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BLOCK-DISTRIBUTION IN RANDOM STRINGS

by Peter J. GRABNER

1. Introduction.

We investigate some properties of infinite sequences of independent random variables, which take the values 0 and 1 with probabilities p and q respectively (Bernoulli's scheme). It is one of the basic results of probability theory that the limit relation

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N - k : x_n x_{n+1} \dots x_{n+k} = a_1 \dots a_k\}}{N} = \mu_k(A)$$

holds in probability for all blocks $A = a_1 \dots a_k$ of a given constant length k ($\mu_k(A)$ is the k -fold product measure generated by $\mu(\{0\}) = p$ and $\mu(\{1\}) = q$). This result can also be naturally imbedded into ergodic theory : consider the infinite product space $X = \{0, 1\}^{\mathbb{N}}$ equipped with the infinite product measure μ_∞ generated by μ . Then the shift operator S (Bernoulli shift) on X defined by $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$ is an ergodic transformation on X (cf. e.g. [Wa]) and the above relation is a consequence of Birkhoff's ergodic theorem.

It is now natural to ask how fast (depending on N) k could grow such that this relation persists. In order to answer this question we introduce a

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special notion of discrepancy (cf. [HI], [KN]) :

$$(1.1) \quad D_N^k(x_1, \dots, x_N) = \max_{A \in \{0,1\}^k} \frac{1}{\sqrt{p^k \mu_k(A)}} \left| \frac{\#\{1 \leq n \leq N-k : x_n x_{n+1} \dots x_{n+k} = a_1 \dots a_k\}}{N} - \mu_k(A) \right|.$$

The following calculations will show that this is a proper measure for the distribution behaviour of the sequence x_1, x_2, \dots . Note that this definition agrees with the definition in [FKT] for $p = q = \frac{1}{2}$.

DEFINITION. — A sequence x_1, x_2, \dots is called $k(N)$ -distributed with respect to μ if

$$\lim_{N \rightarrow \infty} D_N^{k(N)}(x_1, \dots, x_N) = 0.$$

Our Theorem will show under which conditions almost all sequences are $k(N)$ -distributed. Without loss of generality assume that $p \leq q$. The notation $\text{lp } n$ is the logarithm to base $\frac{1}{p}$: $\text{lp } n = \log_{\frac{1}{p}} n$.

THEOREM. — Let $k(N)$ be a non-decreasing sequence of positive integers. Then the following 0-1-law holds

$$\mu_\infty \left(\lim_{N \rightarrow \infty} D_N^{k(N)}(x_1, \dots, x_N) = 0 \right) = \begin{cases} 1 & \text{if } \text{lp } n - \text{lp } \text{lp } n - k(n) \rightarrow \infty \\ 0 & \text{otherwise.} \end{cases}$$

It clearly follows from Kolmogoroff's 0-1-law or the fact that the set

$$\left\{ \lim_{N \rightarrow \infty} D_N^{k(N)}(x_1, \dots, x_N) = 0 \right\}$$

is invariant under the (ergodic) shift S , that the only possible values for the above probability are 0 and 1. The proof of this theorem will use bivariate correlation polynomials, which are a generalization of Guibas' and Odlyzko's correlation polynomials in one variable (cf. [GO]). Using these polynomials we are able to compute the probability generating functions of the events we are interested in.

2. Generating Functions.

Throughout this section let $A = a_1 a_2 \dots a_k$ be a 0-1-string of length k . We are interested in the cardinalities of the following subsets of the set

$\mathcal{S}_{r,s}$ of strings containing r digits 0 and s digits 1 :

$$(2.1) \begin{aligned} f_A(r,s) &= \#\{B \in \mathcal{S}_{r,s} : B \text{ contains } A \text{ only at the end}\} \\ g_A(r,s) &= \#\{B \in \mathcal{S}_{r,s} : B \text{ contains } A \text{ only at the beginning and at the end}\} \\ h_A(r,s) &= \#\{B \in \mathcal{S}_{r,s} : B \text{ does not contain } A\}. \end{aligned}$$

