

C. DENSON HILL

MAURO NACINOVICH

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THE TOPOLOGY OF STEIN CR MANIFOLDS AND THE LEFSCHETZ THEOREM

by C.D. HILL & M. NACINOVICH

1. Preliminaries.

An abstract CR manifold is a triple (M, H, J) where M is a connected paracompact smooth real manifold, H is an even dimensional subbundle of the tangent bundle TM , and J a partial pseudocomplex structure on H ; i.e. $J : H \rightarrow H$ is a smooth fiber preserving bundle isomorphism with $J^2 = -I$. We also require that J be formally integrable; i.e. that we have

$$(1.1) \quad [\tau^{0,1}M, \tau^{0,1}M] \subset \tau^{0,1}M,$$

where

$$\tau^{0,1}M = \{X + iJX | X \in \Gamma(M, H)\} \subset \Gamma(M, \mathbf{C}TM),$$

with Γ denoting smooth sections.

Let m be the real dimension of M and $2n$ be the real dimension of the fiber of H . Then n is called the CR dimension of M and $k = m - 2n$ is its CR codimension. In this case we say that M is of type (n, k) .

A CR map of a CR manifold (M_1, H_1, J_1) into a CR manifold (M_2, H_2, J_2) is a differentiable map $\varphi : M_1 \rightarrow M_2$ such that

$$(1.3) \quad d\varphi(H_1) \subset H_2,$$

$$(1.4) \quad d\varphi(J_1X) = J_2d\varphi(X) \quad \text{for } X \in H_1.$$

A CR map $f : M \rightarrow \mathbb{C}$ is called a CR function.

If, in addition, φ is a diffeomorphism and φ^{-1} is CR, then we say that φ is a CR isomorphism.

A CR manifold of the form (M, TM, J) is a complex manifold (type $(n, 0)$) by the Newlander-Nirenberg theorem, and one of the form $(M, 0, 0)$ is simply a real differentiable manifold (type $(0, k)$).

Let M be a real submanifold of a complex manifold \widetilde{M} , with complex structure \widetilde{J} . For $p \in M$ set

$$(1.5) \quad H_p M = T_p M \cap \widetilde{J}T_p M.$$

Then $(M, HM, \widetilde{J}|_{HM})$ is a CR manifold provided that the spaces $H_p M$ have constant dimension. In this case the embedding map $\iota : M \rightarrow \widetilde{M}$ is a CR map of $(M, HM, \widetilde{J}|_{HM})$ into $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$.

An *embedding* (resp. *immersion*) φ of a CR manifold (M, H, J) into a complex manifold $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$ is a CR map $\varphi : M \rightarrow \widetilde{M}$ which is an embedding (resp. immersion).

An immersion (resp. embedding) of a CR manifold (M, H, J) of type (n, k) into a complex manifold \widetilde{M} of complex dimension $n + k$ is said to be *generic*.

DEFINITION 1. — A *Stein CR manifold* is a CR manifold (M, H, J) such that

$$(1.6) \quad (M, H, J) \text{ has a CR embedding as a closed CR submanifold of some Stein manifold } X.$$

Because of the known results about the embedding of Stein manifolds [3] we could just as well replace (1.6) by the equivalent condition

$$(1.7) \quad (M, H, J) \text{ has a CR embedding as a closed CR submanifold of } \mathbb{C}^N, \text{ for some } N.$$

For example suppose Ω is a domain of holomorphy in \mathbb{C}^ℓ , with complex structure \widetilde{J} , and let $(M, HM, \widetilde{J}|_{HM})$ be a closed CR submanifold of Ω . Then it is a Stein CR manifold. Let (M, H, J) be a CR manifold of type (n, k) . Let $H^0 \subset T^*M$ be the annihilator bundle of the bundle H . We consider the bundle $T^{(0,1)}M = \{X + iJX | X \in H\}$. Then the Levi form of (M, H, J) at $\omega \in H^0_p$ is the Hermitian form on $T^{(0,1)}_p M$:

$$(1.8) \quad L(\omega, Z) = \text{id } \tilde{\omega}(Z, \bar{Z}) = -i\omega([\mathbf{Z}, \tilde{\mathbf{Z}}]),$$

where $\tilde{\omega} \in \Gamma(M, H^0)$ satisfies $\tilde{\omega}(P) = \omega$ and $\mathbf{Z} \in \tau^{0,1}M$ satisfies $\mathbf{Z}(P) = Z$.

