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ON INTEGER POINTS IN POLYGONS

by Maxim SKRIGANOV

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1. The results.

Let $P \subset \mathbb{R}^2$ be a compact region, $\mathbb{Z}^2 \subset \mathbb{R}^2$ be the integer point lattice and

$$n(P) = \operatorname{card} P \cap \mathbb{Z}^2$$

be the number of integer points lying in P. We set by definition

$$n(P) = \operatorname{area} P + r(P).$$

Let $P + X$ be the translation of P by a vector $X \in \mathbb{R}^2$ and tP be the dilatation of P by a factor $t > 0$. We set

$$R(P) = \max_{X \in \mathbf{R}^2} |r(P + X)|.$$

The classical lattice point problem deals with behavior of *the error* $R(tP)$ as $t \to \infty$. In the present paper we shall study this problem under the assumption that P is a polygon with finitely many sides. At first sight, the problem under this assumption seems trivial. Indeed, the trivial bound $R(tP) \ll t$ is, obviously, best possible here if at least one of the normals to the sides of P is proportional to a vector of $\mathbb{Z}^2$. However, in the present paper we wish to examine the opposite situation.

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We shall use the following definitions. The polygon P is said to be *irrational* if the normals to its sides are proportional to the vectors with coordinates $(1, \alpha_j)$, $j = 1, \ldots, h$ (h is the number of sides of P) and all of *the slopes* $\alpha_1, \ldots, \alpha_h$ are irrational numbers. If all of the slopes $\alpha_1, \ldots, \alpha_h$ are irrational algebraic numbers the polygon P is said to be *algebraic.* We shall use the notation P_A for the polygon with a given set of slopes $A = (\alpha_1, \ldots, \alpha_h)$.

It turns out that for irrational polygons the error term can be *logarithmically small.* Thus the asymptotic behavior of the error is extremely *unstable* with respect to small variations of the polygon. The phenomenon of such amazingly instability as well as the possibility of logarithmically small errors in the lattice point problem were first studied in the author's papers [S1], [S2].(*)

We wish to state one of our results from [S2]. Let $\alpha = [a_0; a_1, a_2, a_3, \cdots]$, $a_j \in \mathbb{Z}$, be the continued fraction expansion of an irrational α with partial quotients $a_j \geq 1$, $j \geq 1$ (*cf.* [L] and [P] for all details concerning continued fractions). We introduce the following *arithmetic function*

$$q_\alpha(\rho) = \sum_{1 \leq j \leq \rho} a_j$$

for real $\rho \geq 1$.

THEOREM 1 (*cf.* [S2], Th. 6.1). — *For any irrational polygon P_A one has the bound*

$$R(tP_A) \ll \sum_{j=1}^{h} q_{\alpha_j}(5\ell n\ t).$$

Using this theorem one can easily give a lot of examples of polygons with extremely small errors. For this purpose it is sufficient to choose natural partial quotients as we like, since the corresponding continued fraction always converges. We mention here only two examples.

Example 1. — Let the partial quotients of all of the slopes $A = (\alpha_1, \ldots, \alpha_h)$ be bounded by a constant, say, all of the slopes $A = (\alpha_1, \ldots, \alpha_h)$ be quadratic irrationalities. Then for the corresponding irrational polygon P_A one has the bound

$$R(tP_A) \ll \ell n\ t.$$

(*) In connection with this, the following works should be also mentioned : the well known memoir [HL] by Hardy and Littlewood which is rather closed to the matter of our study and the paper [R2] by Randol where a logarithmic bound was given for the mean value of the error over all rotations of the polygon.