In order to compute the generating functions of these quantities we introduce the bivariate autocorrelation polynomial $[AA](z,w)$:

$$[z^r w^s][AA](z,w) = \begin{cases} 1 & \text{if } a_1 a_2 \dots a_{k-r-s} = a_{r+s+1} a_{r+s+2} \dots a_k \text{ and the} \\ & \text{string } a_1 a_2 \dots a_{r+s} \text{ contains } r \text{ digits 0 and } s \\ & \text{digits 1} \\ 0 & \text{otherwise,} \end{cases}$$

where $[z^r w^s]P(z,w)$ as usual denotes the coefficient of $z^r w^s$ in $P(z,w)$. We are now ready to formulate

PROPOSITION 1. — *The generating functions of the combinatorial expressions (2.1) are given by*

$$\begin{aligned} F_A(z,w) &= \sum_{r,s=0}^{\infty} f_A(r,s) z^r w^s = \frac{z^{0(A)} w^{1(A)}}{z^{0(A)} w^{1(A)} + (1-z-w)[AA](z,w)} \\ G_A(z,w) &= z^{0(A)} w^{1(A)} + \frac{(z+w-1)z^{0(A)} w^{1(A)}}{z^{0(A)} w^{1(A)} + (1-z-w)[AA](z,w)} \\ H_A(z,w) &= \frac{[AA](z,w)}{z^{0(A)} w^{1(A)} + (1-z-w)[AA](z,w)}, \end{aligned}$$

where $0(A)$ and $1(A)$ denote the number of 0's and 1's in A respectively.

The proof of this proposition is analogous to the proof of the corresponding results for ordinary generating functions (cf. [GO]).

Remark 1. — Obviously these results can be generalized to any finite alphabet.

As in [FKT] we use these functions to compute the probability generating function (p.g.f.) of all strings containing the substring A exactly r times :

$$\begin{aligned} \Phi_A^{(r)}(z) &= \frac{z^{-kr}}{\mu_k(A)} F_A(pz, qz)^2 G_A(pz, qz)^{r-1} \text{ for } r \geq 1 \\ \Phi_A^{(0)}(z) &= H_A(pz, qz). \end{aligned}$$

Inserting the results of Proposition 1 and setting

$$(2.2) \quad P(z) = \frac{1}{\mu_k(A)} [AA](pz, qz)$$

yields

$$\Phi_A^{(r)}(z) = \frac{z^k \left((1-z)(P(z) - \frac{1}{\mu_k(A)}) + z^k \right)^{r-1}}{\mu_k(A) \left((1-z)P(z) + z^k \right)^{r+1}}$$

$$\Phi_A^{(0)}(z) = \frac{P(z)}{(1-z)P(z) + z^k}.$$

3. Proof of the Theorem.

We split the proof into two parts; first we show that almost all sequences are $k(N)$ distributed if $\text{lp } n - \text{lp } n - k(n) \rightarrow \infty$. Using our p.g.f. results we can write

(3.1)

$$\mu_\infty(\#\{0 \leq n \leq N - k : x_{n+1} \dots x_{n+k} = a_1 \dots a_k\} = r) = p_A^{(r)}(N) = [z^N] \Phi_A^{(r)}(z) = \frac{1}{2\pi i} \oint_C \Phi_A^{(r)}(z) \frac{dz}{z^{N+1}}.$$

In order to be able to estimate the integral we need information on the the zeros of the polynomial $(1 - z)P(z) + z^k$.

LEMMA 1. — *The zero of smallest modulus z_0 of $(1 - z)P(z) + z^k$ is real and positive and satisfies the estimate*

$$z_0 > 1 + C\mu_k(A)$$

for a positive constant C only depending on p .

Proof. — As $F_A(pz, qz)$ is a p.g.f. and $(1 - z)P(z) + z^k$ is the denominator of this rational function the zero of smallest modulus has to be positive and ≥ 1 . Investigation of the derivative shows the existence of the constant C .

