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QUANTIZATIONS AND SYMBOLIC CALCULUS
OVER THE $p$-ADIC NUMBERS

by Shai HARAN

INTRODUCTION

We shall be concerned with functions $f : V_0 \to \mathbb{C}$, defined in some vector space $V_0$ over the $p$-adic numbers $\mathbb{Q}_p$ and taking values in the complex numbers $\mathbb{C}$. One of the most basic problems encountered when trying to imitate the classical theory — where the domain $V_0$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$ — is the lack of derivatives. Indeed, the derivation $\partial / \partial x$ is nothing but $\mathcal{F}^{-1} x \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform, and $\langle x \rangle$ is multiplication by the function $x$ which is an additive homomorphism from $V_0$ to $\mathbb{C}$; and there are no such homomorphisms from $V_0$ to $\mathbb{C}$ when $V_0$ is a vector space over $\mathbb{Q}_p$.

This problem repeatedly makes its appearance in various disguises; for example, given a unitary representation of a $p$-adic analytic group on a Hilbert space, one cannot associate with it the derived representation of a $p$-adic Lie algebra. From a different perspective, the derivatives $(\partial / \partial x)^n$ correspond to the extra poles of the $\infty$-component of the zeta function, while the $p$-components have a unique pole.

There are thus no differential operators over $\mathbb{Q}_p$. But as we will show in this paper, there is a meaningful theory of pseudodifferential operators over the $p$-adics, which parallels the classical theory over the real numbers $\mathbb{R}$. In fact, the theory over $\mathbb{Q}_p$ is better behaved than the one over $\mathbb{R}$, in as much as in all estimates the numbers $2 = [\mathbb{C} : \mathbb{R}]$ for the reals can be replaced by $\infty = [\mathbb{Q}_p : \mathbb{Q}_p]$ for the $p$-adics, and one encounters here the phenomenon expounded in [15], [16].

Key words : $p$-adic integers – Heisenberg groups – Weyl quantization – Besov spaces – Elliptic operators – Spectral asymptotics.
The pseudodifferential operator $\hat{\rho}(f)$ associated with the «symbol» $f$, a function defined on the «phase space» $V = V_0 \oplus V_0$, is given via the action of the Heisenberg group (cf. 0.1.10). The composition of operators correspond to «twisted multiplication» of their symbols: $\hat{\rho}(f_1)\hat{\rho}(f_2) = \hat{\rho}(f_1 \# f_2)$. The main goal of the symbolic calculus is to control the twisted multiplication over appropriately defined symbol classes, thus enabling the comparison of the resulting function algebra and operator algebra. Classically, the symbol classes are defined using the Mihlin condition which incorporates the global growth condition with the local smoothness condition on the symbols. To define symbol classes in the $p$-adic setting we encounter once again the lack of derivatives. Fortunately, this time the problem can be overcome if one uses in place of derivatives a Besov condition.

Chapter 0 is devoted to the Heisenberg group and its fundamental representation. For the analogous theory over $\mathbb{R}$ see [6], [9], [21], [23], [32]; some of the material over $\mathbb{Q}_p$ can be found in [11], [36]. In paragraph 0, we fix our notations and recall the Schrödinger model, the Weyl quantization procedure and twisted multiplication. In paragraph 0.2, we review the lattice model and the Bargmann isomorphism [1], and we describe the Wick symbols [4]. In paragraph 0.3, with an eye towards connection with [14], [15], we offer a third realization of the fundamental representation in the «positive» subspace of $L_2$ of the $p$-adic circle, or the space of «$p$-adic particles», giving explicit formulas for the Hermite functions, and explaining why the Toeplitz operators over $\mathbb{Q}_p$ have index zero. In paragraph 0.4, we give a «mannerized» version of the lattice model, and we present the Wigner transform, the uncertainty principle and the wave-packet-expansions [7] in the $p$-adic setting.

Chapter 1 is devoted to the exposition of the various functions spaces over $\mathbb{Q}_p$. For the analogous theory over $\mathbb{R}$, we refer to [3], [25], [33], [34]; a good reference for some of the material over $\mathbb{Q}_p$ can be found in [31]. In paragraph 1.1, we recall the Bessel potentials and Sobolev spaces. In paragraph 1.2, we explain the problematics of these — the lack of the Leibniz rule for differentiation, and offer a way out via Besov spaces. After reviewing the elementary properties of the ordinary Besov spaces, we present the concept of a metric-covering, and the associated «mannerized» Besov spaces. In paragraph 1.3, we define the «smooth» functions over $\mathbb{Q}_p$, and the (analytic) Schwartz spaces. We quote the Schwartz kernel theorems and define our symbol classes (in analogy with [18]). In paragraph 1.4, we collect the basic properties of temperate metric-coverings needed in the
sequel, and offer as examples the Toeplitz symbols (discussed over the reals in [12], [17], [21], [30], [35]) and the pseudodifferential symbols.

The Toeplitz symbols $\Sigma^\alpha(V)$ consist of functions $f$ on $V$ which satisfy

$$|f(x)| \leq C_0 \cdot |1, px|^{\alpha},$$

and for all $\beta > 0$,

$$|f(x) - f(x + y)| \leq C_\beta \cdot |1, px|^{\alpha-\beta} \cdot |y|^\beta$$

for $|y| \leq |1, px|$. We note that our (analytic) Schwartz space, $S$, which can be defined as the intersection over all $\alpha$ of $\Sigma^\alpha$, is different from the usual (algebraic) Schwartz space of locally constant compactly supported functions that one encounters in the literature, though it shares much of its nice properties such as nuclearity, closure under multiplication, convolution and Fourier transform.

Chapter 2 is devoted to the proof of the main theorem of the calculus (2.2.8) and the basic boundedness theorems, along the lines of [18], [19]. In paragraph 2.1, we prepare the grounds by giving the short range estimate (2.1.6), the error estimate (2.1.11) and the Long range estimate (2.1.24) and (2.1.27). Then in paragraph 2.2, we put these together and prove the main theorem of the calculus (2.2.8). In paragraph 2.3, we establish the continuity of our operators on $S$ and $S'$, (2.3.1), and their boundedness on $L_2$ (2.3.9), and give a proof of the Calderón-Vaillancourt estimate [5] (along the lines of [21]).

In chapter 3 we establish the basic properties of elliptic operators. In paragraph 3.1, we prove the sharp Gårding inequality (3.1.11) and the improved Fefferman-Phong inequality (3.1.9) (cf. [8]), as consequences of the wave-packets theory (cf. [7]). In paragraph 3.2, we show that every elliptic symbol defines a Fredholm operator of index zero (3.2.9). In paragraph 3.3, we consider positive elliptic symbols, show that they give rise to self-adjoint operators with discrete spectra (3.3.1); study their complex powers (3.3.21) (in analogy with [28]); and determine the precise asymptotic behavior of their eigenvalues (3.3.14) (see [10], [13], [17], [20], [27], [30] for some of the literature on the analogous theory over $\mathbb{R}$).

Thus the operators on $L_2(V_0)$ corresponding to symbols in $\Sigma^\alpha(V)$ are bounded for $\alpha \leq 0$, Hilbert-Schmidt for $\alpha < -\dim V_0$, and trace-class for $\alpha < -2\dim V_0$. For $f$ in $\Sigma^\alpha(V)$, we can write $f = \tilde{f} + f_0$, where $\tilde{f}(x)$ is the average of $f$ over the unit ball centered at $x$, and $f_0$ is in the
Schwartz space. The operator \( \hat{\rho}(f) \) is thus the sum of \( \hat{\rho}(\tilde{f}) \), an explicitly diagonalizable operator, and of \( \hat{\rho}(f_0) \), a trace-class operator (in fact, a «smoothening» operator in the sense that it maps \( S' \) into \( S \)). As a corollary we see that if \( f \geq 0 \) is positive, then the operator \( \hat{\rho}(f) \) is positive modulo trace-class operators and is bounded from below. If \( f \) in \( \Sigma^\alpha(V) \) is elliptic, in the sense that \( |f(x)| \geq C \cdot |1, px|^\alpha \) for \( x \) outside a compact set, then \( \hat{\rho}(f) \) is a Fredholm operator of index zero; if \( f \) is real valued, then \( \hat{\rho}(f) \) is an essentially self-adjoint operator, if, moreover, \( \alpha > 0 \), then \( \hat{\rho}(f) \) has a discrete point spectrum tending to infinity and there is an orthonormal basis for \( L_2(V_0) \) consisting of \( \hat{\rho}(f) \)-eigenvectors all of which are in the Schwartz space. Assuming further that both \( f \) and \( \hat{\rho}(f) \) are positive, we can defined their zeta functions

\[
\zeta_{\rho(f)}(s) = \text{tr}(\hat{\rho}(f)^{-s}) \quad \text{and} \quad \zeta_f(s) = \int_V f(x)^{-s} \, dx
\]

both of which are holomorphic in the right half plane \( \text{Re}(s) > \frac{1}{\alpha} \dim V \), have meromorphic continuation and moreover, their difference is entire. Denoting by \( N(\lambda) \) the number of eigenvalues \( \leq \lambda \), we have the following precise estimate:

\[
N(\lambda) = \text{vol}\{ x \mid f(x) \leq \lambda \} + O(\lambda^\varepsilon), \quad \text{for any } \varepsilon > 0.
\]

Our motivation in developing the theory of pseudodifferential operators in an arithmetical situation was derived from [14], [15], where it was shown that such a theory could have bearings on the problem of the Riemann hypothesis. Much of what we do here carries over to the global situation of the adeles, where the zeta function not only controls the symbolic calculus, but to some extent is controlled by it.

We note that the content of this paper generalizes easily to the case of an arbitrary local field. Other future applications of our theory could include the local Howe’s duality conjectures. The theory can also be made to work over varieties where it might be useful as a tool in defining the zeta functions of such varieties by analytic means (a basic ingredient for trying to do Tate’s thesis for such varieties). Since we wrote this paper, we have found some attempts to construct quantum mechanics over the \( p \)-adics in the literature of physics, e.g. [42], but these are always wrong and lead to strange phenomenon (e.g. the existence of two vacuum states), because the physicists were insensitive to the basic difference between the \( p \)-adics and the reals – the two dimensional absolute value, which for the reals is the \( L_2 \).
one : \( |x, y| = (|x|^2 + |y|^2)^{1/2} \), should be replaced in the \( p \)-adic setting by its \( L_\infty \) analog : \( |x, y| = \max(|x|, |y|) \).

In the one dimensional theory, \( V_0 = \mathbb{Q}_p, V = \mathbb{Q}_p \oplus \mathbb{Q}_p \), we can take as a basic group of symmetries the group \( G = \text{Sp}(1, \mathbb{Q}_p) = \text{SL}(2, \mathbb{Q}_p) \), and identify \( V \backslash \{(0,0)\} \) with the symmetric space \( G/N \), where \( N \) is the subgroup of matrices of the form \( \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \), the action of \( G \) on \( V \), and the resulting action on symbols, correspond to the metaplectic representation on \( L_2(\mathbb{Q}_p) \). In a similar fashion we can follow the Unterbergers, cf. [39], [40], [41], and take as our phase space \( G/K \), where \( K = \text{SL}(2, \mathbb{Z}_p) \) is a maximal compact subgroup, obtaining quantizations which are relevant to the other discrete series representations of \( G \); or we can take as phase space \( G/T = \mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_p) \backslash \) diagonal, where \( T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \) is a maximal torus, obtaining quantizations relevant to the principal and the complementary series. Again, these constructions could be carried in an adelic situation, and will ultimately lead to the quantization of \( G(\mathbb{A})/G(\mathbb{Q}) \), i.e. quantization of automorphic forms.

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0. THE HEISENBERG GROUP

0.1. The Schrödinger model and twisted multiplication.

We denote by \( \psi \) the additive character of \( \mathbb{Q}_p \), given by

\[
\psi : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \simeq \mathbb{Z}[p^{-1}]/\mathbb{Z} \xrightarrow{\exp(2\pi ix)} \mathbb{C}^*.
\]

We let \( dx \) denote Haar measure on \( \mathbb{Q}_p^{\otimes d} \), self-dual with respect to \( \psi(x \cdot y) = \psi(\sum_{i=1}^d x_i \cdot y_i) \), so the Fourier transform

\[
\mathcal{F}_d \varphi(y) = \int \varphi(x) \psi(xy) \, dx
\]

satisfy \( \mathcal{F}_d \mathcal{F}_d \varphi(x) = \varphi(-x) \). The symbol \( \phi \) will denote the characteristic function of \( \mathbb{Z}_p^{\otimes d} \), so that \( \mathcal{F}_d \phi = \phi \). We shall usually assume \( p \neq 2 \), and leave it to the reader to make the easy modifications needed for \( p = 2 \).
We let $V$, $\langle \cdot, \cdot \rangle$ denote a symplectic vector space over $\mathbb{Q}_p$, and $\mathbb{H} = V \times \mathbb{Q}_p$ the associated Heisenberg group, with multiplication law given by

$$(0.1.2) \quad (v_1, t_1) \cdot (v_2, t_2) = (v_1 + v_2, t_1 + t_2 + \frac{1}{2}(v_1, v_2)), \quad v_i \in V, \ t_i \in \mathbb{Q}_p.$$ 

$\mathbb{H}$ is two-step nilpotent group, with center $\mathbb{Q}_p$, and the Stone-von Neumann theorem states that it has a unique irreducible representation with central character $\psi$ (up-to-equivalence). The Schrödinger model for this representation, $\rho : \mathbb{H} \to U(L_2(V_0))$, is associated with any complete polarization $V = V_0 \oplus V_1$, $V_i$ Lagrangian subspaces. Fixing coordinates we can focus our attention on the case

$$V_0 = V_1 = \mathbb{Q}_p^d, \quad \langle (x_1, y_1), (x_2, y_2) \rangle = x_1 y_2 - x_2 y_1,$$

so that $\rho$ is given by

$$(0.1.3) \quad \rho(x, y, t) \varphi(z) = \psi(t + x(z - \frac{1}{2}y)) \cdot \varphi(z - y).$$

The integrated representation, $\rho : L_1(\mathbb{H}) \to B(L_2(V_0))$, factors through the algebra

$$(0.1.4) \quad L_1(\mathbb{H}, \varphi) = \left\{ f \in L_1(\mathbb{H}) \mid f(h \cdot t) = \bar{\varphi}(t) \cdot f(h) \right\}$$

for $t \in \mathbb{Q}_p = \text{center}(\mathbb{H})$.