Example 2. — The famous *Euler's* formula (*cf.* [L], Chap. 5, Sect. 1 and [P], §64) gives the following continued fraction expansion

$$\tanh \frac{1}{k} = \frac{e^{2/k} - 1}{e^{2/k} + 1} = [0; k, 3k, 5k, 7k, 9k, \cdots]$$

where e is the base of natural logarithms and $k = 1, 2, \cdots$. Now, let all of the slopes $A = (\alpha_1, \ldots, \alpha_h)$ be transcendental numbers of the indicated form. Then one has the bound

$$R(tP_A) \ll \ell n^2\, t.$$

It is well known that for almost all $\alpha \in \mathbb{R}^1$ the arithmetic function $q_\alpha(\rho)$ satisfies the estimate $q_\alpha(\rho) \ll \rho\varphi(\ell n\, \rho)$ as $\rho \to \infty$, where $\varphi(t)$, $t > 0$, is an arbitrary nondecreasing positive function such that the series

$$\sum_{j=1}^{\infty} \frac{1}{\varphi(j)}$$

converges (*cf.* [Kh], p. 123). Taking this statement into account we derive from Theorem 1 the following "*metrical*" result.

THEOREM 2. — *Let the number h of sides of polygons P_A be fixed. Then for almost all slopes $A = (\alpha_1, \ldots, \alpha_h) \in \mathbb{R}^h$ one has the bound*

$$R(tP_A) \ll \ell n\, t \cdot \varphi(2\ell n\, \ell n\, t),$$

where the function $\varphi(\cdot)$ is indicated above.

In particular (if we take $\varphi(t) = t^{1+\varepsilon}$), for almost all slopes $A = (\alpha_1, \ldots, \alpha_h) \in \mathbb{R}^h$ one has the bound

$$R(tP_A) \ll \ell n\, t \cdot (\ell n\, \ell n\, t)^{1+\varepsilon}$$

with arbitrarily small $\varepsilon > 0$.

Theorem 2 shows that the appearance of logarithmically small errors on the set of all polygons is quite typical.

Continued fractions that had been used in Theorem 1 are not always convenient, say, for general algebraic polygons, since, nothing is known about the growth properties of the partial quotients for algebraic irrationalities of degree > 2. In the present paper we wish to evaluate the error for a polygon P_A directly in the terms of *diophantine properties* of its slopes $A = (\alpha_1, \ldots, \alpha_h)$.

To state our results we recall some definitions (*cf.* [KN], Chap. 2, Sect. 3). For a real number t, let $\langle t \rangle$ denote the distance from t to the nearest integer, namely,

$$\langle t \rangle = \min_{n \in \mathbf{Z}} |t - n| = \min[\{t\}, 1 - \{t\}],$$

where $\{t\}$ is the fractional part of t. Let $\Psi(t)$, $t > 0$, be a nondecreasing positive function. The irrational α is said to be *of type* $< \Psi$ if the inequality

$$q\langle \alpha q\rangle > \frac{1}{\Psi(q)}$$

holds for all positive integers q. This definition was given by Lang (*cf.* [L], Chap. 2, Sect. 1).

The irrational polygon P_A is said to be *of type* $< \Psi$ if all of its slopes $A = (\alpha_1, \dots, \alpha_h)$ are of type $< \Psi$.

THEOREM 3. — *For any irrational polygon P_A of type $< \Psi$ one has the bound*

$$R(tP_A) \ll t\rho^{-\theta} + \Psi(\rho)\ell n^2\ \rho,$$

where $0 < \theta < 1$ is arbitrarily fixed and $\rho \geq 1$ is an arbitrary parameter; moreover the constant implied by $\ll$ is independent of ρ.

The proof of this theorem will be given in the next section. Note that the bound of Theorem 3 can be slightly improved, namely, the factor $\ell n^2\ \rho$ can be replaced by $\ell n\ \rho$. However, this improvement requires more complicated techniques and is not given here.

The famous theorem of *Thue-Siegel-Roth* says that every irrational algebraic number α is of type $< \Psi_\varepsilon$, where $\Psi_\varepsilon(t) = c_\varepsilon t^\varepsilon$ with arbitrarily small $\varepsilon > 0$ and a suitable $c_\varepsilon = c_\varepsilon(\alpha) > 0$ (*cf.* [Sch]). Taking this statement into account and choosing $\rho^\theta = t$ we derive from Theorem 3 the following result.