Let now

$$k(n) = \text{lp } n - \text{lp } n - \text{lp } \psi(n),$$

where $\psi(n) \rightarrow \infty$. We need estimates for the probability that the number of occurrences $Z_N(A)$ of a block A deviates too far from the mean value :

(3.2)

$$L_N(\delta_A) = \mu_\infty(Z_N(A) < N\mu_k(A)(1 - \delta_A)) \quad \text{and}$$

$$U_N(\delta_A) = \mu_\infty(Z_N(A) > N\mu_k(A)(1 + \delta_A)).$$

These probabilities are sums of the $p_A^{(r)}(N)$ defined in (3.1) :

$$(3.3) \quad \begin{aligned} L_N(\delta_A) &= \sum_{r < N\mu_k(A)(1-\delta_A)} p_A^{(r)}(N) \quad \text{and} \\ U_N(\delta_A) &= \sum_{r > N\mu_k(A)(1+\delta_A)} p_A^{(r)}(N). \end{aligned}$$

We will use the integral representation (3.1) to estimate these quantities.

For convenience we now introduce some notations

$$(3.4) \quad \begin{aligned} Q(z) &= (1-z)P(z) + z^k \\ a(z) &= \frac{z^k}{Q(z)^2}, \quad b(z) = 1 + \frac{z-1}{\mu_k(A)Q(z)}. \end{aligned}$$

This gives

$$\Phi_A^{(r)}(z) = \frac{1}{\mu_k(A)} a(z)b(z)^{r-1}$$

for $r \geq 1$. Observe further that

$$(3.5) \quad \begin{aligned} a(1 \pm \varepsilon) &= 1 + O\left(\frac{1}{\mu_k(A)}\varepsilon\right) \\ b(1 \pm \varepsilon) &= 1 \pm \frac{\varepsilon}{\mu_k(A)} + O\left(\frac{\varepsilon^2}{\mu_k(A)^2}\right) \\ b^j(1 \pm \varepsilon) &= \exp\left(\pm \frac{\varepsilon j}{\mu_k(A)} + O\left(\frac{\varepsilon^2 j}{\mu_k(A)^2}\right)\right) \\ (1 \pm \varepsilon)^{-n} &= \exp(\mp n\varepsilon + O(n\varepsilon^2)). \end{aligned}$$

We can now write

$$U_N(\delta_A) = \frac{1}{2\pi i} \oint_C \frac{1}{\mu_k(A)} a(z) \frac{b^j(z)}{1-b(z)} \frac{dz}{z^{N+1}},$$

where $j = \lceil N\mu_k(A)(1+\delta_A) \rceil$. As all the power series involved have positive coefficients and because of Lemma 1 we can estimate

$$U_N(\delta_A) \leq \frac{1}{\mu_k(A)} a(1-\varepsilon) \frac{b^j(1-\varepsilon)}{1-b(1-\varepsilon)} (1-\varepsilon)^{-N}$$

for every positive $\varepsilon < C\mu_k(A)$. Using (3.5) yields

$$U_N(\delta_A) \leq \frac{1}{\varepsilon} \frac{1+O\left(\frac{\varepsilon}{\mu_k(A)}\right)}{1+O\left(\frac{\varepsilon}{\mu_k(A)}\right)} \exp\left(\left(N - \frac{j}{\mu_k(A)}\right)\varepsilon + O\left(\frac{\varepsilon^2 j}{\mu_k(A)^2}\right) + O(N\varepsilon^2)\right).$$

Inserting $\varepsilon = \left(\mu_k(A) \frac{\text{lp } N}{N}\right)^{\frac{1}{2}}$ into the above inequality yields

$$(3.6) \quad U_N(\delta_A) \leq \exp(-\delta_A (N\mu_k(A) \text{lp } N)^{\frac{1}{2}} + C_1 \log N).$$

