

# ANNALES DE L'INSTITUT FOURIER

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*Annales de l'institut Fourier*, tome 43, n° 2 (1993), p. 359-364

[http://www.numdam.org/item?id=AIF\\_1993\\_\\_43\\_2\\_359\\_0](http://www.numdam.org/item?id=AIF_1993__43_2_359_0)

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## ON ACTIONS OF $C^*$ ON ALGEBRAIC SPACES

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A theorem of Luna [L] says that any torus embedding which is a smooth complete algebraic space (i.e. a smooth Moisozon space) is an algebraic variety. This result is a consequence of the following theorem proved in the present paper.

**THEOREM.** — *Let  $C^*$  act on a smooth complete algebraic space  $X$ . Let  $X_1$  be the source of the action. If  $X_1$  is an algebraic variety, then  $X_1$  is contained in the set of all schematic points of  $X$ .*

As a corollary of the theorem we obtain not only the theorem of Luna, but also a result saying that any smooth and complete algebraic space with an action of a reductive group  $G$ , such that there exists only one closed  $G$ -orbit in  $X$ , is a projective variety.

For basic properties of algebraic spaces see [Kn].

We begin with the following

**LEMMA 1.** — *Let an algebraic group  $G$  act on a complete algebraic space  $X$ . Then the action is meromorphic.*

*Proof.* — Let  $X_0$  be a projective model of the field  $C(X)$  of meromorphic functions on  $X$ . Then the action of  $G$  on  $X$  leads to an action of  $G$  on  $C(X)$  and to the induced action of  $G$  on  $X_0$ . Moreover by Hironaka Resolution Theorem we may assume that  $X_0$  is smooth and that we have

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<sup>(\*)</sup> supported by Polish KBN Grant GR-87.

Key words : Actions of  $C^*$  - Algebraic spaces - Toric varieties.

A.M.S. Classification : 14L30 - 32M05.

a holomorphic  $G$ -equivariant map  $X_0 \rightarrow X$ . Let  $G_1$  be a projective variety containing  $G$  as an open subset. Since  $X_0$  is projective with an action of  $G$ , the action of  $G$  on  $X_0$  is meromorphic and the graph of the action  $\Gamma_0 \subseteq G \times X_0 \times X_0$  has an analytic subvariety  $\Gamma_1$  in  $G_1 \times X_0 \times X_0$  as its closure. Let  $\Gamma$  be the closure in  $G_1 \times X \times X$  of the graph of the action of  $G$  on  $X$ . Now, the map  $X_0 \rightarrow X$  induces a map  $\Gamma_1 \rightarrow \Gamma$ . Since the image of a compact analytic subvariety is an analytic subvariety,  $\Gamma$  is an analytic subvariety of  $G_1 \times X \times X$  and thus the proof is complete.

Assume now that we have a meromorphic action of  $C^*$  on a compact manifold  $X$ . Let  $X_1 \cup \dots \cup X_r$  be the decomposition of the fixed point set of the action of  $C^*$  on  $X$  into connected components. For  $i = 1, \dots, r$ , let  $X_i^+ = \{x \in X; \lim_{t \rightarrow 0} tx \in X_i\}$ . It follows from [B-BS] Appendix to §0, that there exists exactly one  $i = 1, \dots, r$ , such that  $X_i^+$  is open and Zariski dense in  $X$ .  $X_i$  with this property is called the source of the action. Assume that  $X_1$  is the source. Again by [B-BS] Appendix to §0 the map  $\tau : X_1^+ \rightarrow X_1$  defined by  $x \rightarrow x_0 = \lim_{t \rightarrow 0} tx$  is holomorphic.