THEOREM 4. — *For any algebraic polygon P one has the bound*

$$R(tP) \ll t^\varepsilon$$

with arbitrarily small $\varepsilon > 0$.

Note that the upper bounds given in Theorems 1–4 are rather sharp. For example, one can prove that for every parallelogram Π the following *omega-theorem*

$$R(t\Pi) = \Omega(\ell n\ t)$$

holds.

Perhaps, it is worth to mention also the following simple corollary of Theorem 3.

THEOREM 5. — *For any irrational polygon P one has the bound*

$$R(tP) = o(t).$$

Indeed, any irrational polygon P is of type $< \Psi_P$, with certain $\Psi_P(t)$. Choosing the parameter $\rho = \rho(t)$ in Theorem 3 such that $\rho \to \infty$ as $t \to \infty$, but at the same time $\Psi_P(\rho)\ell n^2 \rho = o(t)$, we complete the proof. The bound of Theorem 5 cannot be improved on the set of all irrational polygons (*cf.* [HL]).

The reader can ask the question : are there regions besides polygons with logarithmically small errors terms in the lattice point problem? We conjecture that the answer is negative and the set of polygons is the only class of regions with logarithmically small errors. I checked this conjecture only for regions bounded by finitely many pieces of analytic curves. For general regions this problem requires, probably, more deep analysis of the Fourier transform of the characteristic functions of planar sets in the spirit of works by Randol [R2] and Colin de Verdière [CV]. The author hopes to touch this task on a suitable occasion.

The foregoing results give rather complete picture of the behavior of the errors terms for polygons in the plane. These results form a part of much more complete theory of *anomaly small errors* in the lattice point problem that can be developed in arbitrary dimensions and that will be given in consequent author's papers (see [S3] and [S4]).

It gives me pleasure to express my sincere appreciation to professor Colin de Verdière for his hospitality in Institut Fourier where this paper had been written and given as a lecture.

2. Proof of Theorem 3.

In proving we shall need the following ingredients.

i. General bounds for the number of integer points in compact regions (*cf.* [CV], [R1], here we follow [S2], Sect. 3).

We recall the definition : given a compact region P and a number τ, $0 < \tau \leq 1$; we say that the compact regions $P_\tau^\pm$ form a *τ-approximation* of P if $P_\tau^- \subset P \subset P_\tau^+$ and the points of the boundaries $\partial P_\tau^\pm$ are at a distance $\geq \tau$ from the boundary ∂P.

Let $\chi(\mathcal{P}, X)$, $X \in \mathbb{R}^2$, denote the characteristic function of a region $\mathcal{P}$ and let

$$\widetilde{\chi}(\mathcal{P}, Y) = \iint_{\mathcal{P}} \exp(2\pi i Y \cdot X) dX \ , \quad i = \sqrt{-1},$$

be the Fourier transform of $\chi(\mathcal{P}, X)$. We use the notation $Y \cdot X$ for the inner product of the vectors Y and X, and the notation $|X|$ for the norm.

Fix a non-negative function $G(X)$, $X \in \mathbb{R}^2$, of the class C^∞ with the support inside the disk $|X| \leq 1$, and assume that $\int G(X)dX = 1$. We set $G_\tau(X) = \tau^{-2} G(\tau^{-1} X)$, $0 < \tau < 1$. Taking into account that the Fourier transform $\widetilde{G}_\tau(Y) = \widetilde{G}(\tau Y)$ we obtain the estimates

$$|\widetilde{G}_\tau(Y)| \leq C_B(\tau |Y|)^{-B} \ , \quad |\widetilde{G}_\tau(Y)| \leq 1 \tag{2.1}$$

with any $B > 1$ and some $C_B > 0$.

LEMMA 1. — *Let a compact region $P \subset \mathbb{R}^2$ be given. Then for any τ-approximation $P_\tau^\pm$ of P one has the bound for the error $R(P)$*

$$R(P) \leq \operatorname{area} P_\tau^+ - \operatorname{area} P_\tau^- + \sum_{\gamma \in \mathbf{Z}^2}{}' \Big(|\widetilde{\chi}(P_\tau^+, \gamma| + |\widetilde{\chi}(P_\tau^-, \gamma)| \Big) |\widetilde{G}(\tau\gamma)|,$$

where the prime over $\sum$ means that the summand with $\gamma = 0$ is omitted.