Identifying $V$ with the subspace of $\mathbb{H}$, via the isotopic cross-section $v \mapsto (v, 0)$, we can identify $L_1(\mathbb{H}, \varphi)$ with $L_1(V)$ via restriction. Thus for $f \in L_1(V)$ and $\varphi \in L_2(V_0),$

$$(0.1.5) \quad \rho(f) \varphi(z) = \int_V dv f(v) \rho(v, 0) \varphi(z) = \int_{V_0} dy K_f(z, y) \varphi(y)$$

with associated kernel given by

$$(0.1.6) \quad K_f(z, y) = \int dx f(x, z - y) \psi \left( \frac{1}{2} x(z + y) \right).$$

We have $\rho(f_1) \rho(f_2) = \rho(f_1 \triangleright f_2)$ with twisted convolution $\triangleright$ given by

$$(0.1.7) \quad f_1 \triangleright f_2(v) = \int_V dv' f_1(v') f_2(v - v') \psi \left( \frac{1}{2} (v', v - v') \right).$$

Since (0.1.6) is essentially a partial Fourier transform, we see that $\rho$ induces a unitary isomorphism from the algebra $L_2(V)$, $\triangleright$ onto the algebra
of Hilbert-Schmidt operators $\mathcal{K}_2(L_2(V_0))$; hence it takes $L_1(V)$ into the algebra of compact operators $\mathcal{K}(L_2(V_0))$. We have

$$\begin{align*}
\|f_1 \# f_2\|_{L_1} &\leq \|f_1\|_{L_1} \cdot \|f_2\|_{L_1}, \\
\|f_1 \# f_2\|_{L_2} &\leq \|f_1\|_{L_2} \cdot \|f_2\|_{L_2}, \\
\|f_1 \# f_2\|_{L_\infty} &\leq \|f_1\|_{L_2} \cdot \|f_2\|_{L_2}.
\end{align*}$$

(0.1.8)

We denote the symplectic Fourier transform, or isotropic symbol, by

$$\mathcal{F} f(v) = \hat{f}(v) = \int dv' f(v') \psi(\langle v', v \rangle).$$

(0.1.9)

$\mathcal{F}$ is a unitary operator on $L_2(V)$, satisfying $\mathcal{F}^2 = \text{id}$. The Weyl quantization procedure associates with a function $f$ on $V$ the operator $\hat{\rho}(f) = \rho(\hat{f})$ on $L_2(V_0)$, which is given by

$$\hat{\rho}(f) \varphi(x) = \iint dy \, d\xi \, f(\xi, \frac{1}{2}(x + y)) \psi(\xi(x - y)) \varphi(y).$$

(0.1.10)

We have $\hat{\rho}(f_1) \hat{\rho}(f_2) = \hat{\rho}(f_1 \# f_2)$, where twisted multiplication $\#$ is given by

$$f_1 \# f_2(v) = \mathcal{F}(\mathcal{F} f_1 \# \mathcal{F} f_2)(v)$$

$$= \iint dv_1 \, dv_2 \, f_1(v + v_1) \, f_2(v + v_2) \, \psi(2\langle v_1, v_2 \rangle)$$

$$= \psi(Q(D)) \, f_1 \otimes f_2(v, v).$$

Here $f_1 \otimes f_2(v_1, v_2) = f_1(v_1) \cdot f_2(v_2)$, and $\psi(Q(D))$ is the unitary operator on $L_2(V \oplus V) = L_2(V) \otimes V_2(V)$ given by

$$\psi(Q(D)) = \mathcal{F} \otimes \mathcal{F} \psi(Q(v_1, v_2)) \mathcal{F} \otimes \mathcal{F}$$

(0.1.12)

with $Q(v_1, v_2) = \frac{1}{2}\langle v_1, v_2 \rangle$ the quadratic form on $V \oplus V$ associated with the symplectic form $\langle \ , \rangle$ on $V$.

The adjoint operator $\hat{\rho}(f)^*$ is given by:

$$\hat{\rho}(f)^* = \hat{\rho}(\bar{f}), \quad \bar{f} \text{ the complex conjugate of } f.$$  

(0.1.13)

If $f(\xi, x) = f(x)$ is independent of $\xi$, then $\hat{\rho}(f) \varphi(x) = f(x) \cdot \varphi(x)$.

(0.1.14)

If $f(\xi, x) = f(\xi)$ is independent of $x$, then $\hat{\rho}(f) \varphi(x) = \mathcal{F}_d^{-1} f(\xi) \mathcal{F}_d \varphi(x)$.

(0.1.15)
Setting, for example,
\[ \Phi_{x_0, y_0}^{a, b}(x, y) = p^{-(a+b)d} \phi(p^a(x-x_0)) \phi(p^by-y_0)), \]
\[ \tilde{\Phi}_{x_0, y_0}^{a, b}(x, y) = \phi(p^{-b}(x-x_0)) \phi(p^{-a}(y-y_0)) \tilde{\psi}((x-x_0)(y-y_0)), \]
a straightforward calculation gives:

\[
\Phi_{x_1, y_1}^{a_1, b_1} \Phi_{x_2, y_2}^{a_2, b_2} =
\begin{cases}
\Phi_{x_1, y_1}^{a_1, b_1} \Phi_{x_2, y_2}^{a_2, b_2} & \text{if } a_1 + b_2, b_1 + a_2 \geq 0, \\
\Phi_{x_1, y_1}^{a_1, b_2} \Phi_{x_2, y_2}^{a_2, b_1} & \text{if } a_1 + b_2 \geq 0 \geq b_1 + a_2, \\
\tilde{\Phi}_{x_1, y_1}^{a_1, b_2} \tilde{\Phi}_{x_2, y_2}^{a_2, b_1} & \text{if } a_1 + b_2 \leq 0 \leq b_1 + a_2, \\
\tilde{\Phi}_{x_1, y_1}^{a_1, b_2} \tilde{\Phi}_{x_2, y_2}^{a_2, b_1} & \text{if } a_1 + b_2, b_1 + b_2 \leq 0.
\end{cases}
\]

For a lattice \( \mathcal{O} \subseteq V \), we denote the dual lattice by
\[
\mathcal{O}^\vee = \{ v \in V \mid \langle v, \mathcal{O} \rangle \subseteq \mathbb{Z}_p \}
\]
and say that \( \mathcal{O} \) is certain if \( \mathcal{O}^\vee \subseteq \mathcal{O} \). If \( \mathcal{O} \) is a certain lattice, and \( f_1, f_2 \) are \( \mathcal{O} \)-locally-constant, i.e., they factor through \( V/\mathcal{O} \), then an easy calculation shows that \( f_1 \neq f_2 = f_1 \cdot f_2 \), i.e., twisted multiplication reduces to ordinary multiplication. More generally, if \( V = \bigsqcup_j \mathfrak{A}_j \) is written as a disjoint union of translates of certain lattices, \( \mathfrak{A}_j = x_j + \mathcal{O}_j, \mathcal{O}_j \supseteq \mathcal{O}_j^\vee \), and \( f_1, f_2 \) are constant on each \( \mathfrak{A}_j \), then again
\[
f_1 \neq f_2 = f_1 \cdot f_2.
\]

0.2. The lattice model.

If \( \mathcal{O} \subseteq V \) is a certain lattice, then we can always find a self-dual lattice \( \mathcal{O}^0 = (\mathcal{O}^0)^\vee \), such that \( \mathcal{O}^\vee \subseteq \mathcal{O}^0 \subseteq \mathcal{O} \). The representation \( \rho \) can be realized in the \( \mathcal{O} \)-lattice model
\[
\rho \cong \rho_\psi \overset{\text{def}}{=} \text{ind}_{\mathcal{O}^0 \times \mathbb{Q}_p}^\mathbb{H} (\psi) = \text{ind}_{\mathcal{O} \times \mathbb{Q}_p}^\mathbb{H} (\rho_\mathcal{O})
\]
where \( \psi \) is viewed as a character of the abelian subgroup \( \mathcal{O}^0 \times \mathbb{Q}_p/\mathbb{Z}_p \subseteq \mathbb{H}/\mathbb{Z}_p \), via \( \psi(v, t) = \psi(t) \), and \( \rho_\mathcal{O} \) is the representation \( \text{ind}_{\mathcal{O}^0 \times \mathbb{Q}_p}^{\mathcal{O} \times \mathbb{Q}_p} (\psi) \) of \( \mathcal{O} \times \mathbb{Q}_p \), which is finite dimensional of dimension
\[
[\mathcal{O} : \mathcal{O}^0] = [\mathcal{O} : \mathcal{O}^\vee]^{1/2}.
\]
In particular for $V = \mathbb{Q}^{\mathbb{Z}_{2d}}$, $\mathcal{O}_0 = \mathbb{Z}^{\mathbb{Z}_{2d}}$, we obtain a realization of $\rho$ in $L_2(V)$ where the subscript $\psi$ indicates that we are looking on the subspace of $L_2(V)$ consisting of $\psi$-holomorphic functions, i.e., functions $f \in L_2(V)$ satisfying the following "Cauchy-Riemann equation"

\[(0.2.3) \quad f(v + v_0) = f(v) \cdot \psi \left( \frac{1}{2} \langle v, v_0 \rangle \right) \quad \text{for all } v \in V, \ v_0 \in \mathcal{O}_0.\]

The representation $\rho_\psi : \mathbb{H} \to U(L_2(V))$ is given by

\[(0.2.4) \quad \rho_\psi(v, t) f(v') = \psi(t + \frac{1}{2} \langle v, v' \rangle) f(v' - v).\]

To obtain the Bargmann isomorphism $B : L_2(V) \to L_2(V_0)$, intertwining $\rho_\psi$ and $\rho$, we notice that $L_2(V)$ is also gotten by taking the coefficients of $\rho$ with respect to $\phi \in L_2(V_0)$.

\[(0.2.5) \quad B f(x) = \rho(f) \phi(x) = (f, \rho(x) \phi(x))_{L_2(V)}.\]

\[(0.2.6) \quad B^{-1} \varphi(v) = (\varphi, \rho(v) \phi)_{L_2(V_0)}.\]

For any $v \in V$ there is a unique function $\Phi_v \in L_2(V)$ which is supported in $v + \mathcal{O}_0$, and normalized by $\Phi_v(v) = 1$; the functions $B\Phi_v = \rho(v)\phi$, $v \in V/\mathcal{O}_0$, form an orthonormal basis for $L_2(V_0)$; in particular we have

\[(0.2.7) \quad \Phi_0 = B^{-1} \phi = \phi \otimes \phi = \text{characteristic function of } \mathcal{O}_0.\]

For $f \in L_2(V)$ we have $f(v) = (f, \Phi_v)$, i.e., $L_2(V)$ is a reproducing-kernel-Hilbert space. The orthogonal projection $P_\psi : L_2(V) \to L_2(V)$ is given by

\[(0.2.8) \quad P_\psi f(v) = f \cdot \Phi_0(v) = \int_{\mathcal{O}_0} dv_0 f(v - v_0) \psi \left( \frac{1}{2} \langle v, v_0 \rangle \right).\]

For a function $f \in L_\infty(V)$, letting $M_f$ denote multiplication by $f$, we obtain the operator $P_\psi M_f$ on $L_2(V)$ having anti-Wick symbol $f$; but an easy calculation gives

\[(0.2.9) \quad P_\psi M_f = M_f, \quad \hat{f}(v) = \int_{\mathcal{O}_0} dv_0 f(v - v_0),\]

i.e. the algebra of such operators reduces to the commutative algebra $L_\infty(V/\mathcal{O}_0)$ acting on $L_2(V)$ by multiplication.
For any bounded operator $T \in \mathcal{B}(L_2(V)_\psi)$, we denote its Wick symbol by

$$ET(v) = (T\Phi_v, \Phi_v).$$

$E$ is a norm-decreasing positive projection $\mathcal{B}(L_2(V)_\psi) \to L_\infty(V/\mathcal{O}_0)$. The integrated representation associated with $\rho_\psi$ is given by

$$\rho_\psi(f) f_0 = f \upharpoonright f_0, \quad f_0 \in L_2(V)_\psi,$$

and the Weyl quantization $\hat{\rho}_\psi(f) = \rho_\psi(\hat{\rho}) = B^{-1}\hat{\rho}(f)B$ is given by

$$\hat{\rho}_\psi(f) f_0 = f \# f_0, \quad f_0 \in L_2(V)_\psi.$$

The relations between the Weyl, Wick, and anti-Wick symbols are easy to obtain

$$\begin{cases} E\hat{\rho}_\psi(f)(v) = \tilde{f}(v), \\ M_f = \hat{\rho}_\psi(f) \quad \text{for } f \in L_\infty(V/\mathcal{O}_0). \end{cases}$$

0.3. The Toeplitz model.

In this section we take $d = 1$, so that we are dealing with operators on $L_2(\mathbb{Q}_p)$ and $L_2(\mathbb{Q}_p \oplus \mathbb{Q}_p)_\psi$. Choose $\varepsilon \in \mathbb{Z}_p^* \setminus (\mathbb{Z}_p^*)^2$, and identify $V \cong \mathbb{Q}_p[\sqrt{\varepsilon}]$, the unramified quadratic extension of $\mathbb{Q}_p$, via $(x, y) \leftrightarrow x + \sqrt{\varepsilon} y$; thus $\mathcal{O}_0 = \mathbb{Z}_p[\sqrt{\varepsilon}]$ is the ring of integers of $V$, and the associated norm is given by

$$|x, y| = |x + \sqrt{\varepsilon} y| = |x^2 - \varepsilon y^2|^{1/2} = \max\{|x|, |y|\}.$$

$$\begin{align*}
|x, y| &= |x + \sqrt{\varepsilon} y| = |x^2 - \varepsilon y^2|^{1/2} = \max\{|x|, |y|\}.
\end{align*}$$

(0.3.2) Let $U_1 = \ker\{N : V^* \to \mathbb{Q}_p^*\}

= \{x + \sqrt{\varepsilon} y \in \mathcal{O}_0^* \mid x^2 - \varepsilon y^2 = 1\} = \mu_{p+1} \times \exp(\sqrt{\varepsilon} p \mathbb{Z}_p).