The equality of the last two expressions shows that they do not depend on the choice of $\tilde{\omega}$ and \mathbf{Z} , and therefore L is a function defined on the direct sum of the bundles H^0 and $T^{0,1}M$.

DEFINITION 2. — *A weakly q -concave CR manifold ($0 \leq q \leq n$) is a CR manifold (M, H, J) such that, for every $\omega \in H^0 - \{0\}$, the Levi form $L(\omega, \cdot)$ has at least q eigenvalues that are ≤ 0 .*

Replacing the requirement ≤ 0 above by < 0 we arrive at the standard definition of a q -concave CR manifold. Hence every q -concave CR manifold is a fortiori weakly q concave. Note that weak 0-concavity involves no condition at all on the manifold. We adopt the convention that a real differentiable manifold (type $(0, k)$) is Stein (Whitney embedding theorem) and is also weakly 0-concave. A complex manifold (type $(n, 0)$) is weakly n -concave. Any Levi flat ($L \equiv 0$) CR manifold of type (n, k) is also weakly n -concave. It follows from the result of [2] that a weakly 1-concave Stein CR manifold cannot be compact.

1. The topology of weakly q -concave CR manifolds.

THEOREM 1. — *Let (M, H, J) be a weakly q -concave Stein CR manifold of type (n, k) . Then M has the homotopy type of a CW-complex of dimension $\leq 2n + k - q$. In particular*

(2.1)
$$H_j(M; \mathbb{Z}) = 0 \text{ for } j > 2n + k - q$$

and

(2.2)
$$H_{2n+k-q}(M; \mathbb{Z}) \text{ has no torsion.}$$

Note that the above theorem interpolates between two classical results :

1. (type $(0, k)$) Any real k -dimensional differentiable manifold has the homotopy type of a CW-complex of dimension $\leq k$ [4], and

2. (type $(n, 0)$) Any complex n -dimensional Stein manifold has the homotopy type of a CW-complex of dimension $\leq n$. Thus we obtain the classical result of Andreotti-Frankel [1].

Proof. — Assume for a moment the first assertion of the theorem. Then we have (2.1) as well as

(2.3)
$$H_j(M; K) = 0 \text{ for } j > 2n + k - q,$$

where K is an arbitrary field. The universal coefficient theorem

$$(2.4) \quad H_j(M; K) = H_j(M; \mathbb{Z}) \otimes K + \text{Tor}[H_{j-1}(M; \mathbb{Z}), K]$$

then yields (2.2).

Since M is Stein we can assume that M is embedded as a closed submanifold of \mathbb{C}^N . Following Andreotti-Frankel we take, for $P_0 \in \mathbb{C}^N - M$, the square of the Euclidean distance

$$(2.5) \quad \varphi(P) = |P - P_0|^2, \quad P \in M.$$

By a standard argument using Sard's theorem, we choose the point P_0 so that $\varphi(P)$ is a Morse function on M ; i.e. φ has only isolated nondegenerate critical points. By Morse theory (see [4], p. 20) M has the homotopy type of a CW-complex obtained by attaching an r -cell for each critical point having Morse index r . Hence it will suffice to show that φ has no critical point with Morse index $r > 2n + k - q$.

Let $P \in M$ be a critical point of φ . By an affine orthogonal change of coordinates we may assume $P = 0$ and that M is described in a neighborhood of 0 by equations of the form

$$(2.6) \quad \begin{cases} y_j = h_j(x_1, \dots, x_k, w_1, \dots, w_n) & 1 \leq j \leq k \\ \zeta_s = g_s(x_1, \dots, x_k, w_1, \dots, w_n) & 1 \leq s \leq \ell \end{cases}$$

where $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ are real coordinates, and $z = x + iy$, $w = (w_1, \dots, w_n)$, $\zeta = (\zeta_1, \dots, \zeta_\ell)$ are complex holomorphic coordinates, with $N = k + n + \ell$. Here the smooth functions h_j are real valued, the g_s are complex valued, and all vanish to second order at 0. The tangent space to M at 0 is $T_0M = \{(x, w, 0)\}$. As the line segment $\overline{PP_0}$ is orthogonal to M , the point P_0 has coordinates $P_0 = (ia_1, \dots, ia_k, 0, \dots, 0, \lambda_1, \dots, \lambda_\ell)$ with the $a_j \in \mathbb{R}$ and the $\lambda_s \in \mathbb{C}$. A point $Q \in M$ near $P = 0$ has coordinates $Q = (x + ih(x, w), w, g(x, w))$. We have

$$\left\{ \begin{aligned} \varphi(Q) &= \sum_1^k x_j^2 + \sum_1^k (h_j - a_j)^2 \\ &+ \sum_1^n |w_r|^2 + \sum_1^\ell |g_s - \lambda_s|^2 \\ &= \varphi(0) + \sum_1^k x_j^2 + \sum_1^n |w_r|^2 - 2 \sum_1^k a_j h_j \\ &- 2\text{Re} \sum_1^\ell \bar{\lambda}_s g_s + O(3). \end{aligned} \right.$$