Proof of Lemma 1 is identical with the proof of Lemma 3.3 in [S2].

ii. The Fourier transform of the characteristic function of a polygon (here we follow [S4]).

Let P be a convex polygon with counterclockwise oriented boundary

$$\partial P = \sum_{j=1}^{h} S_j,$$

where $S_j = [A_j, A_{j+1}]$ are oriented sides joining the vertices A_j and A_{j+1}; moreover $A_{h+1} = A_1$.

Let N_j be normals to S_j and L_j be the unit vectors in the direction of S_j.

LEMMA 2. — *In the above notation one has the following estimates for the Fourier transform of the characteristic function $\widetilde{\chi}(P, Y)$ of the polygon P*

$$|\widetilde{\chi}(P, Y)| \leq \frac{1}{2\pi |Y|} \quad \operatorname{perimeter} P,$$

$$|\widetilde{\chi}(P, Y)| \leq \frac{1}{2\pi^2 |Y|^2} \sum_{j=1}^{h} \frac{|Y \cdot N_j|}{|Y \cdot L_j|}.$$

Proof. — The statements of Lemma 2 are direct corollaries of the following explicit formulas for $\widetilde{\chi}(P,Y)$. Let Δ be the Laplacian in variables $X \in \mathbb{R}^2$ and $\dfrac{\partial}{\partial N}$ be the normal derivative at a given point $X \in \partial P$. One has

$$-4\pi^2|Y|^2\widetilde{\chi}(P,Y) = \iint\limits_P \Delta \exp(2\pi i Y \cdot X)dX.$$

Integration by parts gives

$$\begin{aligned}-4\pi^2|Y|^2\widetilde{\chi}(P,Y) &= \int_{\partial P} \frac{\partial}{\partial N} \exp(2\pi i Y \cdot X)dX \\ &= \sum_{j=1}^{h} \int_{S_j} \frac{\partial}{\partial N} \exp(2\pi i Y \cdot X)dX \\ &= \sum_{j=1}^{h} 2\pi i Y \cdot N_j \int_{S_j} \exp(2\pi i Y \cdot X)dX.\end{aligned}$$

Now, first estimate of lemma follows from this formula at once. For the second estimate we have to calculate in this formula the integrals over the segments S_j. As a result

$$-4\pi^2|Y|^2\widetilde{\chi}(P,Y) = \sum_{j=1}^{h} \frac{Y \cdot N_j}{Y \cdot L_j} \Big[\exp(2\pi i Y \cdot A_{j+1}) - \exp(2\pi i Y \cdot A_j)\Big].$$

This completes the proof.

iii. Some results from diophantine approximations (*cf.* [HL], [L], [Sch], here we follow [KN] and [L]).

Let α be an irrational number. We introduce *the arithmetic function*

$$\xi_\alpha(\rho) = \sum_{1 \le q \le \rho} \frac{1}{q\langle \alpha q \rangle}$$

for real $\rho \geq 1$.

LEMMA 3. — *Let α be of type $< \Psi$. Then one has the estimate*

$$\xi_\alpha(\rho) \ll \Psi(2\rho)\ell n^2\, \rho.$$

For the proof of Lemma 3 we refer to [KN], Chap. 3, Sect. 3, Lemma 3.3. Note that the bounds for such sums were first given in [HL] in terms of continued fractions and in [L], Chap. 3, Sect. 2, in terms of the type of α.

We introduce another *arithmetic function*

$$\sigma_\alpha(\rho) = \sum_{1\le q\le\rho} \frac{1}{q} \sum_{1\le p\le\rho} \frac{1}{|p-\alpha q|}$$

for irrational α and real $\rho \ge 1$.