By Hilbert lemma 90, we have a commutative diagram with exact rows

$$
\begin{array}{c}
* \to U_1 \to \mathcal{O}_0^* \xrightarrow{N_u = u \cdot \tilde{u}} \mathbb{Z}_p^* \to * \\
\downarrow u^2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
Hence we may also identify

\[(0.3.4) \quad U_1 \cong \mathbb{O}_0^*/\mathbb{Z}_p^* = V^*/\mathbb{Q}_p^* = \mathbb{P}^1(\mathbb{Q}_p).\]

\[U_1\] may also be identified with the isometry group of the Hertmitian form on \(V\), given by

\[(0.3.5) \quad \left\{ \begin{array}{l}
\bar{v}_1 \cdot v_2 = q_\epsilon(v_1, v_2) + \sqrt{\epsilon} \langle v_1, v_2 \rangle \\
\text{with symmetric part } q_\epsilon(x_1 + \sqrt{\epsilon} y_1, x_2 + \sqrt{\epsilon} y_2) = x_1x_2 - \epsilon y_1y_2, \\
\text{and symplectic part } \langle v_1, v_2 \rangle = \text{tr}((2\sqrt{\epsilon})^{-1} \bar{v}_1v_2).
\end{array} \right.\]

Note that the metric induced on \(U_1\) via \(U_1 \hookrightarrow V\) is given, under the above identification, by any of the following metrices

\[(0.3.6) \quad \delta(\xi_1 e^{\sqrt{\epsilon} p u_1}, \xi_2 e^{\sqrt{\epsilon} p u_2}) = \left\{ \begin{array}{ll}
1 & \text{if } \xi_1 \neq \xi_2, \\
|p u_1 - p u_2| & \text{if } \xi_1 = \xi_2,
\end{array} \right. \quad \text{where } U_1 \cong \mu_{p+1} \times \exp(\sqrt{\epsilon} p \mathbb{Z}_p);
\]

\[\delta(z_1, z_2) = |z_1/z_1 - z_2/z_2|, \quad \text{where } U_1 \cong V^*/\mathbb{Q}_p^*,
\]

\[\delta(x_1 : y_1, x_2 : y_2) = \frac{|x_1y_2 - x_2y_1|}{|x_1, y_1| \cdot |x_2, y_2|}, \quad \text{where } U_1 \cong \mathbb{P}^1(\mathbb{Q}_p).
\]

The group \(\text{Sp}(V) = \text{SL}_2(\mathbb{Q}_p)\) acts on \(\mathbb{H}\) as automorphisms inducing the identity on the center, hence by the uniqueness of the representation \(\rho\), we obtain a projective representation \(R\) of \(\text{Sp}(V)\) such that

\[(0.3.7) \quad R_A \rho(v) R_A^{-1} = \rho(Av), \quad A \in S_p(V), \ v \in V.
\]

\(R\) in fact lifts to a true representation of the double cover of \(\text{Sp}(V)\), the \textit{metaplectic group} \(\widehat{\text{Sp}}(V)\). We have an embedding

\[(0.3.8) \quad U_1 \hookrightarrow \text{SL}_2(\mathbb{Z}_p), \quad x + \sqrt{\epsilon} y \longmapsto \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix},
\]

by which we view \(U_1\) as a maximal \textit{abelian} subgroup of \(\text{Sp}(V)\); in the terminology of \([22]\) : \((U_1, U_1)\) is a d\textit{ual} \textit{reductive pair}. The metaplectic group splits over \(\text{SL}_2(\mathbb{Z}_p)\), hence we obtain a true representation \(R\) of \(\text{SL}_2(\mathbb{Z}_p)\), and a posteriori of \(U_1\). This can be summarized in the following diagram :

\[(0.3.9) \quad U_1 \hookrightarrow \text{SL}_2(\mathbb{Z}_p) \hookrightarrow \text{Sp}(V)\]
Explicitly, \( R : U_1 \to U(L_2(\mathbb{Q}_p)) \) is given by the Mehler kernel

\[
R_{x+y}(z_1, z_2) = \frac{(x, y)}{\gamma(y) \cdot |y|^{1/2}} \psi \left( \frac{x}{2y} (z_1^2 + z_2^2) - \frac{z_1 z_2}{y} \right)
\]

for \( x + y \in U_1 \setminus \{ \pm 1 \} \) and \( R_{\pm 1} \varphi(z) = \varphi(\pm z) \).

Here \((x, y)\) denotes the quadratic symbol, \( \gamma(y) \) denotes Weil's root of unity, and they are given by \((p \neq 2)\)

\[
(p^{j_1} \xi^k_1, p^{j_2} \xi^k_2) = (-1)^{\frac{1}{2}(p-1)} j_1 j_2 + k_1, j_1 k_2,
\]

\[
\gamma(p^j \xi^k) = \begin{cases} 
1 & \text{if } j \text{ is even}, \\
(-i)^{\frac{1}{2}(p-1)+2k} & \text{if } j \text{ is odd}.
\end{cases}
\]

Denote the dual group of \( U_1 \) by \( \mathcal{Z} = (U_1)^\vee \cong (\mu_{p+1})^\vee \times \mathbb{Q}_p/\mathbb{Z}_p \), and write a typical character \( \chi \in \mathcal{Z} \) as

\[
\chi(\xi e^{\sqrt{p}w}) = \chi_t(\xi) \cdot \tilde{\psi}(\frac{1}{2} q_x pw),
\]

where \( \chi_t \in (\mu_{p+1})^\vee \) and \( q_x \in \mathbb{Q}_p/p^{-1}\mathbb{Z}_p \).

We put \( \mathcal{Z} = \bigoplus_{N \geq 0} \mathcal{Z}^{(N)}, \mathcal{Z}^{(0)} = \{ \chi^0 \} \) the identity, \( \mathcal{Z}^{(1)} = \{ \chi_t \neq \chi^0 \} \) the non-trivial tame characters, and for \( N \geq 2 : \mathcal{Z}^{(N)} = \{ \chi; |q_x| = p^N \} \). A straightforward calculation gives that \( L_2(\mathbb{Q}_p) \) decomposes under the \( U_1 \) action given by \( R \) as

\[
L_2(\mathbb{Q}_p) = \bigoplus_{\chi \in \mathcal{Z}^+} \mathbb{C} \cdot \phi^\chi, \quad R_u \phi^\chi = \chi(u) \cdot \phi^\chi,
\]

where \( \mathcal{Z}^+ = \bigcup_{N \geq 0} \mathcal{Z}^{(2N)} \). In particular, \( \phi^{\chi^0} = \phi \) is the unique \( U_1 \)-invariant vector. For \( \chi \in \mathcal{Z}^{(2N)}, N \geq 1 \), the Hermite function \( \phi^\chi \) is the unique \( \chi \)-covariant vector (up-to a constant of absolute value one), and is given explicitly by

\[
\phi^\chi = (1 + p^{-1})^{1/2} p^{N/2} \int_{U_1} d^* u \, \bar{\chi}(u) \rho(u \cdot z_\chi) \phi
\]

where \( q_\chi = Nz_\chi \), and \( d^* u \) is Haar measure on \( U_1 \) of total mass 1.
Comparing the orthonormal basis \( \{ \phi^x \mid x \in \mathbb{Z}^+ \} \) with the orthonormal basis \( \{ \rho(x,y) \phi \mid x,y \in \mathbb{Q}_p/\mathbb{Z}_p \} \) of the lattice model, we note that

\[
\#\mathbb{Z}^{(2N)} = p^{2N}(1 - p^{-2})
\]

\[
= \# \left\{ (x,y) \in \mathcal{O}_0 \mid |x,y| = p^N \right\}, \quad N \geq 1,
\]

and each \( \phi^x \) is a linear combination of \( p^N(1 + p^{-1}) \rho(x,y)\phi \)'s, and vice versa. More precisely,

\[
\Phi^x(u) = B^{-1} \phi^x(v) = (\phi^x, \rho(v)\phi)
\]

is supported in \( V^x = \{ v \in V \mid Nv \equiv q_x \pmod{p^{-N}\mathbb{Z}_p} \} = U_1 \cdot z_x + \mathcal{O}_0 \), where it is given by

\[
\Phi^x(u \cdot z_x + v_0) = ((1 + p^{-1})p^N)^{-1/2} \cdot \chi(u) \cdot \psi\left( \frac{1}{2} \langle u \cdot z_x, v_0 \rangle \right),
\]

with \( u \in U_1 \) and \( v_0 \in \mathcal{O}_0 \).

Using (0.3.14) we obtain the T"oeplitz model \( \rho_\tau : \mathbb{H} \to U(L_2(U_1)^+) \), where \( L_2(U_1)^+ \subseteq L_2(U_1) \) is the subspace corresponding to \( \ell_2(\mathbb{Z}^+) \subseteq \ell_2(\mathbb{Z}) \) under Fourier transform \( L_2(U_1) \simeq \ell_2(\mathbb{Z}) \). The orthogonal projection \( P^+ : L_2(U_1) \to L_2(U_1)^+ \) is easily seen to be given by the singular integral

\[
P^+ f(u_0) = \text{p.v.} \int_{U_1} \delta^* u f(u) \delta(u, u_0)^{-1},
\]

where \( \delta(u, u_0) = (-p)^{-N} \) if \( \delta(u, u_0) = p^{-N} \).

If \( f \in C(U_1)^* \) is a continuous non-vanishing function, the associated T"oeplitz operator \( P^+ M_f \) is Fredholm of index zero. Indeed, any such function is homotopic to \( 1 \), the constant function 1, since \( U_1 \) is totally disconnected and compact. This can be partially explained by considering the problem from a dual point of view : if \( \chi \in \mathbb{Z}^{(N)} \), translation by \( \chi \) followed by the projection \( \widehat{P^+} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}^+) \) induces a Fredholm operator \( \widehat{P^+} \chi \) on \( \ell_2(\mathbb{Z}^+) \), which has index zero since,

\[
\dim \ker \widehat{P^+} \chi = \dim \text{coker} \widehat{P^+} \chi = p^{N-1}, \quad N \geq 1.
\]

Remark. — The material in paragraph 0.3. works equally well for the case of a totally ramified quadratic extension of \( \mathbb{Q}_p \).
0.4. Wave packets.

We can consider more general coefficients of the representation \( \rho \) than those of \( \phi \). Thus for \( \varphi_1, \varphi_2 \in L_2(V_0) \) let \( c_{\varphi_1, \varphi_2}(v) = (\varphi_2, \rho(v) \varphi_2) \). After Fourier inversion we obtain

\[
(0.4.1) \quad c_{\varphi_1, \varphi_2}(x, y) = \int dz \bar{\psi}(xz) \varphi_1(z + \frac{1}{2} y) \varphi_2(z - \frac{1}{2} y).
\]

A direct calculation gives

\[
(0.4.2) \quad (c_{\varphi_1, \varphi_2}, c_{\varphi_3, \varphi_4})_{L_2(V)} = (\varphi_1, \varphi_3)_{L_2(V_0)} \cdot (\varphi_2, \varphi_4)_{L_2(V_0)}
\]

showing that \( \rho \) is square integrable modulo the center. The Schwartz inequality gives

\[
(0.4.3) \quad \|c_{\varphi_1, \varphi_2}\|_{L_2} \leq \|\varphi_1\|_{L_2} \cdot \|\varphi_2\|_{L_2}.
\]

Hence the map \( \varphi_1 \otimes \varphi_2 \mapsto c_{\varphi_1, \varphi_2} \) induces

\[
\varphi : L_2(V_0) \otimes \overline{L_2(V_0)} \longrightarrow L_2(V) \cap C_0(V).
\]

Using (0.4.2) we get \( \varphi', \rho(c_{\varphi_1, \varphi_2}) \varphi) = (c_{\varphi', \varphi}, c_{\varphi_1, \varphi_2}) = (\varphi', (\varphi, \varphi_2) \cdot \varphi_1) \), i.e.,

\[
(0.4.4) \quad \rho(c_{\varphi_1, \varphi_2}) \text{ is the rank } 1 \text{ operator } \rho(c_{\varphi_1, \varphi_2}) \varphi = (\varphi, \varphi_2) : \varphi_1.
\]

Hence \( c_{\varphi_1, \varphi_2} \leq c_{\varphi_3, \varphi_4} = (\varphi_2, \varphi_3) \cdot c_{\varphi_1, \varphi_4} \).

Taking the symplectic Fourier transform of \( c_{\varphi_1, \varphi_2} \), we obtain after Fourier inversion

\[
(0.4.5) \quad \hat{c}_{\varphi_1, \varphi_2}(\xi, x) = \int dz \bar{\psi}(\xi z) \varphi_1(x + \frac{1}{2} z) \varphi_2(x - \frac{1}{2} z).
\]

\( \hat{c}_{\varphi_1, \varphi_2} \) satisfies (0.4.2), (0.4.3), and again induces

\[
\hat{c} : L_2(V_0) \otimes \overline{L_2(V_0)} \longrightarrow L_2(V) \cap C_0(V).
\]

Moreover, by direct verification we have

\[
(0.4.6) \quad \hat{c}_{\mathcal{F}_d \varphi_1, \mathcal{F}_d \varphi_2}(\xi, x) = \hat{c}_{\varphi_1, \varphi_2}(x, -\xi),
\]

\[
(0.4.7) \quad \hat{c}_{\varphi_1, \varphi_2} = \hat{c}_{\varphi_2, \varphi_1}.
\]
We define the Wigner transform by $W_{\varphi} = \check{\varphi} \in L_2(V) \cap C_0(V)$, and by (0.4.7) it is real valued. It also satisfies the following properties:

(0.4.8) $\int d\xi W_{\varphi}(\xi, x) = |\varphi(x)|^2$,

(0.4.9) $\int dx W_{\varphi}(\xi, x) = |\mathcal{F}_d\varphi(\xi)|^2$,

(0.4.10) $W_{\varphi_1} = W_{\varphi_2} \iff \varphi_1 = \lambda \cdot \varphi_2$ for some $\lambda, |\lambda| = 1$.