In the same way we treat the lower tail. Let now $j = \lfloor N\mu_k(A)(1-\delta_A) \rfloor$. Thus we obtain

$$L_N(\delta_A) = \frac{1}{2\pi i} \oint_C \left(\frac{P(z)}{Q(z)} + \frac{a(z)}{\mu_k(A)} \frac{b^j(z) - 1}{b(z) - 1} \right) \frac{dz}{z^{N+1}}.$$

We can now estimate

$$L_N(\delta_A) \leq \frac{P(1+\varepsilon)}{Q(1+\varepsilon)}(1+\varepsilon)^{-N} + \frac{1}{\mu_k(A)} j b^j(1+\varepsilon) a(1+\varepsilon)(1+\varepsilon)^{-N}.$$

Using the same value for ε as above yields

$$(3.7) \quad L_N(\delta_A) \leq \exp\left(-\delta_A(N\mu_k(A) \text{lp } N)^{\frac{1}{2}} + C_2 \log N\right).$$

Combining this with (3.6) yields

$$(3.8) \quad \begin{aligned} \mu_\infty \left(\left| \frac{Z_N(A)}{N} - \mu_k(A) \right| > \delta_A \mu_k(A) \right) \\ \leq \exp(-\delta_A(N\mu_k(A) \text{lp } N)^{\frac{1}{2}} + C_3 \log N). \end{aligned}$$

Let now $\delta_A = \delta \left(\frac{p^k}{\mu_k(A)} \right)^{\frac{1}{2}}$ and observe that $p^k = \frac{\text{lp } N}{N} \psi(N)$.

Therefore we have

$$(3.9) \quad \begin{aligned} \mu_\infty(D_N^{k(N)}(\omega) > \delta) &\leq 2^{k(N)} \exp(-\delta \psi(N)^{\frac{1}{2}} \text{lp } N + C_3 \log N) \\ &\leq \exp(-\delta \psi(N)^{\frac{1}{2}} \text{lp } N + C' \log N). \end{aligned}$$

We now choose δ as a function of N

$$\delta = \psi(N)^{-\frac{1}{4}}$$

and observe that

$$\sum_{N=1}^{\infty} \exp(-\psi(N)^{\frac{1}{4}} \text{lp } N + C' \log N) < \infty.$$

Thus by the Borel-Cantelli lemma (cf. [Fe]), we obtain the first part of our Theorem.

We now have to prove that almost no series are $k(N)$ -distributed if $\text{lp } n - \text{lp } \text{lp } n - k(n) \not\rightarrow \infty$ (we confine ourselves to the case $p < \frac{1}{2}$, because the case $p = \frac{1}{2}$ has been treated by Grill [Gr]). We introduce a set \mathcal{A} of strings of length k , which have only trivial autocorrelation and do not overlap each other :

$$\mathcal{A} = \left\{ \underbrace{0 \dots 0}_l \underbrace{A}_{l+d(k)-2} \underbrace{1 \dots 1}_l \right\},$$

where $l = \left\{ \frac{k}{3} \right\} + 1$ and $d(k) = k \bmod 3$. We need the p.g.f. $\varphi(z)$ of all strings not containing an element of \mathcal{A} . This function satisfies the equations

$$\begin{aligned} \varphi(z) + \varphi_{A_1}(z) + \dots + \varphi_{A_m}(z) &= z\varphi(z) + 1 \\ \varphi_{A_1}(z) &= z^k \mu_k(A_1) \varphi(z) \\ &\dots \\ \varphi_{A_m}(z) &= z^k \mu_k(A_m) \varphi(z), \end{aligned}$$

where A_1, \dots, A_m are the elements of \mathcal{A} and $\varphi_{A_l}(z) (l = 1, \dots, m)$ is the p.g.f. of the blocks ending with A_l but containing no further occurrence of any element of \mathcal{A} . Solving these equations yields

$$(3.10) \quad \varphi(z) \frac{1}{1 - z + \mu_k(\mathcal{A})z^k}.$$

Note that the simplicity of these equations comes from the trivial overlap structure of the elements of \mathcal{A} .