LEMMA 4. — *Let α be of type $< \Psi$. Then one has the estimate*

$$\sigma_\alpha(\rho) \ll \Psi(2\rho)\ell n^2\ \rho.$$

Proof. — Let us estimate the inner sum in the definition of $\sigma_\alpha(\rho)$. If the number αq is outside of the interval $[1/2; \rho + 1/2]$, then we have the bound

$$\sum_{1\le p\le\rho} \frac{1}{|p-\alpha q|} \le \sum_{1\le \ell\le\rho} \frac{1}{\frac{1}{2}+\ell} \ll \ell n\ \rho < \frac{1}{\langle\alpha q\rangle} + \ell n\ \rho.$$

Let αq be inside the interval $[1/2; \rho + 1/2]$ and let $\alpha q = \langle\alpha q\rangle + p_0$ where $p_0 \in \mathbb{Z} \cap [1, \rho]$, then we have the bound

$$\sum_{1\le p\le\rho} \frac{1}{|p-\alpha q|} = \frac{1}{\langle\alpha q\rangle} + \sum_{\substack{1\le p\le\rho\\ p\ne p_0}} \frac{1}{|p-\alpha q|} \le \frac{1}{\langle\alpha q\rangle} + 2\sum_{1\le\ell\le\rho} \frac{1}{\frac{1}{2}+\ell} \ll \frac{1}{\langle\alpha q\rangle} + \ell n\ \rho.$$

Using these bounds we find that

$$\sigma_\alpha(\rho) \ll \sum_{1\le q\le\rho} \frac{1}{q\langle\alpha q\rangle} + \ell n\ \rho \sum_{1\le q\le\rho} \frac{1}{q} \ll \xi_\alpha(\rho) + \ell n^2\ \rho.$$

The reference to Lemma 3 completes the proof.

Now we are in position to prove Theorem 3. Let P_A be an irrational polygon of type $< \Psi$. Without loss of generality we can assume that P_A is convex. Really, extending the sides of a given polygon P_A we obtain a fragmentation of P_A into a collection of disjoint convex polygons with the same set A of slopes, and it remains to prove Theorem 3 for each of these fragments.

The convex polygon tP_A can be given by the inequalities

$$tP_A = \{X \in \mathbb{R}^2 : X\cdot N_j \le b_j t, \quad j = 1, \ldots, h\}$$

where $b_1, \ldots, b_h$ are some constant independent of t. We choose the following τ-approximations for tP_A

$$P_\tau^\pm(t) = \{X \in \mathbb{R}^2 : X\cdot N_j \le b_j t \pm \tau, \quad j = 1,, \ldots, h\},$$

$0 < \tau \le 1$, $t \ge 1$; moreover

$$\operatorname{area} P_\tau^\pm(t) = t^2 \operatorname{area} P \pm \tau t \operatorname{perimeter} P + O(\tau^2),$$
$$\operatorname{perimeter} P_\tau^\pm(t) = t \operatorname{perimeter} P + O(\tau).$$

Note that the normals $N_1, \dots, N_h$ to the sides and the unit vectors $L_1, \dots, L_h$ in the directions of the sides are the same for all polygons tP_A, $P_\tau^\pm(t)$ and they are of the form

$$N_j = c_j(1, \alpha_j), L_j = c_j(-\alpha_j, 1), c_j^2 = \frac{1}{1+\alpha_j^2}, j = 1, \dots, h.$$

From Lemma 1 we derive the following bound

$$R(tP) \le 2\tau t \operatorname{perimeter} P + O(\tau^2) + \sum_{\gamma \in \mathbb{Z}^2}{}' \Big(|\widetilde{\chi}(P_\tau^+(t), \gamma)| + |\widetilde{\chi}(P_\tau^-(t), \gamma)| \Big) |\widetilde{G}(\tau\gamma)|. \tag{2.2}$$