Note that $W_{\varphi}(\xi, x) \neq 0$ implies that there exists $z$ such that $x \pm z \in \text{supp} \varphi$, and hence $x \in \Delta(\text{supp} \varphi) \overset{\text{def}}{=} \{ \frac{1}{2} (y_1 + y_2) \mid y_i \in \text{supp} \varphi \}$, thus we have

(0.4.11) $\text{proj}_x(\text{supp} W_{\varphi}) \subseteq \Delta(\text{supp} \varphi)$,

and using (0.4.6),

(0.4.12) $\text{proj}_\xi(\text{supp} W_{\varphi}) \subseteq \Delta(\text{supp} \mathcal{F}_d \varphi)$.

We have the following inequalities:

(0.4.13) $\|W_{\varphi}\|_{L_\infty} \leq \|\varphi\|_{L_2}^2 = \int \int dx d\xi W_{\varphi}(\xi, x) \leq \|W_{\varphi}\|_{L_\infty} \cdot \text{vol}(\text{supp} W_{\varphi})$

from which we obtain the uncertainty principle

(0.4.14) $\text{vol}(\text{supp} W_{\varphi}) \geq \frac{\|\varphi\|_{L_2}^2}{\|W_{\varphi}\|_{L_\infty}} \geq 1$.

We have further

(0.4.15) $W_{\varphi} = c_{\varphi, \varphi} = \Phi_0 = \text{characteristic function of } O_0$,

(0.4.16) $W(\rho(v_0)\varphi)(v) = W_{\varphi}(v - v_0)$,

(0.4.17) $W(\mathcal{R}_A\varphi)(v) = W_{\varphi}(A^{-1}v)$, $A \in \text{Sp}(V)$, by (0.3.7).

Since $\text{Sp}(V)$ acts transitively on the collection of self-dual lattices, we deduce using (0.4.15), (0.4.16), (0.4.17) that for any translate of a self-dual lattice $\mathfrak{a} = v_0 + O, O^\vee = O$, there exists a unit vector $\varphi_{\mathfrak{a}} \in L_2(V_0)$ such
that $W_\varphi = \text{characteristic function of } \mathfrak{A}$; by (0.4.10) $\varphi$ is unique up to a constant of absolute value one.

For any unit vector $\varphi_0 \in L_2(V_0)$, the map $C_{\varphi_0}(\varphi) = c_{\varphi, \varphi_0}$ is by (0.4.2) an isometry $c_{\varphi_0} : L_2(V_0) \to L_2(V)$. The adjoint map

$$C_{\varphi_0}^* : L_2(V) \to L_2(V_0)$$

is given by $C_{\varphi_0}^* f(x) = \rho(f) \varphi_0(x)$, $f \in L_2(V) \cap L_1(V)$. Since $C_{\varphi_0}^* C_{\varphi_0} = \text{id}_{L_2(V_0)}$, we obtain the wave packet expansion of $\varphi$ as superposition of the $\rho(v) \varphi_0$'s

$$\varphi(x) = \int dv \left( \varphi, \rho(v) \varphi_0 \right) \cdot \rho(v) \varphi_0(x).$$

(0.4.18)

For $f \in L_\infty(V) \cap L_1(V)$, denoting by $M_f$ multiplication by $f$, we get the operator $C_{\varphi_0}^* M_f C_{\varphi_0}$ on $L_2(V_0)$. We have

$$\langle C_{\varphi_0}^* M_f C_{\varphi_0} \varphi_1, \varphi_2 \rangle_{L_2(V_0)}$$
$$= \langle M_f C_{\varphi_0} \varphi_1, C_{\varphi_0} \varphi_2 \rangle_{L_2(V)}$$
$$= \int dv \int f(v) \varphi_1, \varphi_0(v) \overline{\varphi_2, \varphi_0(v)}$$
$$= \int dv \int f(v) \left( \varphi_1, \rho(v) \varphi_0 \right) \cdot \left( \varphi_2, \rho(v) \varphi_0 \right)$$
$$= \int dv \int f(v) \left( \varphi_1, \rho(v) \varphi_0 \right) \cdot \left( \varphi_2, W(\rho(v) \varphi_0) \right) \quad \text{by (0.4.2)}$$
$$= \int dv \int dv_1 f(v) \varphi_1, \varphi_0(v_1) W(\varphi_0(v_1 - v)) \quad \text{by (0.4.16)}$$
$$= \int f_1 W(\varphi_0(v_1)) \varphi_1, \varphi_0(v_1)$$
$$= \int dv \mathcal{F}(f \ast W \varphi_0)(-v)(\varphi_1, \rho(v) \varphi_2)$$
$$= \int dv \mathcal{F}(f \ast W \varphi_0)(v)(\rho(v) \varphi_1, \varphi_2) = \langle \hat{\rho}(f \ast W \varphi_0) \varphi_1, \varphi_2 \rangle.$$

(0.4.19) Hence: $C_{\varphi_0}^* M_f C_{\varphi_0} = \hat{\rho}(f \ast W \varphi_0)$.

(0.4.20) If $V = \bigsqcup_j \mathfrak{A}_j$ is written as a disjoint union of translates of self-dual lattices, $\mathfrak{A}_j = x_j + \mathcal{O}_j$, $\mathcal{O}_j^* = \mathcal{O}_j$, and if for each $j$, $\varphi_j \in L_2(V_0)$ is a unit vector such that $W \varphi_j = \Phi_j$ is the characteristic function of $\mathfrak{A}_j$, then
the \( \varphi_j \) form an orthonormal basis for \( L_2(V_0) \). Indeed, we have by (0.4.2), for \( j_1 \neq j_2 \)

\[
| (\varphi_{j_1}, \varphi_{j_2})_{L_2(V_0)} |^2 = (\Phi_{j_1}, \Phi_{j_2})_{L_2(V)} = 0;
\]

moreover, letting \( \varphi_j^0 = \rho(-x_j)\varphi_j \), so that by (0.4.16), \( \Phi_j^0 = W\varphi_j^0 \) is the characteristic function of \( O_j \), \( \Phi_j^0 \ast \Phi_j = \Phi_j \), and for any locally constant compactly supported function \( \varphi \) in \( L_2(V_0) \), we have by (0.4.19)

\[
(0.4.21) \quad \varphi = \hat{\rho}(1) \varphi = \sum_j \hat{\rho}(\Phi_j) \varphi = \sum_j \hat{\rho}(\Phi_j^0 \ast \Phi_j) \varphi
\]

\[
\begin{align*}
&= \sum_j C_{\Phi_j}^* M_{\Phi_j} C_{\varphi_j} \varphi \\
&= \sum_j C_{\varphi_j}^* \left[ \Phi_j^0(v) \cdot (\varphi, \rho(v)\varphi_j)_{L_2(V_0)} \right] \\
&= \sum_j \int_{V} dv \Phi_j^0(v) \cdot (\varphi, \rho(v)\varphi_j)_{L_2(V_0)} \cdot \rho(v)\varphi_j \\
&= \sum_j (\varphi, \varphi_j)_{L_2(V_0)} \cdot \varphi_j.
\end{align*}
\]

If \( f \) is a function on \( V \), assuming a constant value, say \( \lambda_j \), on each of the \( \mathfrak{M}_j \)'s, so that \( f = \sum_j \lambda_j \cdot \Phi_j \), we have as in (0.4.21) : \( \hat{\rho}(f)\varphi_j = \lambda_j \cdot \varphi_j \), i.e., \( \hat{\rho}(f) \) is diagonalizable with respect to the orthonormal basis \( \varphi_j \) with associated eigenvalues the \( \lambda_j \)'s.

1. FUNCTION SPACES OVER \( \mathbb{Q}_p \)

1.1. Sobolev spaces.

The local zeta function of \( \mathbb{Q}_p \) is denoted by

\[
(1.1.1) \quad \zeta(s) = (1 - p^{-s})^{-1} = \int_{\mathbb{Q}_p^*} \phi(x) |x|^s d^*x, \quad \text{Re } s > 0.
\]

It is a meromorphic function of \( s \), \( (2\pi i/\log p)\mathbb{Z} \) periodic, with a unique simple pole at \( s = 0 \). We denote by \( I^\alpha \) the operator of multiplication by the function

\[
(1.1.2) \quad |1, x|^{-\alpha} = \max\{1, |x|\}^{-\alpha}
\]
and we denote by $J^\alpha$ the operator $F^{-1}_1 I^\alpha F_1$, which for $\Re \alpha > 0$ is given by convolution with the $L_1$ function

\begin{equation}
J^\alpha(x) = F_1(|1, \xi|^{-\alpha})(x) = \frac{\zeta(1 - \alpha)}{\zeta(\alpha)} (|x|^{\alpha - 1} - p^{\alpha - 1}) \phi(x),
\end{equation}

where $\Re \alpha > 0$ and $\alpha \not\equiv 1 \mod (2\pi i / \log p) \mathbb{Z}$.

The $J^\alpha$'s form an entire family of distributions, and we have $J^\alpha \ast J^\beta = J^{\alpha + \beta}$. For real $\alpha > 0$, the $J^\alpha$'s form a semi-group of probability measures associated with the negative definite function

\begin{equation}
\log |1, \xi| = \int (1 - \psi(x\xi)) \, d\sigma(x),
\end{equation}

where the Lévi measure $\sigma(x)$ is given by the divisor function

\begin{equation}
\sigma(x) = \log p \cdot \sum_{n \geq 0} p^n \cdot \phi(p^{-n}x).
\end{equation}

Note that, since $J^\alpha$ is supported in $\mathbb{Z}_p$, and $I^\beta$ is $\mathbb{Z}_p$-locally-constant, we have

\begin{equation}
I^\beta J^\alpha = J^\alpha I^\beta.
\end{equation}

All this extends to the higher dimensional case $V_0 = \mathbb{Q}_p^{\otimes d}$; using the metric $|x_1, \ldots, x_d| = \max\{|x_1|, \ldots, |x_d|\}$ we have commuting operators $I^\beta$, $J^\alpha = F^{-1}_d I^\alpha F_d$; $J^\alpha$ forming an entire family of distributions, $J^\alpha \ast J^\beta = J^{\alpha + \beta}$; for $\alpha > 0, J^\alpha$ being a probability measure supported in $\mathbb{Z}_p^{\otimes d}$; and

\begin{equation}
J^\alpha(x) = \frac{\zeta(d - \alpha)}{\zeta(\alpha)} (|x|^{\alpha - d} - p^{\alpha - d}) \cdot \phi(x),
\end{equation}

for $\Re \alpha > 0$, $\alpha \not\equiv d \mod (2\pi i / \log p) \mathbb{Z}$,

\begin{equation}
J^d(x) = \frac{1}{\zeta(d)} (1 - \log_p |x|) \cdot \phi(x).
\end{equation}

Note that

\begin{equation}
J^\alpha \in L_q(V_0) \quad \text{for} \quad \Re \alpha > d \cdot (1 - q^{-1})
\end{equation}

and in particular: $J^\alpha \in L_1$, $\Re \alpha > 0$; $J^\alpha \in L_2$, $\Re \alpha > \frac{1}{2} d$; $J^\alpha \in L_\infty$, $\Re \alpha > d$. 
The Sobolev spaces (or Lebesgue spaces of Bessel potentials [31]) are defined by

\[(1.1.10) \quad L^\alpha_q(V_0) = J^\alpha(L_q(V_0)), \quad \|f\|_{L^\alpha_q} = \|J^{-\alpha}f\|_{L_q}\]

and in a similar way, we can define \(L^\alpha_q(\mathfrak{A})\), where \(\mathfrak{A} \subseteq V_0\) is any domain such that \(\mathfrak{A} + \mathbb{Z}_p^{\otimes d} = \mathfrak{A}\). We also define the weighted Sobolev spaces by

\[(1.1.11) \quad L^{\alpha;\beta}_q(V_0) = J^\alpha I^\beta(L_q(V_0)), \quad \|f\|_{L^{\alpha;\beta}_q} = \|I^{-\beta}J^{-\alpha}f\|_{L_q}.

The spaces \(L^\alpha_q, L^{\alpha;\beta}_q\) are Banach spaces isomorphic to \(L_q\), and they satisfy the following list of basic properties [31] :

\[(1.1.12) \quad \text{Embedding } L^{\alpha;\beta}_q(V_0) \hookrightarrow L^{\alpha;\gamma}_q(V_0) \text{ continuously for } 0 < q_1^{-1} \leq q_2^{-1} < 1, \quad d \cdot (q_1^{-1} - q_2^{-1}) \leq \alpha_1 - \alpha_2;
0 \leq q_2^{-1} \leq q_1^{-1} \leq 1, \quad d \cdot (q_1^{-1} - q_2^{-1}) < \alpha_1 - \alpha_2.

\[(1.1.13) \quad \text{Duality } (L^\alpha_q)' = L^{-\alpha}_q \text{ for } 1 < q < \infty, \quad q^{-1} + q'^{-1} = 1.

\[(1.1.14) \quad \text{Relation with the Riesz potential}
R^\alpha = \mathcal{F}[|\xi|^{-\alpha}] = \frac{\zeta(d-\alpha)}{\zeta(\alpha)} |x|^{d-\alpha}, \quad \text{for } \Re \alpha > 0,
L^\alpha_q(V_0) \cong \text{domain of } R^\alpha \text{ acting on } L_q(V_0), \quad \Re \alpha > 0,
i.e. \|f\|_{L^\alpha_q} \approx \|f\|_{L_q} + \|R^{-\alpha}f\|_{L_q}.