Because of this simple overlap structure it is easy to see that

$$(3.11) \quad \begin{aligned} \phi_{j_1 \dots j_m}(z) &= \frac{(j_1 + \dots + j_m)!}{j_1! \dots j_m!} \mu_k(A_1)^{j_1} \dots \mu_k(A_m)^{j_m} z^{k(j_1 + \dots + j_m)} \varphi(z)^{j_1 + \dots + j_m + 1} \end{aligned}$$

is the p.g.f. of all blocks containing A_l exactly j_l times ($l = 1 \dots m$). As in the first part of the proof we use

$$(3.12) \quad \begin{aligned} M_N(\delta) &= \mu_\infty (|Z_N(A_l) - N \mu_k(A_l)| \leq N \mu_k(A_l) \delta_{A_l}, l = 1 \dots m) \\ &= \frac{1}{2\pi i} \oint_C \sum_{\substack{|j_l - N \mu_k(A_l)| \leq N \mu_k(A_l) \delta_{A_l} \\ l=1, \dots, m}} \varphi_{j_1 \dots j_m}(z) \frac{dz}{z^{N+1}}, \end{aligned}$$

where $\delta_{A_l} = \delta \left(\frac{p^k}{\mu_k(A_l)} \right)^{\frac{1}{2}}$.

We want to treat (3.12) exactly like the corresponding expressions in the first part of the proof. For this purpose we need information on the zeros of the polynomial $1 - z + \mu_k(\mathcal{A})z^k$.

LEMMA 2. — *The zero of smallest modulus z_0 of $1 - z + \mu_k(\mathcal{A})z^k$ is real and satisfies*

$$z_0 > 1 + \mu_k(\mathcal{A}).$$

Proof. — The proof of the first statement is as in the proof of Lemma 1. For the proof of the inequality insert $z = 1 + \mu_k(\mathcal{A})$ into the polynomial.

Observe now that

$$\frac{1}{2\pi i} \oint_C \varphi(z)^{J+1} \frac{dz}{z^{N-kJ+1}} \leq \varphi(1 + \varepsilon)^{J+1} (1 + \varepsilon)^{kJ-N}$$

for $\varepsilon \leq \mu_k(\mathcal{A})$. Inserting $\varepsilon = \mu_k(\mathcal{A}) - \frac{J}{N}$ and performing similar calculations as in the first part of the proof yields

(3.13)

$$\begin{aligned} & \frac{1}{2\pi i} \oint_C \varphi(z)^{J+1} \frac{dz}{z^{N-kJ+1}} \\ & \leq \frac{1}{\mu_k(\mathcal{A})^{J+1}} \exp\left(-\frac{(J - N\mu_k(\mathcal{A}))^2}{2N\mu_k(\mathcal{A})} + O(k\mu_k(\mathcal{A})^2 N)\right). \end{aligned}$$

Let now $n = N\mu_k(\mathcal{A})$, $J = j_1 + \dots + j_m$ and $p_l = \frac{\mu_k(A_l)}{\mu_k(\mathcal{A})}$ and insert

(3.13) into (3.12) to obtain

(3.14)

$$M_N(\delta) \leq \frac{1}{\mu_k(\mathcal{A})} \sum_{\substack{|j_l - np_l| \leq np_l \delta_{A_l} \\ l=1, \dots, m}} \frac{J!}{j_1! \dots j_m!} \prod_{l=1}^m p_l^{j_l} \exp\left(-\frac{(J - n)^2}{2n} + O(k\mu_k(\mathcal{A})n)\right),$$

where $\sum_{l=1}^m p_l = 1$. Thus we have arrived at an expression that we can treat by the normal approximation of the multinomial distribution.