We are going to show the following

**LEMMA 2.** — *Let  $X$  be a smooth algebraic space  $X$ . Then the map  $\tau : X_1^+ \rightarrow X_1$  defined above is a holomorphic bundle with fiber being an affine space  $C^p$  with a linear action of  $C^*$  such that all weights of the action are positive.*

*Proof.* — Take  $x \in X_1$ . Then there exists an open neighborhood  $U$  of 0 in the tangent space  $T_{x,x}$  invariant under the induced action of  $S^1$  (where  $S^1 = \{z \in C^*; |z| = 1\}$ ) and a  $S^1$ -invariant biholomorphic map  $\phi$  of  $U$  onto an open neighborhood of  $x \in X$  (see e.g. [Ka], Satz 4.4). We may assume that  $T_{x,X} = C^n$ , and that the induced action of  $C^*$  is diagonal

$$t(z_1, \dots, z_n) = (\kappa_1(t)z_1, \dots, \kappa_n(t)z_n),$$

where  $\kappa_1, \dots, \kappa_n$  are characters of  $C^*$  and hence can be identified with integers. Moreover we may assume that

$$U = \{z = (z_1, \dots, z_n) \in C^n; |z_i| \leq \varepsilon, \text{ for } i = 1, \dots, n\},$$

where  $\varepsilon$  is a sufficiently small positive real number. Consider an open connected subset  $V$  of  $C^* \times U$  composed of all such points  $(t, u)$  that  $tu \in U$ . On  $V$  we define two holomorphic mappings :

$$(t, u) \mapsto \phi(tu)$$

$$(t, u) \mapsto t\phi(u).$$

For  $t = s \in S^1$ , we have  $\phi(su) = s\phi(u)$ . Hence the above mappings are equal on  $S^1 \times U$ . Since  $V$  is connected and the mappings are holomorphic, they coincide. Thus  $\phi(tu) = t\phi(u)$ , whenever  $u, tu \in U$ . Now define  $\psi : C^* \times U \rightarrow X$  by  $\psi(t, u) = t\phi(u)$ . We claim that  $t_1u_1 = tu$  implies  $\psi(t_1, u_1) = \psi(t, u)$ , i.e. that  $\psi$  induces a holomorphic map on  $C^*U$ . In order to prove this claim notice that if  $t_1u_1 = tu$ , then  $t^{-1}t_1u_1 = u$  and by the above,  $\phi(u) = \phi(t^{-1}t_1u_1) = (t^{-1}t_1)\phi(u_1)$ . Hence  $\psi(t, u) = t\phi(u) = t(t^{-1}t_1)\phi(u_1) = t_1\phi(u_1) = \psi(t_1, u_1)$ .

So we have obtained a holomorphic  $C^*$ -invariant map  $\psi : C^*U \rightarrow X$ . Since  $x$  belongs to the source of  $X$ , the weights  $\kappa_i$ ,  $i = 1, \dots, n$ , are nonnegative and we may assume that  $\kappa_1 \geq \dots \geq \kappa_p > \kappa_{p+1} = \dots = \kappa_n = 0$ . Since on  $U$  the map is an open immersion into  $X$ , it is an open immersion of  $C^*U$  into  $X$ . In fact, assume that  $\psi(tu) = \psi(t_1u_1)$ . Then, since the weights  $\kappa_i$  are nonnegative, there exists  $t_0 \in T$  such that  $t_0tu, t_0t_1u_1 \in U$  and  $\psi(t_0tu) = t_0\psi(tu) = t_0\psi(t_1u_1) = \psi(t_0t_1u_1)$ . Hence  $t_0tu = t_0t_1u_1$  and  $tu = t_1u_1$ .

Let  $\pi : C^n \rightarrow C^{n-p} \subset C^n$  be the projection map  $\pi(z_1, \dots, z_n) = (0, \dots, 0, z_{p+1}, \dots, z_n)$ . Then for  $z = (z_1, \dots, z_n) \in C^*U$ ,  $\psi\pi(z) = \tau\psi(z)$ . Thus  $\tau|_{\psi(C^*U)}$  is a trivial bundle with fiber  $C^p$ . This finishes the proof of the lemma.

The gluing functions of the bundle  $X_1^+ \rightarrow X_1$  have values in the automorphism group  $\text{Aut}_{C^*}(C^p)$  of holomorphic automorphisms of  $C^p$  commuting with the action of  $C^*$ .

**LEMMA 3.** — *Let  $\tau : X_1^+ \rightarrow X_1$  be as in Lemma 2. Then the bundle is algebraic.*

*Proof.* — By theorem 3 in [Se2] (compare also [Se1]), it is enough to show that  $\text{Aut}_{C^*}(C^p)$  is a linear algebraic group. Any  $\alpha \in \text{Aut}_{C^*}(C^p)$  is of the form  $\alpha(z) = (\alpha_1(z), \alpha_2(z), \dots, \alpha_p(z))$ , where  $\alpha_1, \dots, \alpha_p$  are holomorphic functions in  $p$  variables. Moreover since  $\alpha$  commutes with action of  $C^*$ ,  $\alpha_i$ , for  $i = 1, \dots, p$ , is homogeneous of weight  $\kappa_i$  when we attach weight  $\kappa_j$  to variable  $x_j$ , for  $j = 1, \dots, p$ .