In particular, let us note that we have

\[(1.1.15) \quad L^\alpha_2(V_0) \hookrightarrow C_0(V_0), \quad \text{for } \alpha > \frac{1}{2} d,
and that we have quasinuclear (i.e. Hilbert-Schmidt) embeddings

\[(1.1.16) \quad L^{\alpha;1}_2(\mathfrak{A}) \hookrightarrow L^{\alpha;2}_2(\mathfrak{A}), \quad \alpha_1 - \alpha_2 > \frac{1}{2} d, \quad \mathfrak{A} \text{ bounded domain},
(1.1.17) \quad L^{\alpha;1;\beta}_2(V_0) \hookrightarrow L^{\alpha;2;\beta}_2(V_0), \quad \alpha_1 - \alpha_2 > \frac{1}{2} d, \quad \beta_1 - \beta_2 > \frac{1}{2} d.
Recalling the Bargmann isomorphism $B : L_2(V_\psi) \overset{\sim}{\rightarrow} L_2(V_0)$, where $V = V_0 \oplus V_0$, (0.2.5), (0.2.6), we note that using (0.1.14), (0.1.15), and (0.2.13), we have

\begin{align}
B^{-1}I^\alpha B &= M_{[1,x]^{-\alpha}}, \\
B^{-1}J^\alpha B &= M_{[1,\xi]^{-\alpha}}.
\end{align}

It becomes natural to define the Schrödinger operators by

\begin{equation}
B^{-1}K^\alpha B = M_{[1,\xi,x]^{-\alpha}}
\end{equation}

and the associated spaces

\begin{equation}
L_q^{\{\alpha\}}(V_0) = K^\alpha(L_q(V_0)), \quad \|f\|_{L_q^{\{\alpha\}}} = \|K^{-\alpha}f\|_{L_q}
\end{equation}

since $|1,\xi|^{-\alpha} \cdot |1,x|^{-\alpha} \leq |1,\xi,x|^{-\alpha} \leq |1,\xi|^{-\alpha}$ and $|1,x|^{-\alpha}$, for $\alpha > 0$, we have

\begin{equation}
L_q^{\alpha[\alpha]} \hookrightarrow L_q^{\{\alpha\}} \hookrightarrow L_q^{\alpha[0]} \cap L_q^{0[\alpha]}.
\end{equation}

The operators $I^\alpha$, $J^\alpha$, $K^\alpha$ belong to the algebra $BL_\infty(V/O_0)B^{-1}$, for $\text{Re} \alpha \geq 0$, and hence they admit the lattice basis $\{\rho(v)\phi \mid v \in V/O_0\}$ as eigenfunctions. In particular, the operators $I^\beta J^\alpha$, and $K^\alpha$, are trace class in the following domains, with trace

\begin{align}
\text{tr}(I^\beta J^\alpha) &= \frac{\zeta(\alpha - d)}{\zeta(\alpha)} \cdot \frac{\zeta(\beta - d)}{\zeta(\beta)}, \quad \text{Re} \alpha, \text{Re} \beta > d, \\
\text{tr}(K^{2\alpha}) &= \frac{\zeta(2(\alpha - d))}{\zeta(2\alpha)}, \quad \text{Re} \alpha > d.
\end{align}

The operators $K^{2\alpha}$, for $d = 1$, also admit the Hermite functions $\{\phi^\chi\}$ as eigenfunctions:

\begin{equation}
K^{2\alpha} \phi^\chi = (\text{cond} \chi)^{-\alpha} \cdot \phi^\chi
\end{equation}

where $\text{cond} \chi = p^N$ for $\chi \in Z^{(N)}$.

One can further generalize the above spaces by considering more general «metrics» i.e. functions $g : V_0 \rightarrow \{0\} \cup p\mathbb{Z}$ satisfying

\begin{align}
\begin{cases}
g(v_1 + v_2) \leq \max\{g(v_1), g(v_2)\}, & v_i \in V_0, \\
g(\lambda \cdot v) = |\lambda| \cdot g(v), & \lambda \in \mathbb{Q}_p, \quad v \in V_0, \\
g(v) = 0 \iff v = 0.
\end{cases}
\end{align}
Recall that such metrics correspond bijectively to lattices \( \mathcal{O} \subseteq V_0 \), the correspondence given by

\[
\mathcal{O} = \{ v \in V | g(v) \leq 1 \}, \quad g(v) = \inf\{ |\lambda| ; \lambda^{-1} \cdot v \in \mathcal{O} \}.
\]

If \( \mathcal{O}_i \subseteq V_i \) corresponds to \( g_i, i = 1, 2 \), then \( \mathcal{O}_1 \oplus \mathcal{O}_2 \subseteq V_1 \oplus V_2 \) corresponds to

\[
g_1 \otimes g_2(v_1, v_2) = \max\{ g_1(v_1), g_2(v_2) \}.
\]

If \( \mathcal{O} \subseteq V \) corresponds to \( g \), the dual lattice \( \mathcal{O}^\ast \subseteq V^\ast \), defined in the dual vector space by

\[
\mathcal{O}^\ast = \{ v^\ast \in V^\ast | \langle v^\ast, \mathcal{O} \rangle \subseteq \mathbb{Z}_p \}
\]
corresponds to the dual-metric \( g^\ast \), given by

\[
g^\ast(v^\ast) = \inf\{ \frac{|\langle v^\ast, v \rangle|}{g(v)} ; v \in V \}.
\]

This holds in particular when \( V \) is a symplectic vector space identified with its own dual by means of the symplectic form.

Given \( V, g \) we can define the operators

\[
\begin{cases}
I_g^\alpha = \text{multiplication by } \max\{ 1, g(x) \}^{-\alpha}, \\
J_g^\alpha = \mathcal{F}^{-1} I_g^\alpha \mathcal{F}, \\
R_g^\alpha = \mathcal{F}^{-1} g^\ast(\xi)^{-\alpha} \mathcal{F}, \\
K_g^\alpha = BI_g^\ast \otimes g B^{-1},
\end{cases}
\]

and the associated spaces

\[
L_q^\alpha(V, g), \; L_q^{\alpha|\beta}(V, g), \; L_q^\alpha(V, g), \text{ and } L_q^\alpha(\mathfrak{A}, g)
\]

for a domain \( \mathfrak{A} \) such that \( \mathfrak{A} + \mathcal{O} \subseteq \mathfrak{A} \) (note that since \( I_g^\ast \) is \( \mathcal{O}^\ast \)-locally-constant, \( \text{supp} J_g^\alpha \subseteq \mathcal{O} \)). Since any metrics are equivalent by an affine transformation (i.e. given \( g_1, g_2 \) on \( V \), there exists \( A \in \text{GL}(V) \) such that \( g_1(v) = g_2(Av) \)), it is clear that the results (1.1.12) through (1.1.17) remain valid for the spaces (1.1.32).
1.2. Besov spaces.

The problem with operators $J^\alpha$, and the spaces $L^2_q$, is that unlike the real case, where we have $J^{-2N} = (1 + \partial^2/\partial x^2)^N$, we lack in the $p$-adic setting an analogue of the Leibnitz rule

$$
\left( \frac{\partial}{\partial x} \right)^N (f_1 \cdot f_2) = \sum_{0 \leq j \leq N} \binom{N}{j} \left( \frac{\partial}{\partial x} \right)^j f_1 \cdot \left( \frac{\partial}{\partial x} \right)^{N-j} f_2
$$

which follows upon taking the Fourier transform from the Newton rule:

$$(y_1 + y_2)^N = \sum_{0 \leq j \leq N} \binom{N}{j} y_1^j \cdot y_2^{N-j}. \tag{1.2.1}$$

Thus, for example, in the $p$-adic setting $f_1, f_2 \in L^\infty_\infty$ does not imply $f_1 \cdot f_2 \in L^\infty_\infty$. To see the problem more clearly, we notice that we can write for $\Re \alpha > 0$ and $y_i \in \mathbb{Q}_p$,

$$
\begin{align*}
|y_1 + y_2|^\alpha &= \text{p.v.} \int_0^{2\pi/\log p} \frac{\log p}{2\pi} \, dt \cdot \frac{\log p}{2\pi} \\
&\quad \times \left\{ \zeta(it) - \frac{1}{2} \frac{\zeta(1)}{\zeta(1 + \alpha)} + \frac{\zeta(-1)\zeta(-(1 + \alpha))}{\zeta(\alpha)} \right\} \\
&\quad \times \left\{ |y_1|^{\alpha+it} \cdot |y_2|^{-it} + |y_1|^{-it} \cdot |y_2|^{\alpha+it} \right\} \\
&+ \frac{\zeta(1)\zeta(1 + \alpha)}{\zeta(-\alpha)} \sum_{\chi \neq 1} \chi(-1) \cdot (\operatorname{cond} \chi)^{(1+\alpha)} \\
&\quad \times \int_0^{2\pi/\log p} \frac{\log p}{2\pi} \cdot \chi(y_1) |y_1|^{\alpha/2+it} \cdot \bar{\chi}(y_2) |y_2|^{\alpha/2-it} \, dt,
\end{align*}
$$

where $\chi$ varies over all non-trivial characters of $\mathbb{Z}_p^*$, but the sum $\sum_{\chi}$ is actually finite for $y_1 + y_2 \neq 0$. Thus, (1.2.2) like (1.2.1), expresses $|y_1 + y_2|^\alpha$ as "sum" of $\chi_1(y_1) \cdot \chi_2(y_2)$, with $\chi_1 \cdot \chi_2 = \mid \cdot \mid^\alpha$.

To overcome this problem we introduce the Besov spaces $B^\alpha_{q,r}$ (or Lipschitz spaces) of functions that are well approximated by locally-constant-functions. We shall use the following notations: $V$ is a $\mathbb{Q}_q$-vector space with a metric $g$ corresponding to the lattice $\mathcal{O} \subseteq V$, and $\mathcal{O}^\vee \subseteq V^\vee$ is
the dual lattice. We let

\[
\phi_{g,k} = \text{Fourier transform of the characteristic function of } p^{-k}\mathcal{O}, \quad k \geq 0;
\]

\[
\varphi_{g,k} = \phi_{g,k} - \phi_{g,k-1} \quad \text{for } k \geq 1, \varphi_{g,0} = \phi_{g,0};
\]

\[
d_g x = \text{Haar measure in } V \text{ normalized by } d_g x(O) = 1;
\]

\[
L_q = L_q(V, d_g x);
\]

\[
d^*_g y = \zeta(d) \cdot g(y)^{-d} \cdot d_g y, \quad d = \dim V, \text{ so that } d^*_g y \text{ is the «multiplicative Haar measure» with } d^*_g y(p^k \mathcal{O} \setminus p^{k+1} \mathcal{O}) = 1;
\]

\[L^*_r = L_r(O, d^*_g y);
\]

\[\ell_r \text{ is the usual } \ell_r\text{-space on } k \in \{0, 1, 2, \ldots\}.
\]

The Besov space \(B^{\alpha}_{q,r}(V,g)\) consists of functions \(f\) (or distributions, if \(\alpha < 0\)) such that

\[
\|f\|_{B^{\alpha}_{q,r}} = \|p^k \varphi_{g,k} \ast f(x)\|_{L_q} \|_{\ell_r} < \infty.
\]

We shall only consider these spaces for \(1 \leq q \leq \infty, 1 \leq r \leq \infty\). The \(B^{\alpha}_{q,r} = B^{\alpha}_{q,r}(V,g)\) are Banach spaces with the following basic properties:

**Embeddings**

\[
B^{\alpha}_{q,r_1} \hookrightarrow B^{\alpha}_{q,r_2}, \quad r_1 \leq r_2;
\]

\[
B^{\alpha}_{q,\infty} \hookrightarrow B^{\alpha}_{q,1}, \quad \alpha_1 > \alpha_2;
\]

\[
\begin{cases}
B^{\alpha}_{q,q} \hookrightarrow L^\alpha_q \hookrightarrow B^{\alpha}_{q,2}, & 1 < q \leq 2, \\
B^{\alpha}_{q,2} \hookrightarrow L^\alpha_q \hookrightarrow B^{\alpha}_{q,q}, & 2 \leq q < \infty;
\end{cases}
\]

\[
B^{\alpha+d(1/q-1/q_0)}_{q,r} \hookrightarrow B^{\alpha}_{q_0,r}, \quad q \leq q_0;
\]

**Interpolation**

\[
[B^{\alpha}_{q_0,r_0}, B^{\alpha}_{q_1,r_1}]_{\theta} = B^{\alpha}_{q,r}, \text{ for the complex method;}
\]

here \(0 < \theta < 1, \quad q^{-1} = (1 - \theta) \cdot q_0^{-1} + \theta \cdot q_1^{-1},
\]

\[
r^{-1} = (1 - \theta) \cdot r_0^{-1} + \theta \cdot r_1^{-1}, \quad \alpha = (1 - \theta) \cdot \alpha_0 + \theta \cdot \alpha_1 \neq \alpha_1;
\]

\[
(L^\alpha_q, L^\alpha_1)_{\theta,r} = B^{\alpha}_{q,r}, \text{ for the real method;}
\]

\[
(B^{\alpha}_{q,r})' = B^{-\alpha}_{q',r'} \text{ with } q^{-1} + q'^{-1} = 1 = r^{-1} + r'^{-1};
\]
(1.2.12) Lifting $J^\alpha_g : B^{\alpha_0}_{q,r} \xrightarrow{\sim} B^{\alpha_0+\alpha}_{q,r}$;

(1.2.13) Characterization via approximation: for $\alpha > 0$,
\[
\|f\|_{B^{\alpha}_{q,r}} \approx \|f\|_{L^q} + \left\| p^{k\alpha} \cdot (f(x) - \phi_{g,k} \ast f(x)) \right\|_{L^q_{L^\infty}}.
\]

(1.2.14) Characterization via differences: for $\alpha > 0$,
\[
\|f\|_{B^{\alpha}_{q,r}} \approx \|f\|_{L^q} + \left\| g(y)^{-\alpha} \cdot (f(x) - f(x+y)) \right\|_{L^q_{L^\infty}};
\]

(1.2.15) Pointwise multiplier: for $\alpha > 0$,
\[
\|f_1 \cdot f_2\|_{B^{\alpha}_{q,r}} \leq C \|f_1\|_{B^{\alpha}_{\infty,r}} \cdot \|f_2\|_{B^{\alpha}_{q,r}}.
\]

(1.2.16) $B^{d/q+\alpha}_{q,r}$ and $B^{d/q}_{q,r}$ are quasi Banach algebras of continuous functions.