Assume that N runs through a subsequence of \mathbb{N} such that

$$\text{lp } N - \text{lp } N - k(N) \rightarrow \limsup_{N \rightarrow \infty} (\text{lp } N - \text{lp } N - k(N)) = \text{lp } C < \infty.$$

It will suffice to prove our theorem for the case that $\limsup \text{lp } N - \text{lp } N - k(N) > -\infty$, such that $0 < C < \infty$. Observe now that $N\mu_k(A_l)\delta_{A_l} = \delta\sqrt{CN\mu_k(A_l)} \text{lp } N$ and use Stirling's formula

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \leq n! \leq \frac{11}{10} \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

to obtain

(3.15)

$$\begin{aligned}
 M_N(\delta) &\leq \frac{11 \exp(O(k\mu_k(\mathcal{A})n))}{10 \mu_k(\mathcal{A})\sqrt{2\pi}^{m-1}} \sum_{|j_i - np_i| \leq \delta \sqrt{CN\mu_k(A_i)} \text{lp } N} \frac{\sqrt{J}}{\sqrt{j_1 \cdots j_m}} \\
 &\quad \times \frac{J^J}{j_1^{j_1} \cdots j_m^{j_m}} \prod_{l=1}^m p_l^{j_l} \exp\left(-\frac{(J-n)^2}{2n}\right) \\
 &= \frac{11 \exp(O(k\mu_k(\mathcal{A})n))}{10 n^{m-12} \mu_k(\mathcal{A})(2\pi)^{\frac{m-1}{2}} \sqrt{p_1 \cdots p_m}} \sum_{\substack{|x_l| \leq \delta_{A_l} \\ l=1, \dots, m}} \frac{\sqrt{1+\eta}}{\sqrt{(1+x_1) \cdots (1+x_m)}} \\
 &\quad \times \exp\left(-\frac{n}{2} \sum_{l=1}^m p_l x_l^2 + O(n\eta^3) + O\left(n \sum_{l=1}^m p_l x_l^3\right)\right),
 \end{aligned}$$

where $j_l = np_l(1 + x_l)$ and $J = n(1 + \eta)$. The terms in the last exponential come from $(1 + x)^{1+x} = \exp\left(x + \frac{x^2}{2} + O(x^3)\right)$ for $x \rightarrow 0$ and the observation that $j_1 + \cdots + j_m = J$ transforms to $\sum p_l x_l = \eta$. In the following we will use $p < \frac{1}{2}$ which yields $\delta_{A_l} \rightarrow 0$ for our choice of \mathcal{A} (in the case $p = \frac{1}{2}$ we have $\delta_{A_l} = \delta$ and the following arguments cannot be used).

Inserting the definition of δ_{A_l} yields the estimate

(3.16)

$$\begin{aligned}
 \left| n \sum_{l=1}^m p_l x_l^3 \right| &\leq \frac{(\log N)^{\frac{3}{2}}}{\sqrt{n}} \sum_{l=1}^m \frac{1}{\sqrt{p_l}} \\
 &= O\left(N^{-\frac{1}{3} + \frac{1}{6} \frac{\log q}{\log p} + \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)} (\log N)^{\frac{4}{3} - \frac{1}{6} \frac{\log q}{\log p} - \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)}\right)
 \end{aligned}$$

and a similar estimate holds for $n\eta^3$. Using an exponential estimate yields

$$\begin{aligned}
 &\frac{\sqrt{1+\eta}}{\sqrt{(1+x_1) \cdots (1+x_m)}} \\
 &= \exp\left(O\left(N^{-\frac{1}{3} + \frac{1}{6} \frac{\log q}{\log p} + \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)} (\log N)^{\frac{4}{3} - \frac{1}{6} \frac{\log q}{\log p} - \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)}\right)\right).
 \end{aligned}$$

Inserting these inequalities into (3.15) and setting $\alpha = \frac{1}{6} \frac{\log q}{\log p} +$