Since the weights  $\kappa_j$  are strictly positive,  $\alpha_i$  for  $i = 1, \dots, p$ , is a polynomial and there exists an integer  $N$  such that degrees of all polynomials  $\alpha_i$ , for all  $\alpha \in \text{Aut}_{C^*}(C^p)$ , are bounded by  $N$ . On the other hand  $\alpha \in \text{Aut}_{C^*}(C^p)$  if and only if coefficients of the corresponding polynomials  $\alpha_i$  satisfy some fixed polynomial identities. This shows that

$\text{Aut}_{C^*}(C^p)$  is an affine and hence a linear group. The proof is complete.

It follows from Lemma 3 that  $X_1^+ - X_1/C^* \rightarrow X_1$  is an algebraic bundle with fiber  $C^p - \{0\}/C^* - a$  weighted projective space.

**LEMMA 4.** — *Any  $C^*$ -invariant meromorphic function on  $X_1^+$  is meromorphic on  $X$ .*

*Proof.* — The field of  $C^*$ -invariant meromorphic functions on  $X_1^+$  can be identified with the field  $C(X_1^+ - X_1/C^*)$  of meromorphic (hence rational) functions on a complete algebraic variety  $X_1^+ - X_1/C^*$ . On the other hand the field of  $C^*$ -invariant meromorphic functions on  $X$  can be identified with a subfield  $L$  of  $C(X_1^+ - X_1/C^*)$ . Since both have transcendence degree over  $C^*$  equal to  $n - 1$ , the extension  $L \subseteq C(X_1^+ - X_1/C^*)$  is algebraic.

Let  $U \subseteq X$  be an open  $C^*$ -invariant subset of  $X$  composed of all schematic points. Let  $U_1 \subseteq U$  be an open  $C^*$ -invariant algebraic subvariety such that there exists space of orbits  $U_1/C^*$ . Then  $U_1 \cap (X_1^+ - X_1)$  is open dense in  $X$  and  $U_1 \cap (X_1^+ - X_1)/C^*$  is open dense in  $(X_1^+ - X_1)/C^*$ . Rational  $C^*$ -invariant functions on  $U_1$  are meromorphic on  $X$  and separate points of  $U_1/C^*$ . Hence functions from  $L$  separate points belonging to an open dense subset  $U_1 \cap (X_1^+ - X_1)/C^* \subseteq (X_1^+ - X_1)/C^*$ . This shows that the degree of  $C(X_1^+ - X_1)/C^*$  over  $L$  is equal to 1 and hence  $L = C((X_1^+ - X_1)/C^*)$ . The proof of the lemma is finished.

We say that a complex valued function  $g$  defined on a space  $Y$  with an action of  $C^*$  is  $C^*$ -semi-invariant if, for any  $y \in Y$  and  $t \in C^*$ ,  $g(ty) = \kappa(t)g(y)$ , where  $\kappa : C^* \rightarrow C^*$  is a character of  $C^*$ . Then  $\kappa$  is called the weight of the semi-invariant function  $g$ .

**LEMMA 5.** — *Let  $f$  be a  $C^*$ -semi-invariant meromorphic function on  $X_1^+$ . Then  $f$  is a meromorphic function on  $X$ .*

*Proof.* — Let  $U$  be as in the proof of Lemma 4. Then the field  $C(U)$  of rational functions on  $U$  coincides with the field of meromorphic functions on  $X$ . One can find a function  $g \in C(U)$  of the same weight as  $f$ . Then  $f/g$  is  $C^*$ -invariant and meromorphic on  $X_1^+$ . Hence by Lemma 4  $f/g$  is meromorphic on  $X$ . Since  $g$  is meromorphic on  $X$ ,  $f$  is meromorphic on  $X$ .