(1.2.17) Remark. — Note that if $\text{supp} f \subseteq \mathcal{O}$ then $\text{supp} \varphi_{g,k} \ast f \subseteq \mathcal{O}$. Hence we can define $B^{\alpha}_{q,r}(\mathcal{O}, g) = \{ f \in B^{\alpha}_{q,r}(V, g) \mid \text{supp} f \subseteq \mathcal{O} \}$, and similarly we can define $B^{\alpha}_{q,r}(\mathfrak{A}, g)$ for any domain such that $\mathfrak{A} + \mathcal{O} = \mathfrak{A}$. All the above properties remain valid for the spaces $B^{\alpha}_{q,r}(\mathfrak{A}, g)$. When $\mathfrak{A} = x + \mathcal{O}$, we shall write $B^{\alpha}_{q,r}(\mathfrak{A})$ for $B^{\alpha}_{q,r}(\mathfrak{A}, g)$.

(1.2.18) Remark. — The only discrepancy from the analogous spaces over the real numbers is (1.2.14); for the reals, one has to use instead of the difference $\Delta_y f(x) = f(x) - f(x+y)$, the $N$-fold-difference
\[
\Delta_y^N f(x) = \sum_{0 \leq j \leq n} \binom{N}{j} (-1)^j f(x+jy),
\]
where $N$ is an integer greater than $\alpha$. The proofs of (1.2.5) through (1.2.12) are immediate translations of the corresponding proofs in the real case (e.g. [3]).

The proof of the equivalence of the norms (1.2.4), (1.2.13), (1.2.14) follows as in [31], theorem (2.2), p. 180. The proof of (1.2.15), (1.2.16) follows as in the real case (cf. [25]), or as a consequence of (1.2.14).

We shall also need the Besov spaces of a product space given by

\[
\begin{align*}
B^{\alpha_1,\alpha_2}_{q,r}(V_1 \times V_2, g_1 \otimes g_2) & \overset{\text{def}}{=} B^{\alpha_1}_{q,r}(V_1, g_1) \otimes B^{\alpha_2}_{q,r}(V_2, g_2), \\
B^{\alpha_1,\alpha_2}_{q,r}(\mathfrak{A}_1 \times \mathfrak{A}_2, g_1 \otimes g_2) & \overset{\text{def}}{=} B^{\alpha_1}_{q,r}(\mathfrak{A}_1, g_1) \otimes B^{\alpha_2}_{q,r}(\mathfrak{A}_2, g_2),
\end{align*}
\]
with $\mathfrak{A}_i + \mathfrak{O}_i = \mathfrak{A}_i$, where $\otimes$ denotes the completed (in the crossed-product topology) tensor product. These are again Banach spaces, satisfying analogous properties, their norm being given by

$$
\|f\|_{B_{q,r}^{\alpha_1,\alpha_2}} = \left\| p^{k_1\alpha_1 + k_2\alpha_2} (\varphi_{g_1,k_1} \otimes \varphi_{g_2,k_2}) * f \right\|_{L_q} |_{\ell_r} .
$$

(1.2.20)

In particular, we can restate the pointwise multiplier assertions of (1.2.16) as trace theorems:

$$
\text{Res} : B_{q,r}^{d/q+\alpha,d/q+\alpha}(V \times V, g \otimes g) \longrightarrow B_{q,r}^{d/q+\alpha}(V, g), \quad \alpha > 0,
$$

(1.2.21)

$$
\text{Res} : B_{q,1}^{d/q,d/q}(V \times V, g \otimes g) \longrightarrow B_{q,1}^{d/q}(V, g)
$$

(1.2.22)

are continuous, where Res denotes the restriction to the diagonal. The spaces in (1.2.21), (1.2.22) are spaces of continuous functions, and we can view the $\delta$-functions as elements of their dual. We have

$$
\|\delta_{x_1,x_2}\|_{(B_{2,1}^{d/2,d/2})'} = \|\delta_{x_1,x_2}\|_{B_{2,\infty}^{-d/2,-d/2}} = 1.
$$

(1.2.23)

We can also define the weighted Besov space $B_{q,r}^{\alpha,\beta}(V, g)$ given by the norm

$$
\|f\|_{B_{q,r}^{\alpha,\beta}} = \left\| p^{k\alpha} I_g^{-\beta}(\varphi_{g,k} * f)(x) \right\|_{L_q} |_{\ell_r} = \|I_g^{-\beta} f\|_{B_{q,r}^{\alpha}},
$$

(1.2.24)

noting that $I_g^{-\beta}(\varphi_{g,k} * f) = \varphi_{g,k} * (I_g^{-\beta} f)$, since $\text{supp} \varphi_{g,k} \subseteq \mathcal{O}$. $B_{q,r}^{\alpha,\beta}$ will satisfy the analogous properties, e.g.

- interpolation $[B_{q_0,r_0}^{\alpha_0,\beta_0}, B_{q_1,r_1}^{\alpha_1,\beta_1}]_\theta = B_{q,r}^{\alpha,\beta}$, as in (1.2.9) with $\beta = (1 - \theta)\beta_0 + \theta \cdot \beta_1$;
- duality $\left(B_{q,r}^{\alpha,\beta}\right)' = B_{q',r'}^{-\alpha,-\beta}$;
- lifting $I_g^{\beta} : B_{q,r}^{\alpha,\beta_0} \rightarrow B_{q,r}^{\alpha,\beta_0+\beta}$;
- pointwise multiplier $\|f_1 \cdot f_2\|_{B_{q,r}^{\alpha_1+\alpha_2}} \leq C \cdot \|f_1\|_{B_{\infty,r}^{\alpha_1}} \cdot \|f_2\|_{B_{q,r}^{\alpha_2}}$, for $\alpha > 0$;

etc.

A metric-covering of $V$ is an assignment for each $x \in V$ of a metric $g_x$ such that

$$
g_x(y) \leq 1 \implies g_x = g_{x+y}.
$$

(1.2.25)
Letting $O_x$ denote the lattice corresponding to $g_x$, and setting $\mathfrak{A}_x = x + O_x$, i.e.

\[(1.2.26) \quad \mathfrak{A}_x = \{ y \in V \mid g_x(x - y) \leq 1 \} \]

we can choose a sequence $x_j \in V$ such that $V = \bigsqcup_j \mathfrak{A}_x$, a disjoint union. Conversely, given a covering $V = \bigsqcup_j \mathfrak{A}_x$, by disjoint sets $\mathfrak{A}_x$, of the form $\mathfrak{A}_x = x_j + O_j$, $O_j$ a lattice, we obtain a metric covering by setting

\[(1.2.27) \quad g_x = \text{the metric corresponding to } O_j, \text{ for } x \in \mathfrak{A}_x.\]

A $g$-continuous weight is a function $m : V \to \mathbb{R}^+$, such that for some constant $C$,

\[(1.2.28) \quad g_x(y) \leq 1 \implies C^{-1} \cdot m(x) \leq m(x + y) \leq C \cdot m(x).\]

Given such $g$ and $m$, we define the Banach space $B_{q,r}^\alpha(m, g)$, with norm

\[(1.2.29) \quad \|f\|_{B_{q,r}^\alpha(m, g)} = \sup_{x \in V} \frac{1}{m(x)} \cdot \|f_x\|_{B_{q,r}^\alpha(\mathfrak{A}_x, g_x)}\]

where $f_x$ denote the restriction of $f$ to $\mathfrak{A}_x$.

Choosing a sequence $x_j \in V$ such that $V = \bigsqcup_j \mathfrak{A}_x$, $\mathfrak{A}_x = x_j + O_x$, and putting for $x \in \mathfrak{A}_x : \tilde{m}(x) = m(x_j) = m_j$, it follows from (1.2.28) that $B_{q,r}^\alpha(m, g) = B_{q,r}^\alpha(\tilde{m}, g)$. Hence we may as well assume from the start that $m$ is $g$-locally-constant in the sense that

\[(1.2.30) \quad g_x(y) \leq 1 \implies m(x) = m(x + y)\]

and then the norm of $B_{q,r}^\alpha(m, g)$ is given by

\[(1.2.31) \quad \|f\|_{B_{q,r}^\alpha(m, g)} = \sup_j \frac{1}{m_j} \|f_j\|_{B_{q,r}^\alpha(\mathfrak{A}_j, g_j)}\]

where $f_j$ denotes the restriction of $f$ to $\mathfrak{A}_j$.

The spaces $B_{q,r}^\alpha(m, g)$ will inherit many of the properties of $B_{q,r}^\alpha(V)$, in particular:
Embeddings

(1.2.32) \[ B_{q_1,r_1}^\alpha(m,g) \hookrightarrow B_{q_2,r_2}^\alpha(m,g), \quad r_1 \leq r_2; \]
(1.2.33) \[ B_{q_1,\infty}^\alpha(m,g) \hookrightarrow B_{q_1,1}^{\alpha,g}(m,g), \quad \alpha_1 > \alpha_2; \]
(1.2.34) \[ B_{q,r}^{\alpha+d(1/q-1/q_0)}(m,g) \hookrightarrow B_{q_0,r}^{\alpha}(m,g), \quad q \leq q_0; \]
(1.2.35) \[ B_{q,r}^{\alpha}(m_1,g) \hookrightarrow B_{q,r}^{\alpha}(m_2,g), \quad m_1 < C \cdot m_2. \]

Liftings

(1.2.36) \[ J_f : B_{q,r}^{\alpha}(m,g) \xrightarrow{\sim} B_{q,r}^{\alpha+\alpha}(m,g), \]
where \( J_f \) is an algebra under pointwise multiplication, consisting of continuous functions, and that it is a nuclear space.

Pointwise multiplier : for \( \alpha > 0 \),

(1.2.37) \[ \left\| f_1 \cdot f_2 \right\|_{B_{q,r}^{\alpha+\alpha}(m_1 \cdot m_2,g)} \leq C \cdot \left\| f_1 \right\|_{B_{q,r}^{\alpha}(m_1,g)} \cdot \left\| f_2 \right\|_{B_{q,r}^{\alpha}(m_2,g)} \]

and characterizations via approximation and difference similar to (1.2.13) and (1.2.14) of which we note : for \( \alpha > 0 \),

(1.2.38) \[ \left\| f \right\|_{B_{q,r}^{\alpha}(m,g)} \cong \sup_{x \in V} \frac{1}{m(x)} |f(x)| \]
\[ + \sup_{0 < g(x_0)(x_0 - x_1) \leq 1} \frac{1}{m(x_0)} \cdot \frac{|f(x_0) - f(x_1)|}{g(x_0)(x_0 - x_1)^\alpha}. \]

1.3. Symbol classes and the analytic Schwartz space.

For a lattice \( \mathcal{O} \subseteq V \) we denote by \( C_\infty(\mathcal{O}) \) the Fréchet space defined by the norms \( \left\| f \right\|_{L_\infty^\alpha}, \alpha > 0 \). We can also use the norms \( \left\| f \right\|_{L_q^\alpha}, \alpha \geq 0 \), for any \( 1 \leq q \leq \infty \), and similarly we can use the norms \( \left\| f \right\|_{B_{q,r}^{\alpha}}, \alpha \geq 0 \), for any \( 1 \leq q, r \leq \infty \). In particular, we see that \( C_\infty(\mathcal{O}) \) is an algebra under pointwise multiplication, consisting of continuous functions, and that it is a nuclear space.
(1.3.1). — For any open set $\mathcal{A} \subseteq V$, we denote by $C^\infty(\mathcal{A})$ the Fréchet space defined by the norms $\|f\|_{L^q_{\infty}(x+\mathcal{O})}$, $\alpha \geq 0$, $x+\mathcal{O} \subseteq \mathcal{A}$ a translate of a lattice contained in $\mathcal{A}$. Again we can use the norms associated with $L^q_{\infty}(x+\mathcal{O})$, or with $B^q_{r,\rho}(x+\mathcal{O})$, and $C^\infty(\mathcal{A})$ is a nuclear algebra of continuous functions. In particular, we have $C^\infty(\mathcal{A}_1 \times \mathcal{A}_2) = C^\infty(\mathcal{A}_1) \otimes C^\infty(\mathcal{A}_2)$. Just as $C^\infty(V) = \varproj lim C^\infty(\mathcal{O})$, the inverse limit taken over all lattices $\mathcal{O} \subseteq V$ with respect to the restriction mapping $C^\infty(\mathcal{O}_1) \rightarrow C^\infty(\mathcal{O}_2)$, $\mathcal{O}_1 \supseteq \mathcal{O}_2$, we can define the space of compactly supported smooth functions $C^\infty_c(V) = \varproj lim C^\infty(\mathcal{O})$, where now we take the direct limit with respect to the mappings $C^\infty(\mathcal{O}_2) \hookrightarrow C^\infty(\mathcal{O}_1)$, $\mathcal{O}_1 \supseteq \mathcal{O}_2$, given by extending by zero. $C^\infty_c(V)$ is similarly a nuclear algebra of compactly supported continuous functions.

(1.3.2). — We define the (analytic) Schwartz space $S(V)$ to be the Fréchet space defined by the norms $\|f\|_{L^q_{\infty}(x+\mathcal{O})}$, $\alpha, \beta \geq 0$. Equivalently, we can use the norms associated with $L^q_{\infty}(x+\mathcal{O})$, or the norms of $L^q_{\alpha}(x+\mathcal{O})$, $\alpha \geq 0$, associated with the Schrödinger operator (1.1.21); or the norms of $B^q_{r,\rho}(x+\mathcal{O})$, $\alpha, \beta \geq 0$, for any $1 \leq q, r \leq \infty$ and any metric $\rho$ on $V$. $S(V)$ is a nuclear algebra of continuous functions. Fixing some non-degenerate quadratic form, giving an identification $V \cong V^\vee$ of $V$ with its dual, we can view the Fourier transform $\mathcal{F}$ as a mapping $\mathcal{F} : L^2_{\alpha,\beta}(V) \rightarrow L^2_{\beta,\alpha}(V)$, hence $\mathcal{F}$ defines an automorphism of $S(V)$, which consequently forms an algebra also under convolution. We have again $S(V_1 \oplus V_2) = S(V_1) \otimes S(V_2)$. Denoting by $S_{\text{alg}}(V)$ the algebraic Schwartz space of locally-constant compactly-supported functions, we have proper dense inclusions

$$S_{\text{alg}} \subseteq C^\infty_c \subseteq S \subseteq S' \subseteq C'_{\text{alg}} \subseteq S'.$$

We note that $S_{\text{alg}}$, $C^\infty_{\text{alg},c}$, $S$, $C^\infty$, as well as their duals, are all stable under the action of the Heisenberg group given by $\rho$.