The g_s are CR functions on M near 0 since they are the restrictions to M of the holomorphic coordinates ζ_s . Therefore the formal Taylor expansion of each g_s , about 0 is an expansion in terms of the $x_j + ih_j$ and the w_r . Thus

$$(2.8) \quad g_s = \sum_1^k a_s^{ij} x_i x_j + \sum_1^k \sum_1^n b_s^{jr} x_j w_r + \sum_1^n c_s^{rt} w_r w_t + O(3).$$

We write

$$(2.9) \quad \varphi(Q) = \varphi(0) + B(x, w, \bar{w}) + O(3)$$

where B is a quadratic form in x, w, \bar{w} . To prove our contention, it is enough to show that the quadratic form $B(0, w, \bar{w})$ has at most $2n - q$ negative eigenvalues. In view of (2.8) we can write

$$(2.10) \quad B(0, w, \bar{w}) = \operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2$$

where A is a holomorphic quadratic form in w , and L is the Levi form of M at $\omega = \pm 2 \sum_1^k a_j dx_j|_0$.

Let W be a maximal real subspace of \mathbb{C}^n on which $B(0, w, \bar{w})$ is negative definite. On $W \cap \sqrt{-1}W$ the quadratic form $L(\omega, w) + \sum_1^n |w_r|^2$ is negative definite. Indeed if w and $\sqrt{-1}w$ belong to W we have

$$(2.11) \quad \operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2 < 0$$

and

$$(2.11) \quad -\operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2 < 0,$$

as $\operatorname{Re} A(\sqrt{-1}w) = -\operatorname{Re} A(w)$. It follows, using our hypothesis of weak q concavity, that $W \cap \sqrt{-1}W$ has complex dimension $\leq n - q$ and therefore W has dimension $\leq 2n - q$. The proof of the theorem is complete.

3. The Lefschetz theorem on hyperplane sections.

DEFINITION 3. — A projective CR manifold is a CR manifold (M, H, J) which has a closed CR embedding in $\mathbb{C}\mathbb{P}^N$, for some N .

It follows from the results of [2] that a weakly 1-concave projective CR manifold, when embedded in $\mathbb{C}\mathbb{P}^N$, for some N , intersects every hyperplane. Let Σ be such a hyperplane, and set $M_0 = M \cup \Sigma$. We call the closed subset M_0 a hyperplane section of M .

THEOREM 2. — *Let (M, H, J) be an orientable weakly q -concave projective CR manifold of type (n, k) , and M_0 be a hyperplane section of M . Then the natural homomorphism*

$$(3.1) \quad H^j(M; \mathbb{Z}) \rightarrow H^j(M_0; \mathbb{Z})$$

is an isomorphism for $j < q - 1$. It is injective for $j = q - 1$.

THEOREM 2'. — *Dropping the assumption of orientability, the same results are valid with \mathbb{Z}_2 coefficients.*

Proof. — Since M_0 is closed in M , we have the exact cohomology sequence

$$(3.2) \quad \begin{aligned} \cdots \rightarrow H_K^j(M - M_0; \mathbb{Z}) \rightarrow H^j(M; \mathbb{Z}) \rightarrow H^j(M_0; \mathbb{Z}) \rightarrow \\ \rightarrow H_K^{j+1}(M - M_0; \mathbb{Z}) \rightarrow \cdots, \end{aligned}$$

where the subscript K denotes compact supports.

By Poincaré duality we obtain $H_K^j(M - M_0; \mathbb{Z}) \cong H_{2n+k-j}(M - M_0; \mathbb{Z}) = 0$ for $j < q$. Hence the result follows by (3.2). For the case where $M - M_0$ is not orientable, we apply Poincaré duality for \mathbb{Z}_2 coefficients and argue as above.

THEOREM 3. — *Under the same hypotheses as Theorem 2, the natural homomorphism*

$$(3.3) \quad H_j(M_0; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z})$$

is an isomorphism for $j < q - 1$ and is surjective for $j = q - 1$. For the homomorphism

$$(3.4) \quad H_j(M; \mathbb{Z}) \rightarrow H_j(M, M - M_0; \mathbb{Z}),$$

we obtain an isomorphism if $j > 2n + k - q + 1$, and an injection for $j = 2n + k - q + 1$.