*Proof of the theorem.* — Let  $x \in X_1$ . In order to prove that  $x$  is a schematic point in  $X$  it is sufficient to show that in the local ring of

holomorphic functions at  $x$  there exists a system of parameters composed of functions meromorphic on  $X$  (compare [L]). It follows from Lemma 5 that it suffices to find such a system of parameters composed of  $C^*$ -semi-invariant function meromorphic on  $X_1^+$ . Since  $X_1^+$  is an algebraic variety, there exists a system of parameters at  $x$  composed of  $C^*$ -semi-invariant functions which are regular at  $x$  and hence rational on  $X_1^+$ . The functions are then meromorphic on  $X$  and thus the proof is finished.

**COROLLARY 6.** — *Let a compact and smooth algebraic space  $X$  be a torus embedding of a torus  $T$ . Then  $X$  is an algebraic variety.*

Proof follows from the theorem and the fact that (since  $X$  is a torus embedding) any fixed point of the action of  $T$  on  $X$  is a source of the induced action of a one parameter subgroup  $C^* \rightarrow T$  (this can be seen similarly as in the proof of Lemma 1 by considering a  $T$ -invariant birational morphism of a smooth projective variety  $X_0 \rightarrow X$ ). Notice also that any  $T$ -orbit contains a fixed point in its closure.

**COROLLARY 7.** — *Let  $X$  be a smooth and compact algebraic space with an action of a reductive group  $G$ . Assume that there exists only one closed  $G$ -orbit in  $X$ . Then  $X$  is a projective variety.*

*Proof.* — By Sumihiro Theorem [Su] any point of a normal algebraic variety  $X$  with an action of a connected algebraic group is contained in an invariant open quasi-projective subset. Hence if this variety is complete and contains only one closed orbit it has to be projective (the only open invariant subset containing a point from the closed orbit is the whole space). Thus it suffices to show that the space  $X$  is an algebraic variety. Since any  $G$ -orbit contains a closed orbit in its closure and the set of schematic points is open  $G$ -invariant, it suffices to show that the only closed  $G$ -orbit in  $X$  contains a schematic point. Therefore it follows from the theorem that it suffices to prove that the source of a one parameter subgroup in  $G$  is contained in the closed  $G$ -orbit.

Let  $T$  be a maximal torus in  $G$ . Let  $C^* = T_0 \subseteq T$  be a subtorus of  $T$  such that the sets of fixed points of  $T$  and of  $T_0$  coincide. Let  $P$  be the parabolic subgroup corresponding to  $C^* = T_0$ . Let  $x$  belongs to the source of the action of  $T_0$  on  $X$ . Then  $x$  belongs to the source of the action of  $T_0$  on the closure of  $Gx$  in  $X$ . Hence the opposite  $P^-$  of the parabolic  $P$  has to be contained in the stabilizer subgroup of  $x$ . Thus the stabilizer is parabolic and the orbit  $Gx$  is projective. Hence  $Gx$  is the only closed orbit

in  $X$ . It means that source of  $T_0$  in  $X$  is contained in the only closed orbit and the proof is complete.

**COROLLARY 8.** — *Let  $X$  be a smooth algebraic space with an action of  $C^*$ . Let  $X_1$  be the source of the action. If  $x \in X_1$  is schematic in  $X_1$ , then it is schematic in  $X$ .*

*Proof.* — Assume that  $x \in X_1$  is schematic in  $X_1$ . If  $X = X_1$ , then the corollary is trivial. Assume that  $X \neq X_1$ . Then (by [M]) there exists  $\rho_1 : Y_1 \rightarrow X_1$ , where  $Y_1$  is a smooth algebraic variety and  $\rho_1$  is a composition of blow ups of ideals on  $X_1$  and its transforms supported by smooth centers not containing  $x$ . Let  $\rho : Y \rightarrow X$  be the composition of the blow-ups of the corresponding ideals on  $X$  and its transforms. Then  $Y$  is smooth with the induced action of  $C^*$  and  $Y_1$  is the source of the action. Since  $Y_1$  is an algebraic variety, any point of  $Y_1$  (by the theorem) is schematic in  $Y$ . In particular  $x$  is schematic in  $Y$ . Since  $\rho$  restricted to a Zariski open neighborhood of  $x$  is an isomorphism,  $x$  is schematic in  $X$ .

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Manuscrit reçu le 23 avril 1992.

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