We have the Schwartz kernel theorems (cf. [33] for the similar proof over $\mathbb{R}$):

$$\begin{cases}
\{ \text{Hom}(C^\infty(\mathcal{A}_1), C'_\infty(\mathcal{A}_2)) \cong C'_\infty(\mathcal{A}_2 \times \mathcal{A}_1), \\
\text{Hom}(S(V_1), S'(V_2)) \cong S'(V_2 \oplus V_1) \}.
\end{cases}$$

(1.3.3) An operator $A$ on functions on $V_0$ will be called smoothening if

$$A : S'(V_0) \rightarrow S(V_0)$$
continuously, or equivalently if $A$ is given by a kernel $A(x, y) \in S(V_0 \oplus V_0)$.  
A still equivalent condition for $A$ to be smoothing is that $A = \hat{\rho}(f)$ with $f \in S(V_0^\wedge \oplus V_0) = S(V_0^\wedge)$, since $A(x, y)$ and $f(\xi, x)$ are obtained from each other by a partial Fourier transform which takes $S(V_0^\wedge \oplus V_0)$ isomorphically onto $S(V_0^\wedge \oplus V_0)$.

Given a metric-covering $g$, and a $g$-locally-constant weight function $m$, we define the symbol class $S(m, g)$ to be the Fréchet space defined by the norms $\|f\|_{B^\alpha_{\infty, \infty}(m, g)}$, $\alpha > 0$. These norms are given in an equivalent form in (1.2.39), so that $f \in S(m, g)$ if and only if for some constants $C_0, C_\alpha$, we have

\begin{equation}
|f(x)| \leq C \cdot m(x) \quad \text{for all } x \in V,
\end{equation}

\begin{equation}
|f(x_0) - f(x_1)| \leq C_\alpha \cdot m(x_0) \cdot g_{x_0}(x_0 - x_1)^\alpha,
\end{equation}

for all $x_0, x_1 \in V$, $g_{x_0}(x_0 - x_1) \leq 1$, and for any given $\alpha > 0$.

Using the pointwise-multiplier, or by using (1.3.4), (1.3.5), we have:

\begin{equation}
f_i \in S(m_i, g) \implies f_1 \cdot f_2 \in S(m_1 \cdot m_2, g).
\end{equation}

Note that $C_{\infty, c} \subseteq S(m, g) \subseteq C_\infty$, but the first inclusion is not dense, hence a continuous linear form on $S(m, g)$ is not determined by its restriction to $C_{\infty, c}$; such a form will be called weakly continuous if its restriction to a bounded subset of $S(m, g)$ is continuous in the $C_\infty$ topology.

### 1.4. Temperate metric-coverings.

We return to a symplectic vector space $V, \langle \cdot, \cdot \rangle$, and we shall deal with a metric-covering $g_x$, corresponding to $V = \bigsqcup_j \mathfrak{A}_j$, $\mathfrak{A}_j = x_j + O_j$, where $g_x$ is always assumed to be certain : $g_x \leq g^\vee_x$, or equivalently $O_j \supset g^\vee_j \cdot g_x$ will be said to be temperate if there exist $n_0, N \geq 0$, such that

\begin{equation}
g^\vee_{x_0}(x) \leq p^{n_0} \cdot g^\vee_{x_1}(x) \cdot g^\vee_{x_1}(1, x_1 - x_0)^N, \quad x_0, x_1, x \in V,
\end{equation}

where $g^\vee(1, x) = \max\{1, g^\vee(x)\}$. Using the definition (1.1.30) of the dual metric, one can write (1.4.1) in the equivalent form,

\begin{equation}
g_{x_1}(x) \leq p^{n_0} \cdot g_{x_0}(x) \cdot g^\vee_{x_1}(1, x_1 - x_0)^N.
\end{equation}
Taking $x = x_0 - x_1$ in (1.4.1) we have in particular,

\[(1.4.3) \quad g_{x_0}^\gamma (1, x_0 - x_1) \leq p^{n_0} \cdot g_{x_1}^\gamma (1, x_1 - x_0)^{N+1}.\]

From (1.4.1) we may also deduce that

\[
g_{x_1}^\gamma (x_1 - x_0) \leq p^{n_0} \cdot g_{x_1+x_2-x_0}^\gamma (x_1 - x_0) \cdot g_{x_1+x_2-x_0}^\gamma (1, x_2 - x_0)^N,
\]

and similarly with $x_1, x_2$ interchanged, from which we obtain

\[(1.4.4) \quad g_{x_1}^\gamma (x_1 - x_0) \leq p^{(2+N)n_0} \cdot g_{x_2}^\gamma (1, x_1 - x_0)^{N+1} \cdot g_{x_1}^\gamma (1, x_2 - x_0)^{N(N+1)}.
\]

Multiplying (1.4.4) with the similar expression obtained by interchanging $x_1, x_2$ we finally get,

\[(1.4.5) \quad g_{x_1}^\gamma (x_1 - x_0) \cdot g_{x_2}^\gamma (x_2 - x_0) \leq p^{2(2+N)n_0} \cdot g_{x_2}^\gamma (1, x_1 - x_0)^{(N+1)^2} \cdot g_{x_1}^\gamma (1, x_2 - x_0)^{(N+1)^2}.
\]

**LEMMA (1.4.6).** — For $\alpha > d(2N + 1)$,

\[
\sum_j g_{x_j}^\gamma (1, x_j - x)^{-\alpha} \leq p^{2dn_0} \cdot \zeta (\alpha - d(2N + 1)).
\]

**Proof.** — Set $J(k) = \{j \mid g_{x_j}^\gamma (1, x_j - x) \leq p^k\}$. For $j \in J(k)$ we have

\[
g_{x_j}^\gamma (y) \leq p^{n_0+kN} g_{x_j}^\gamma (y)
\]

by (1.4.2), and hence $x_j + p^{n_0+kN} \cdot \mathcal{O}_x \subseteq x_j + \mathcal{O}_j = \mathfrak{A}_j$. Similarly, using $g_x \leq g_{x_j}^\gamma$ and (1.4.1), for $j \in J(k)$,

\[
g_{x}^\gamma (x - x_j) \leq g_{x_j}^\gamma (x - x_j) \leq p^{n_0} \cdot g_{x_j}^\gamma (1, x_j - x)^{N+1} \leq p^{n_0+k(N+1)},
\]

and hence $\mathfrak{A}_j \subseteq x + p^{-[n_0+k(N+1)]} \mathcal{O}_x$. Denoting by vol the Haar measure in $V$ normalized by $\text{vol}(\mathcal{O}_x) = 1$, we get from the above two inclusions:

\[
\# J(k) = \sum_{i \in J(k)} 1 = \sum_{i \in J(k)} \text{vol}(x_j + p^{n_0+kN} \mathcal{O}_x) \cdot p^{d(n_0+kN)} \leq \text{vol} \left( \prod_{j \in J(k)} \mathfrak{A}_j \right) \cdot p^{d(n_0+kN)} \leq \text{vol}(x + p^{-[n_0+k(N+1)]} \mathcal{O}_x) \cdot p^{d(n_0+kN)} = p^{d(n_0+k(N+1))} \cdot p^{d(n_0+kN)} = p^{2dn_0} \cdot p^{d(2N+1)k}.
\]
Hence
\[
\sum_j g_{x_j}^\gamma (1, x_j - x)^{-\alpha} = \sum_{j \geq 0} \#(J(k) \setminus J(k - 1)) \cdot p^{-k \cdot \alpha} \\
\leq p^{2d_0} \cdot \sum_{k \geq 0} p^{(d(2N + 1) - \alpha) \cdot k} \\
= p^{2d_0} \cdot \zeta(\alpha - d(2N + 1)). \quad \square
\]

Using (1.4.5) we deduce the following

**COROLLARY (1.4.7).** — For \( \beta > d \cdot (2N + 1) \cdot (N + 1)^2 \),
\[
\sum_{j_1, j_2} g_{x_{j_1}}^\gamma (1, x_{j_2} - x)^{-\beta} \cdot g_{x_{j_2}}^\gamma (1, x_{j_1} - x)^{-\beta} < \infty.
\]

The weight function \( m \) will be called *temperate* if there exist \( C \) and \( \beta \), such that
\[
(1.4.8) \quad m(x_1) \leq C \cdot m(x_0) \cdot g_{x_1}^\gamma (1, x_1 - x_0)^\beta.
\]

If \( m \) is temperate, so is \( m^\alpha \) for all \( \alpha \geq 0 \), and using (1.4.3) also for \( \alpha \leq 0 \). We also have
\[
(1.4.9) \quad h(x) \overset{\text{def}}{=} \sup_y \frac{g_x(y)}{g_x^\gamma(y)} \text{ is temperate.}
\]

Indeed, using (1.4.1) and (1.4.2), we have
\[
h(x_1) \leq p^{2d_0} \cdot h(x_0) \cdot g_{x_1}^\gamma (1, x_1 - x_0)^{2N}.
\]

Given two temperate weights \( m_1, m_2 \), then \( m_1 \cdot m_2 \) is also temperate; moreover we have
\[
m_1(x_1) \cdot m_2(x_2) \leq C_1 \cdot C_2 \cdot m_1(x_0) m_2(x_0) g_{x_1}^\gamma (1, x_1 - x_0)^{\beta_1} \cdot g_{x_2}^\gamma (1, x_2 - x_0)^{\beta_2}
\]
which by (1.4.5) is smaller than
\[
C_0 \cdot m_1(x_0) \cdot m_2(x_0) \cdot g_{x_1}^\gamma (1, x_1 - x_0)^{\beta_0} \cdot g_{x_2}^\gamma (1, x_2 - x_0)^{\beta_0}
\]
for some new constants \( C_0, \beta_0 \). Hence,
\[
(1.4.10) \quad g_{x_1}^\gamma (1, x_2 - x_0)^{-\beta_0} \cdot g_{x_2}^\gamma (1, x_1 - x_0)^{-\beta_0} \leq C_0 \cdot \frac{m_1(x_0) \cdot m_2(x_0)}{m_1(x_1)m_2(x_2)}.
\]
When $g$ and $m$ are temperate, we have $1/m(x) \leq C/m(0) \cdot g_0^\beta(1,x)$, and using (1.4.2) we easily obtain $B^\alpha_{\infty,\infty}(V,g_0) \hookrightarrow B^\alpha_{\infty,\infty}(m,g)$, and hence $S(V) \hookrightarrow S(m,g)$. Conversely, given $g$, the functions $m_0(x) = g_0(1,x)^{-1}$, $m_1(x) = g_x(1,x)^{-1}$, are temperate, and an easy estimate gives $B^\alpha_{\infty,\infty}(m_0^\beta \cdot m_1^{N-\alpha},g) \hookrightarrow B^\alpha_{\infty,\infty}(V,g_0)$. Thus we conclude that

$$S(V) = \bigcap_m S(m,g),$$

where the intersection is taken over all temperate weights $m$, i.e. the topology of $S(V)$ is defined by all the norms $B^\alpha_{\infty,\infty}(m,g)$, $\alpha > 0$, $m$ temperate.

Beside the uniform metric-covering $g_x = g_0 \leq g_0^\alpha$, which we studied in paragraph 0.2, basic metrics are the Toeplitz metric and the pseudodifferential metric which we next describe. The Toeplitz symbols are defined by

$$\Sigma^\alpha = S(m^\alpha, g), \quad g_x(y) = |1,px|^{-1} \cdot |y|, \quad m^\alpha(x) = |1,px|^{\alpha}.$$  

It is easily verified that $g_x$ is a metric-covering (i.e. it satisfies (1.2.25)), that it is certain (i.e. $g_x(y) \leq g_x^\alpha(y) = |1,px| \cdot |y|$) with $h(x) = |1,px|^{-2} = m^{-2}(x)$, and that it is temperate; indeed, we have a stronger inequality than (1.4.1)

$$g_x^\beta(x) \leq g_x^\alpha(1,p(x_0 - x_1)),$$

and (1.4.13)

$$\left\{ \begin{array}{l}
g_x^\alpha(x) \leq g_x^\alpha(1,x) \cdot |1,p(0 - x_1)|, \\
|1,p(x_0 - x_1)| \leq g_x^\alpha(1,x_1 - x_0).
\end{array} \right.$$ 

This follows from the basic inequality

$$|1,y_0| \leq |1,y_1| \cdot |1,y_0 - y_1|.$$ 

Explicitly, $f \in \Sigma^\alpha$ if and only if

$$|f(x)| \leq C_0 \cdot |1,px|^\alpha$$

for all $x \in V$, and for all $\beta > 0$:

$$|f(x) - f(x + y)| \leq C_\beta \cdot |1,px|^{\alpha-\beta} \cdot |y|^\beta \quad \text{for } |y| \leq |1,px|. $$
Letting \( C^\alpha_\infty(V) = \bigcap_{\beta > 0} B^{\beta+\alpha}_\infty(V) \) denote the Fréchet space defined by the norms of \( B^{\beta+\alpha}_\infty, \beta > 0 \), we note that since

\[
\|f\|_{B^{\beta+\alpha}_\infty(m, g)} = \sup_x [1, px] \cdot |f(x)| + \sup_{x,y,|y| \leq 1} [1, px] \cdot |f(x) - f(x + y)|
\]

we have

(1.4.16) \( \Sigma^\alpha \longrightarrow C^{-\alpha}_\infty(V) \).

In particular, \( \Sigma^0 \) consists of bounded smooth functions, and \( \Sigma^\alpha \) for \( \alpha < 0 \) consists of smooth functions vanishing at infinity. Letting \( \Sigma^{-\infty} = \bigcap_\alpha \Sigma^\alpha \) denote the Fréchet space defined by the norms of all the \( \Sigma^\alpha \)'s, we have by (1.4.11) and (1.4.16):

(1.4.17) \( \Sigma^{-\infty} = \mathcal{S}(V) \).