Proof. — We consider the exact homology sequence for the pair (M, M_0)

$$(3.5) \quad \begin{aligned} \cdots \rightarrow H_{j+1}(M, M_0; \mathbb{Z}) \rightarrow H_j(M_0; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z}) \rightarrow \\ \rightarrow H_j(M, M_0; \mathbb{Z}) \rightarrow \cdots \end{aligned}$$

But the Lefschetz duality theorem asserts that

$$H_j(M, M_0; \mathbb{Z}) \cong H^{2n+k-j}(M - M_0; \mathbb{Z}),$$

and the latter group is zero for $j < q$, again by Theorem 1.

Next we consider the exact homology sequence for the pair $(M, M - M_0)$

$$(3.6) \quad \begin{aligned} \cdots \rightarrow H_j(M - M_0; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z}) \rightarrow H_j(M, M - M_0; \mathbb{Z}) \rightarrow \\ \rightarrow H_{j-1}(M - M_0; \mathbb{Z}) \rightarrow \cdots \end{aligned}$$

By Theorem 1, $H_j(M - M_0; \mathbb{Z}) = 0$ for $j > 2n + k - q$. Hence the desired conclusion follows.

Remark. — When M is a smooth projective algebraic variety (type $(n, 0)$) we recover the classical Lefschetz theorem on hyperplane sections along the lines of the Morse theoretic proof given by Andreotti-Frankel [1].

4. Homotopy of projective CR manifolds.

Following Milnor [4] we prove

THEOREM 4. — *Let (M, H, J) be a weakly q -concave projective CR manifold of type (n, k) , and M_0 be a hyperplane section of M . Then*

$$(4.1) \quad \pi_j(M, M_0) = 0 \quad \text{for } j < q.$$

Proof. — (We fix a base point $x_0 \in M_0$.) We have a closed CR embedding of M in $\mathbb{C}P^N$, for some N , and $M_0 = M \cap \Sigma$ with Σ given by $z_0 = 0$ in homogeneous coordinates. Let $\mathcal{N}(\Sigma, \delta)$ and $\mathcal{N}(M, \varepsilon)$ be tubular neighborhoods of Σ and M , of radius δ and ε , respectively, with respect to the Fubini-Study metric. For $\delta_0 > 0$ sufficiently small, the geodesic flow determines a deformation retract

$$(4.2) \quad F_\delta : I \times \mathcal{N}(\Sigma, \delta) \rightarrow \mathcal{N}(\Sigma, \delta)$$

of $\mathcal{N}(\Sigma, \delta) \rightarrow \Sigma$ if $0 < \delta < \delta_0$. For $\varepsilon > 0$ sufficiently small, we have a retraction

$$(4.3) \quad g : \mathcal{N}(M, \varepsilon) \rightarrow M.$$

Since $\mathcal{N}(M, \varepsilon)$ is open, $F_\delta^{-1}(\mathcal{N}(M, \varepsilon))$ is open in $I \times \mathcal{N}(\Sigma, \delta)$, and contains $I \times M_0$. Then we can find $\eta, 0 < \eta < \delta$, such that $F_\delta(I \times M \cap \mathcal{N}(\Sigma, \eta)) \subset \mathcal{N}(M, \varepsilon)$.

We have $\mathbf{C}\mathbf{P}^N - \Sigma \cong \mathbf{C}^N$ and take φ as in (2.5). Set

$$\psi = \begin{cases} 0 & \text{on } \Sigma \\ \frac{1}{\varphi} & \text{on } M - \Sigma. \end{cases}$$

Then ψ is continuous on M and is a Morse function on $\psi^{-1}([r, \infty))$ for $r > 0$. The critical points of ψ all have Morse index $\geq q$. Therefore M has the homotopy type of $\psi^{-1}([0, r])$ with finitely many cells of dimension $\geq q$ attached. Then, for $j < q$, every continuous pointed map $f : (I^j, \partial I^j) \rightarrow (M, M_0)$ can be deformed to a continuous pointed map $f_1 : (I^j, \partial I^j) \rightarrow (\psi^{-1}([0, r], M_0))$. If r is sufficiently small, then $\psi^{-1}([0, r]) \subset M \cap \mathcal{N}(\Sigma, \eta)$, and $g \circ F_\delta(t, f_1(p))$ gives a homotopy of f_1 to a continuous pointed map $f_2 : (I^j, \partial I^j) \rightarrow (M, M_0)$. The proof is complete.

