(g-1)\pi^2 \alpha, \]
\[ (\Phi^*_{\overrightarrow{g}}(\alpha g v + \beta \overrightarrow{v})) (\Sigma_g) = -4(g+1)(\log \lambda)^2 \beta. \]
From Theorem in §1, \( \Psi_{\overrightarrow{g}} \) is topologically conjugate to \( \Phi_{\overrightarrow{g}} \). So, if \( \alpha g v + \beta \overrightarrow{v} \)
is a topological invariant, then
\[ -8(g-1)\pi^2 \alpha = -4(g+1)(\log \lambda)^2 \beta \text{(for all } g \geq 2). \]
Therefore,
\[ \beta = \frac{2(g-1)\pi^2}{(g+1)(\log \lambda)^2} \alpha. \]
On the other hand,
\[ \lim_{g \to \infty} \frac{g-1}{g+1} \cdot \frac{1}{(\log \lambda)^2} = 0. \]
This implies that \( \beta = 0 \), therefore, \( \alpha = 0 \). This contradicts the assumption, \( \alpha^2 + \beta^2 \neq 0 \). \( \square \)
3. Some remarks on $\Psi_g$ and $\Phi_g$.

In previous sections, we use the presentation of $\pi_1(\Sigma_g)$ whose generators are $2g + 2$ loops $a_1, a_2, a_3, \ldots, a_{2g+2}$. The fundamental domain of $\Sigma_g$ corresponding to this presentation is a $4g + 4$ gon in the Poincaré disk $D$, where $D = \{z \in \mathbb{C}; |z| < 1\}$ with the Poincaré metric

$$ds = \frac{2|dz|}{1 - |z|^2}.$$

Now we will construct a symmetric fundamental domain $R$ as is shown in Figure 3.
Let $P$ be the regular orthogonal $2g + 2$ gon whose center coincides with $o \in D \subset C$ and whose edges are a part of geodesics called $s_1, s_2, s_3, \ldots, s_{2g+2}$ in the clockwise order. $\frac{l}{2}$ denotes the length of edges of $P$. Let $e_i$ and $e'_i (i = 1, 2, 3, \ldots, 2g + 2)$ be the geodesics satisfying the following conditions (see Figure 3):

(1) they are outside $P$ and orthogonal to $s_i$, 

(2) the distance between $e_i (e'_i)$ and $P$ is $\frac{l}{4}$.

Then, the $4g + 4$ gon surrounded by $e_1, e'_1, e_2, e'_2, \ldots, e_{2g+2}, e'_{2g+2}$ is the desired fundamental domain $R$. In order to obtain $\Sigma_g$ from $R$, we identify $e_i$ with $e'_i$ by the translation by the length $l$ along $s_i$, for $i = 1, 2, 3, \ldots, 2g + 2$. Hence, we have the presentation of $\pi_1(\Sigma_g)$ (see [Ma]).

**Proposition 2.**

$$\pi_1(\Sigma_g) = \langle a_1, a_2, a_3, \ldots, a_{2g+2}; \\
 a_1a_2a_3 \ldots a_{2g+2} = a_1a_3 \ldots a_{2g+1} = a_2a_4 \ldots a_{2g+2} = 1 \rangle.$$ 

Let $h_g \in PSL(2, \mathbb{R})$ be the hyperbolic element corresponding to the translation by the length $l$ along $s_1$ such that $h_g (e'_1 \cap s_1) = e_1 \cap s_1$. Then we obtain the next proposition.

**Proposition 3.** — The total holonomy of the unstable foliation of the geodesic flow $F_t$

$$\Psi_g : \pi_1(\Sigma_g) \rightarrow PSL(2, \mathbb{R})$$

is defined as follows:

$$\Psi_g(a_i) = T\left(-\frac{i-1}{2(g+1)}\right) \circ h_g \circ T\left(\frac{i-1}{2(g+1)}\right) (i = 1, 2, 3, \ldots, 2g + 2).$$

Moreover, $h_g = h \circ f_g \circ h^{-1}$, where $f_g$ and $h$ are homeomorphism of $S^1$ obtained in §1.

In the rest of this section, we will show that $\Phi_g$ factors through

$$\phi_g : \Gamma_g \rightarrow PL_+(S^1),$$
where $\Gamma_g$ is the triangle group (see [Mi])

$$\Gamma(g + 1, 2g + 2, 2g + 2) = \langle \tau_1, \tau_2, \tau_3; \\
(\tau_1)^{g+1} = (\tau_2)^{2g+2} = (\tau_3)^{2g+2} = \tau_1\tau_2\tau_3 = 1 \rangle.$$ 

**Proposition 4.**

(1) A quadrangle $\omega_1u_1v_2$ is the fundamental domain for some action of the group $\Gamma_g$ on $D$. The quotient space $D/\Gamma_g$ is the 2-dimensional sphere $\Sigma(g+1, 2g+2, 2g+2)$ with three elliptic points of order $g+1, 2g+2, 2g+2$. $\Gamma_g$ also acts on the unit tangent bundle $T_1D$ of $D$ and the quotient space $T_1D/\Gamma_g$ is the Brieskorn manifold $M(g+1, 2g+2, 2g+2) = \{(z_1, z_2, z_3) \in \mathbb{C}^3; z_1^{g+1} + z_2^{2g+2} + z_3^{2g+2} = 0, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}.$

(2) $\overline{\pi} : M(g + 1, 2g + 2, 2g + 2) \rightarrow \Sigma(g + 1, 2g + 2, 2g + 2)$ is a Seifert fibration with a transverse foliation $\mathcal{F}'$. $\mathcal{F}'$ is induced from the bundle foliation of $e : T_1D \rightarrow \partial D$ where $e(v) \in \partial D(v \in T_1D)$ is the end point of the geodesic starting at the base point of $v$ in the direction of $-v$.