$\frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)$ yields

$$\begin{aligned}
 M_N(\delta) &\leq \frac{\exp(O(N^{-\frac{1}{3} + \alpha} (\log N)^{\frac{4}{3} - \alpha})) n^{\frac{m+1}{2}} \sqrt{p_1 \cdots p_m}}{\mu_k(\mathcal{A})(2\pi)^{\frac{m-1}{2}}} \\
 &\quad \times \sum_{|x_l| \leq \delta_{A_l}} \exp\left(-\frac{n}{2} \sum_{l=1}^m p_l x_l^2\right) \frac{1}{(np_1) \cdots (np_m)}.
 \end{aligned}$$

The sum in the last line can be interpreted as a lower Riemann sum for the integral

$$\int_{|x_l| \leq \delta_{A_l}} \exp\left(-\frac{n}{2} \sum_{l=1}^m p_l x_l^2\right) dx_1 \cdots dx_m$$

using the lattice

$$\left\{ (x_1, \dots, x_m) \mid x_l = \frac{j_l}{np_l} - 1, \quad |x_l| \leq \delta_{A_l}, \quad l = 1, \dots, m \right\}.$$

Thus we obtain

$$(3.17) \quad M_N(\delta) \leq \exp(O(N^{-\frac{1}{3} + \alpha}(\log N)^{\frac{4}{3} - \alpha}) + O(\log N)) (\Phi(\delta \sqrt{C \log N}))^m,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-\frac{t^2}{2}} dt \sim 1 - \sqrt{\frac{2}{\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}$$

for $x \rightarrow \infty$. Therefore we can estimate

$$(3.18) \quad \begin{aligned} M_N(\delta) &\leq \exp\left(-\sqrt{\frac{2}{\pi}} \frac{m}{\delta \sqrt{C \log N} e^{\frac{1}{2} \delta^2 C \log N}} + O(N^{-\frac{1}{3} + \alpha}(\log N)^{\frac{4}{3} - \alpha}) + O(\log N)\right). \end{aligned}$$

Observe now

$$(3.19) \quad \begin{aligned} m &\asymp \left(\frac{N}{\log N}\right)^{\frac{1}{3} \log 2} \\ \mu_k(\mathcal{A}) &\asymp \left(\frac{\log N}{N}\right)^{-\frac{1}{3} \log p q} \\ n &\asymp N^{1 + \frac{1}{3} \log p q} (\log N)^{-\frac{1}{3} \log p q}. \end{aligned}$$

Inserting these estimates into (3.18) yields

$$M_N(\delta) \leq \exp\left(-D \frac{N^{\frac{1}{3} \log 2 - \frac{1}{2} \delta^2 C \log e}}{\delta (\log N)^{\frac{1}{2} + \frac{1}{3} \log 2}} + O(N^{-\frac{1}{3} + \alpha}(\log N)^{\frac{4}{3} - \alpha}) + O(\log N)\right),$$

where $D > 0$ is a constant implied by (3.19). The right hand side tends to 0 for sufficiently small $\delta > 0$, because $\frac{1}{3} \log 2 > -\frac{1}{3} + \alpha$ holds for $p < \frac{1}{2}$.

Note that

$$\mu_\infty(D_N^{k(N)}(\omega) < \delta) \leq M_N(\delta).$$

Thus the proof is complete. □

Remark 2. — Modifying (1.1) one can also investigate discrepancies

$$D_N^{k,\phi}(\omega) = \max_{A \in \{0,1\}^k} \sqrt{\frac{\phi(k)}{\mu_k(A)}} \left| \frac{\#\{1 \leq n \leq N - k : x_n x_{n+1} \dots x_{n+k} = A\}}{N} - \mu_k(A) \right|,$$

where ϕ is a monotonically increasing function. Then the same calculations as above yield

$$\mu_\infty \left(\lim_{N \rightarrow \infty} D_N^{k(N),\phi}(\omega) = 0 \right) = \begin{cases} 1 & \text{if } \lim_{N \rightarrow \infty} \frac{N\phi(k(N))}{\log N} = \infty \\ 0 & \text{otherwise.} \end{cases}$$

This answers a question posed by Flajolet, Kirschenhofer and Tichy [FKT], Remark 2.

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