We consider next the exact homotopy sequence of the pair (M, M_0) :

$$(4.5) \quad \begin{aligned} \cdots \rightarrow \pi_j(M_0) \rightarrow \pi_j(M) \rightarrow \pi_j(M, M_0) \rightarrow \pi_{j-1}(M_0) \rightarrow \cdots \\ \cdots \rightarrow \pi_1(M, M_0) \rightarrow \pi_0(M_0) \rightarrow \pi_0(M). \end{aligned}$$

Therefore if M_0 is a hyperplane section of a weakly q -concave projective CR manifold (M, H, J) , then the natural map

$$(4.6) \quad \pi_j(M_0) \rightarrow \pi_j(M)$$

is an isomorphism for $j < q - 1$, and is surjective for $j = q - 1$. In particular, for $q \geq 2$, every hyperplane section of M is arcwise connected (as M is connected), and for $q \geq 3$, every hyperplane section of M has the same fundamental group as M . Finally we remark that a generically chosen hyperplane section M_0 of M is a smooth submanifold.

5. A remark on the embedding dimension of projective CR manifolds.

Let (M, H, J) be a projective CR manifold of type (n, k) . Consider a closed CR embedding of it into $\mathbf{C}\mathbf{P}^N$, for some N . Then it may be possible to reduce the embedding dimension N as follows :

THEOREM 5. — *With (M, H, J) as above we have*

($k = 1$) : It has a global closed CR embedding in $\mathbf{C}\mathbf{P}^{2n+2}$, and a global closed CR immersion in $\mathbf{C}\mathbf{P}^{2n+1}$.

($k \geq 2$) : It has a global closed CR embedding in $\mathbf{C}\mathbf{P}^m$, where $m = [2n + (3/2)k]$ (greatest integer in).

Remark. — For compact Stein CR manifolds of type (n, k) , precisely the same results hold, with complex projective space replaced by complex Euclidean space. For non compact Stein CR manifolds of type (n, k) , the same results hold, with the word “closed” removed, and the word “embedding” replaced by “one-to-one immersion”.

Proof. — Let $M' = \{(p, q) \in M \times \mathbb{C}\mathbb{P}^N \mid q \in \mathbb{C}\mathbb{P}^N \text{ tangent to } M \text{ at } p\}$. This is a smooth submanifold of $M \times \mathbb{C}\mathbb{P}^N$ of real dimension $4n + 3k$. The map

$$(5.1) \quad M' \ni (p, q) \mapsto q \in \mathbb{C}\mathbb{P}^N$$

is smooth. By Sard’s theorem its image has measure zero in $\mathbb{C}\mathbb{P}^N$ if $N > 2n + (3/2)k$. Choosing a point $Q_0 \notin \{\text{its range}\} \cup M$, and projecting from this point into a hyperplane Σ not containing Q_0 , we obtain a CR closed immersion into a $\mathbb{C}\mathbb{P}^{N-1}$.

Next we consider $M'' = \{(p, q, r) \mid (p, q) \in M \times M - \Delta, r \in \mathbb{C}\mathbb{P}^N \text{ and } p, q, r \text{ are collinear}\}$. It is a smooth manifold of real dimension $4n + 2k + 2$. The map

$$(5.2) \quad M'' \ni (p, q, r) \mapsto r \in \mathbb{C}\mathbb{P}^N$$

is smooth and, again by Sard’s theorem, its image has measure zero if $N > 2n + k + 1$. If N satisfies both inequalities, the above CR immersion can be chosen to be globally one-to-one.

BIBLIOGRAPHY

- [1] A. ANDREOTTI and T. FRANKEL, The Lefschetz theorem on hyperplane sections, *Ann. Math.*, 69 (1959), 713–717.
- [2] C.D. HILL and M. NACINOVICH, A necessary condition for global Stein immersion of compact CR manifolds, to appear in *Riv. Mat. Parma*.
- [3] L. HÖRMANDER, *An Introduction to Complex Analysis in Several Complex Variables*, Van Nostrand, Princeton, 1966.
- [4] J. MILNOR, *Morse Theory*, *Ann. Math. Studies*, Princeton, 51 (1963).

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C.D. HILL,
Department of Mathematics
SUNY at Stony Brook
Stony Brook, NY 11794 (U.S.A.)
&
M. NACINOVICH,
Dipartimento di Matematica
Università di Pisa
Via F. Buonarroti, 2
56127 Pisa (Italy).