(3) The commutative diagram

$$\begin{array}{ccc}
T_1\Sigma_g & \xrightarrow{p_0} & M(g + 1, 2g + 2, 2g + 2) \\
\pi \downarrow & & \overline{\pi} \downarrow \\
\Sigma_g & \xrightarrow{p_1} & \Sigma(g + 1, 2g + 2, 2g + 2)
\end{array}$$

is held where $p_0$ is a $(2g + 2)$-fold covering and $p_1$ is a $(2g + 2)$-fold ramified covering. Moreover, $\mathcal{F} = p_0^*(\mathcal{F}')$ is the unstable foliation of the geodesic flow $F_t$.

The fundamental group $\pi_1(M(g + 1, 2g + 2, 2g + 2))$ (resp. $\pi_1(T_1\Sigma_g)$) is the central extension of $\Gamma_g$ (resp. $\pi_1(\Sigma_g)$) by the infinite cyclic group, i.e.,

$$\pi_1(M(g + 1, 2g + 2, 2g + 2)) = \overline{\Gamma}_g = \langle \tau_1, \tau_2, \tau_3, z; \tau_1z = z\tau_1(i = 1, 2, 3), (\tau_1)^{g+1} = (\tau_2)^{2g+2} = (\tau_3)^{2g+2} = \tau_1\tau_2\tau_3 = z \rangle,$$

(resp. $\pi_1(T_1\Sigma_g) = \langle a_1, a_2, a_3, \ldots, a_{2g+2}, z; a_1z = za_i(i = 1, 2, \ldots, 2g + 2), a_1a_2a_3 \ldots a_{2g+2} = z^{2g+2}, a_1a_3 \ldots a_{2g+1} = z^2, a_2a_4 \ldots a_{2g+2} = z^2 \rangle$)

where $z$ is the class of a general fibre.

Let

$$p_{0*} : \pi_1(T_1\Sigma_g) \rightarrow \overline{\Gamma}_g$$
denote the homomorphism induced by $p_0$.

**Lemma 6.**

$$p_{0*}(a_i) = (r_2)^{1-i} r_1 (r_2)^{i+1} (i = 1, 2, 3, \ldots, 2g + 2) \text{ and } p_{0*}(z) = z.$$

By Theorem 3.5 in [EHN], there exist homomorphisms

$$\widetilde{\Phi}_g : \pi_1(T_1 \Sigma_g) \to PL_+^\sim(S^1)$$

and

$$\widetilde{\phi}_g : \overline{T}_g \to PL_+^\sim(S^1)$$

corresponding to transverse foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively, where $PL_+^\sim(S^1) \subset \text{Homeo}^\sim_+(S^1)$ is the universal covering group of $PL_+(S^1)$. In Proposition 5, we see that $\mathcal{F}$ is induced from $\mathcal{F}'$. In fact, there exists a lift $\tilde{f}_g \in PL_+^\sim(S^1)$ of $f_g \in PL_+(S^1)$ satisfying that

$$\widetilde{\Phi}_g(a_i) = T \left( -\frac{i - 1}{2(g + 1)} \right) \circ \tilde{f}_g \circ T \left( \frac{i - 1}{2(g + 1)} \right)$$

$$(i = 1, 2, 3, \ldots, 2g + 2), \widetilde{\Phi}_g(z) = T(1)$$

and

$$\widetilde{\phi}_g(\tau_1) = \tilde{f}_g \circ T \left( -\frac{1}{g + 1} \right), \widetilde{\phi}_g(\tau_2) = T \left( \frac{1}{2(g + 1)} \right), \widetilde{\phi}_g(z) = T(1).$$

Consequently,

**Lemma 7.**

$$\overline{\Phi}_g = \widetilde{\phi}_g \circ p_{0*}.$$

We can consider a homomorphism $\phi_g$ satisfying the next commutative diagram,

$$\overline{T}_g \xrightarrow{\widetilde{\phi}_g} PL_+^\sim(S^1) \quad \xrightarrow{\phi_g} \quad PL_+(S^1),$$

i.e., defined by

$$\phi_g(\tau_1) = f_g \circ T \left( -\frac{1}{g + 1} \right), \phi_g(\tau_2) = T \left( \frac{1}{2(g + 1)} \right).$$
\( p_0^* \) induces the homomorphism

\[
p_0^* : \pi_1(\Sigma_g) = \pi_1(T_1 \Sigma_g)/\langle z \rangle \to \Gamma_g = T_g/\langle z \rangle.
\]

From Lemma 7, we have the next proposition which says that \( \Phi_g \) factors through \( \phi_g \).

**Proposition 7.**

\[
\Phi_g = \phi_g \circ p_0^*.
\]

**BIBLIOGRAPHIE**


Manuscrit reçu le 10 avril 1991,

Norikazu HASHIGUCHI,
Department of Mathematics
University of Tokyo
Hongo
Tokyo, 113 (Japan).
Present address :
Department of Mathematics
College of Science and Technology
Nihon University
Kanda-Surugadai
Tokyo, 101 (Japan).