Further we shall assume that $\tau = \rho^{-\theta}$, where $\rho \ge 1$ is large parameter and $\theta = \dfrac{B-1}{B+1}$, moreover B is taken from (2.1). Let $K_\rho \subset \mathbb{R}^2$ be the square

$$K_\rho = \{X = (x, y) \in \mathbb{R}^2 : |x| < \rho, |y| < \rho\}$$

and $Q_\rho = \mathbb{R}^2 \setminus K_\rho$ be its complement. In the bound (2.2) we divide the sum over $\gamma \in \mathbb{Z}^2$ into a sum over $\gamma \in \mathbb{Z}^2 \cap Q_\rho$ and a sum over $\gamma \in \mathbb{Z}^2 \cap K_\rho$. In the first sum over $\gamma \in \mathbb{Z}^2 \cap Q_\rho$ we replace $|\widetilde{\chi}(P_\tau^\pm(t), Y)|$ and $|\widetilde{G}(\tau Y)|$ by first bounds from Lemma 2 and from the formula (2.1), respectively. In the second sum over $\gamma \in \mathbb{Z}^2 \cap K_\rho$ we use similarly second bounds from Lemma 2 and from the formula (2.1). As a result we obtain the inequality

$$R(tP) \le 2\tau t \operatorname{perimeter} P + O(1) + \mathcal{D}_1 + \mathcal{D}_2, \tag{2.3}$$

here

$$\mathcal{D}_1 = \frac{C_B}{2\pi} (\operatorname{perimeter} P) t \tau^{-B} \sum_{\gamma \in \mathbb{Z}^2 \cap Q_\rho} |\gamma|^{-B-1}$$

$$\mathcal{D}_2 = \sum_{\gamma \in \mathbb{Z}^2 \cap K_\rho}{}' \frac{1}{2\pi^2 |\gamma|^2} \sum_{j=1}^h \frac{|\gamma \cdot N_j|}{|\gamma \cdot L_j|} = \frac{1}{2\pi^2} \sum_{j=1}^h H_{\alpha_j}(\rho),$$

where we introduced the arithmetic function

$$H_\alpha(\rho) = \sum_{|q|, |p| \le \rho}{}' \frac{1}{q^2 + p^2} \frac{|q + \alpha p|}{|p - \alpha q|}. \tag{2.4}$$

Recall that the prime over the sums means that the summand with $\gamma = (q, p) = (0, 0)$ is omitted.

The bound for $\mathcal{D}_1$ is very simple, indeed,

$$\mathcal{D}_1 \ll t\tau^{-B}\rho^{1-B} = t\rho^{-\theta},$$

since

$$\sum_{\gamma\in\mathbf{Z}^2\cap Q_\rho} |\gamma|^{-B-1} \ll \rho^{1-B}.$$

Grouping in the sum (2.4) separately terms with $q = 0$, $p \neq 0$ with $q \neq 0$, $p = 0$ and with $q \neq 0$, $p \neq 0$, and using the inequality

$$\frac{|q+\alpha p|}{q^2+p^2} \leq \frac{1}{2}(2+|\alpha|)\frac{1}{|q|}$$

we can evaluate the sum (2.4) as follows

$$\begin{aligned} H_\alpha(\rho) &= 2|\alpha| \sum_{1\leq p\leq\rho} \frac{1}{p} + \frac{2}{|\alpha|} \sum_{1\leq q\leq\rho} \frac{1}{q} \\ &+ (2+|\alpha|) \sum_{1\leq q,p\leq\rho} \frac{1}{q}\Big(\frac{1}{|p-\alpha q|} + \frac{1}{|p+\alpha q|}\Big) \\ &\ll \ell n\ \rho + \sigma_\alpha(\rho) + \sigma_{-\alpha}(\rho) \ll \Psi(2\rho)\ell n^2\ \rho, \end{aligned}$$

where, at the last step, Lemma 4 had been used; note also that the numbers α and $-\alpha$ are of the same type. Therefore

$$\mathcal{D}_2 \ll \Psi(2\rho)\ell n^2\ \rho.$$

Substituting the above bounds for $\mathcal{D}_1$ and $\mathcal{D}_2$ into the inequality (2.3) we complete the proof of Theorem 3.

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