The pseudodifferential symbols are defined by

(1.4.18)

\[
\begin{align*}
S^\alpha &= S(m, g), \\
g_{\xi,x}(\eta, y) &= \max\{|1, p\xi|^{-1} \cdot |\eta|, |y|\}, \\
m^\alpha(\xi, x) &= |1, p\xi|^\alpha,
\end{align*}
\]

where \((\xi, x)\) and \((\eta, y)\) refer to the symplectic coordinates associated with the complete polarization \( V = V_0 \oplus V_1 \). Again it is easy to verify that \( g_{\xi,x} \) is a metric-covering, certain with \( h(\xi, x) = |1, p\xi|^{-1} = m^{-1}(\xi, x) \), and that it is temperate with in fact a stronger inequality

(1.4.19)

\[
\begin{align*}
g(\xi_0, x_0)(\eta, y) &\leq g(\xi_1, x_1)(\eta, y) \cdot |1, p(\xi_0 - \xi_1)|, \\
|1, p(\xi_0 - \xi_1)| &\leq g(\xi_1, x_1)(1, \xi_0 - \xi_1, x_0 - x_1).
\end{align*}
\]

Explicitly, \( f \in S^\alpha \) if and only if

(1.4.20)

\[
\begin{align*}
|f(\xi, x)| &\leq C_0 \cdot |1, p\xi|^\alpha \quad \text{for all } (\xi, x) \in V, \\
&\quad \text{and for all } \beta > 0 : |f(\xi, x) - f(\xi + \eta, x + y)| \\
&\quad \leq C_\beta \cdot |1, p\xi|^\alpha \cdot \max\{|1, p\xi|^{-1} \cdot |\eta|, |y|\}^\beta
\end{align*}
\]

for \(|\eta| \leq |1, p\xi| \) and \(|y| \leq 1\).
For the norm associated with (1.4.20) for a fixed \( \beta \), we have

\[
\sup_{x,\xi} \left| 1, p\xi \right|^{-\alpha} \cdot \left| f(\xi, x) \right| \\
+ \sup_{x,\xi,\eta} \left| 1, p\xi \right|^{-\alpha} \max \left\{ \left| 1, p\xi \right|^{-1} |\eta|, |y| \right\}^{-\beta} \cdot \left| f(\xi, x) - f(\xi + \eta, x + y) \right|
\]

\[
\geq \sup_{x,\xi} \left| 1, p\xi \right|^{-\alpha} \left| f(\xi, x) \right|
+ \sup_{x,\xi,\eta} \left| 1, p\xi \right|^{-\alpha} \cdot |\eta|^{-\beta} \cdot \left| f(\xi, x) - f(\xi + \eta, x + y) \right|
\]

\[
= \| I^{\alpha}_{\xi} f \|_{B^{\beta}_{\infty,\infty}},
\]

with \( I^{\alpha}_{\xi} f(\eta, y) = |1, p\eta|^{-\alpha} \cdot f(\eta, y) \). Hence, letting

\[
C_{\infty}^{(-\alpha, 0)}(V) = \bigcap_{\beta > 0} I^{\alpha - \beta}_{\xi} B^{\beta}_{\infty,\infty},
\]

the Fréchet space defined by the norms \( \| I^{\alpha}_{\xi} f \|_{B^{\beta}_{\infty,\infty}}, \beta > 0 \), we have

(1.4.21) \( S^{\alpha} \hookrightarrow C_{\infty}^{(-\alpha, 0)}(V) \).

In particular, \( S^{0} \subseteq C_{\infty}^{0} \) consists of bounded smooth functions, and \( S^{\alpha} \) for \( \alpha < 0 \) consists of smooth functions vanishing at \( \infty \) in the \( \xi \)-direction.

2. THE SYMBOLIC CALCULUS

2.1. Estimates for twisted multiplication.

We fix a temperate, certain, metric-covering \( g_{x} \) in our \( 2d \)-dimensional symplectic vector space \( V \), so \( V = \bigsqcup_{j} \mathfrak{A}_{j}, \mathfrak{A}_{j} = x_{j} + \mathcal{O}_{j}, \mathcal{O}_{j} \supseteq \mathcal{O}'_{j} \). In \( V \oplus V \) we have the metric covering \( g \otimes g \) corresponding to

\[
V \oplus V = \bigsqcup_{j_{1}, j_{2}} \mathfrak{A}_{j_{1}, j_{2}} = \mathfrak{A}_{j_{1}} \times \mathfrak{A}_{j_{2}} = (x_{j_{1}}, x_{j_{2}}) + \mathcal{O}_{j_{1}} \oplus \mathcal{O}_{j_{2}}.
\]

We begin by giving a short range estimate for the restriction to \( \mathfrak{A}_{j} \) of

(2.1.1) \( f_{1} \# f_{2}(x) = \psi(Q(D)) f_{1} \otimes f_{2}(x, x) \)

\[
= \mathcal{F} \otimes \mathcal{F} \psi(Q(v_{1}, v_{2})) \mathcal{F} \otimes \mathcal{F} f_{1} \otimes f_{2}(x, x),
\]
where \( f_1, f_2 \in B_{\infty, \infty}(\mathfrak{A}_j) \), \( x \in \mathfrak{A}_j \), and \( Q \) is as in (0.1.12). We have for \( x_1, x_2 \in \mathfrak{A}_j \)

\[
\begin{align*}
&\left| \psi(Q(D))f_1 \otimes f_2(x_1, x_2) \right| \\
&\quad \leq \left\| \psi(Q(D))f_1 \otimes f_2 \right\|_{B^{d+\alpha}_{2,1}(V \oplus V, g_j \otimes g_j)} \quad \text{by (1.2.23)}, \\
&\quad \leq \left\| f_1 \otimes f_2 \right\|_{B^{d+\alpha}_{2,1}(V \oplus V, g_j \otimes g_j)} \quad \text{(since } \psi(Q(D)) \text{ is unitary on } L_2 \text{ and commutes with convolution)} \\
&\quad = \left\| f_1 \otimes f_2 \right\|_{B^{d+\alpha}_{2,1}(\mathfrak{A}_j, j)} \quad \text{since supp } f_i \subseteq \mathfrak{A}_j \\
&\quad = \left\| f_1 \right\|_{B^{d+\alpha}_{2,1}(\mathfrak{A}_j)} \cdot \left\| f_2 \right\|_{B^{d+\alpha}_{2,1}(\mathfrak{A}_j)}.
\end{align*}
\]

Using (2.1.2) we also obtain

\[
\begin{align*}
&\left\| \varphi_{g_{j,k_1}, \varphi_{g_{j,k_2}}} \otimes \psi(Q(D))f_1 \otimes f_2 \right\|_{L_\infty(\mathfrak{A}_j)} \\
&\quad = \left\| \psi(Q(D))\varphi_{g_{j,k_1}} \otimes \varphi_{g_{j,k_2}} \otimes f_2 \right\|_{L_\infty(\mathfrak{A}_j)} \\
&\quad \leq \left\| \varphi_{g_{j,k_1}} \otimes f_1 \right\|_{B^{d+\alpha}_{2,1}(\mathfrak{A}_j)} \cdot \left\| \varphi_{g_{j,k_2}} \otimes f_2 \right\|_{B^{d+\alpha}_{2,1}(\mathfrak{A}_j)} \cdot \left\| \varphi_{g_{j,k_1}} \otimes f_2 \right\|_{L_2(\mathfrak{A}_j)} \\
&\quad = p^{k_1 \cdot d} \left\| \varphi_{g_{j,k_1}} \otimes f_1 \right\|_{L_2(\mathfrak{A}_j)} \cdot \left\| \varphi_{g_{j,k_2}} \otimes f_2 \right\|_{L_2(\mathfrak{A}_j)}.
\end{align*}
\]

Multiplying (2.1.3) by \( p^{(k_1+k_2)\alpha} \), and taking \( \ell_r \)-norm, we conclude that

\[
\begin{align*}
&\left\| \psi(Q(D))f_1 \otimes f_2 \right\|_{B_{\infty, r}(\mathfrak{A}_j)} \leq \left\| f_1 \right\|_{B^{d+\alpha}_{2, r}(\mathfrak{A}_j)} \cdot \left\| f_2 \right\|_{B^{d+\alpha}_{2, r}(\mathfrak{A}_j)}.
\end{align*}
\]

Using (1.2.21), we obtain from (2.1.4)

\[
\begin{align*}
&\left\| f_1 \# f_2 \right\|_{B_{\infty, r}(\mathfrak{A}_j)} \leq c_\alpha \cdot \left\| f_1 \right\|_{B^{d+\alpha}_{2, r}(\mathfrak{A}_j)} \cdot \left\| f_2 \right\|_{B^{d+\alpha}_{2, r}(\mathfrak{A}_j)}.
\end{align*}
\]

Note that Haar measure is normalized by (1.2.3) as \( dx(\mathfrak{A}_j) = 1 \), so that \( L_{\infty}(\mathfrak{A}_j) \hookrightarrow L_2(\mathfrak{A}_j) \) with norm 1, and hence \( \left\| f \right\|_{B_{2, r}(\mathfrak{A}_j)} \leq \left\| f \right\|_{B_{\infty, r}(\mathfrak{A}_j)} \), so that (2.1.5) gives

\[
\begin{align*}
&\left\| f_1 \# f_2 \right\|_{B_{2, r}(\mathfrak{A}_j)} \\
&\quad \leq c_\alpha \cdot \left\| f_1 \right\|_{B^{d+\alpha}_{2, r}(\mathfrak{A}_j)} \cdot \left\| f_2 \right\|_{B^{d+\alpha}_{2, r}(\mathfrak{A}_j)}.
\end{align*}
\]

To estimate the «error» , that is the difference between \( f_1 \# f_2 \) and \( f_1 \cdot f_2 \) we notice that for \( y \in \mathbb{Q}_p \) we have

\[
\begin{align*}
\psi(y) - 1 \leq \begin{cases} 
2 & \text{if } y \notin \mathbb{Z}_p, \\
0 & \text{if } y \in \mathbb{Z}_p,
\end{cases}
\end{align*}
\]
and hence in particular for any $\gamma \geq 1$

\begin{equation}
|\psi(y) - 1| \leq |y|^\gamma.
\end{equation}

Also, using $g_j^{\gamma} = g_j$, we have

\begin{equation}
\sup_{g_j^{\gamma}(x_1) = p^{k_1}} \frac{|\langle x_1, x_2 \rangle|}{g_j^{\gamma}(x_1) \cdot g_j^{\gamma}(x_2)} = p^{(k_1 + k_2)} \sup_{x_1, x_2} \frac{|\langle x_1, x_2 \rangle|}{g_j(x_1) \cdot g_j(x_2)} = p^{(k_1 + k_2)} \cdot h_j
\end{equation}

where $h_j$ is defined as in (1.4.9) with the metric $g_j$.

\begin{equation}
\text{(2.1.10)} \quad \text{Applying these remarks we can now estimate}
\end{equation}

\begin{equation}
\left| \psi(Q(D)) f_1 \otimes f_2(x_1, x_2) - f_1 \otimes f_2(x_1, x_2) \right|
\end{equation}

\begin{equation}
\leq \left\| (\psi(Q(D)) - \delta) f_1 \otimes f_2 \right\|_{B^{d+d}(V \oplus V, g_j \otimes g_j)} \quad \text{by (1.2.23)}
\end{equation}

\begin{equation}
= \sum_{k_1, k_2 \geq 0} p^{(k_1 + k_2)d} \left\| \tilde{\varphi}_{g_j, k_1} \otimes \tilde{\varphi}_{g_j, k_2} \cdot (\psi(\frac{1}{2} (x_1, x_2)) - 1) \right\|_{L^2(V \oplus V)}
\end{equation}

(by the definitions (1.2.4), (0.1.12) and Plancherel)

\begin{equation}
\leq h_j^\gamma \sum_{k_1, k_2 \geq 0} p^{(k_1 + k_2)(d+\gamma)} \left\| \tilde{\varphi}_{g_j, k_1} \otimes \tilde{\varphi}_{g_j, k_2} \cdot F f_1 \otimes F f_2 \right\|_{L^2(V \oplus V)}
\end{equation}

(by using (2.1.8) and (2.1.9), with any $\gamma \geq 1$),

\begin{equation}
= h_j^\gamma \cdot ||f_1||_{B_{d+\gamma}(\mathfrak{a_j})} \cdot ||f_2||_{B_{d+\gamma}(\mathfrak{a_j})}
\end{equation}

(by Plancherel again, and the definition (1.2.4)).

Using (2.1.10), we can now deduce as in (2.1.3), (2.1.4), (2.1.5) and (2.1.6), that we have

\begin{equation}
\text{(2.1.11)}
\end{equation}

\begin{equation}
\begin{cases}
\|f_1 \# f_2 - f_1 \cdot f_2\|_{B_{2,r}^d(\mathfrak{a_j})} \\
\leq \|f_1 \# f_2 - f_1 \cdot f_2\|_{B_{2,r}(\mathfrak{a_j})} \\
\leq c_{\alpha} \cdot h_j^\gamma \cdot \|f_1\|_{B_{d+\gamma+\alpha}^d(\mathfrak{a_j})} \cdot \|f_2\|_{B_{d+\gamma+\alpha}^d(\mathfrak{a_j})} \\
\leq c_{\alpha} \cdot h_j^\gamma \cdot \|f_1\|_{B_{d+\gamma+\alpha}(\mathfrak{a_j})} \cdot \|f_2\|_{B_{d+\gamma+\alpha}(\mathfrak{a_j})}.
\end{cases}
\end{equation}
We next give the long range estimate for the restriction to $\mathfrak{A}_j$ of $f_1 \neq f_2$, where $\text{supp } f_i \subset \mathfrak{A}_{j_i}$, and at least one of the $j_i$'s is different from $j$. We recall from (0.1.11) that we can write $\psi(Q(D)) f_1 \otimes f_2$ also as a convolution

$$\psi(Q(D)) f_1 \otimes f_2(x_1, x_2) = \iint_{\mathfrak{A}_{j_1} \times \mathfrak{A}_{j_2}} dx' dx'' f_1(x') f_2(x'') \bar{\psi}(2(x' - x_1, x'' - x_2)).$$

We set

$$\tilde{g}_{j_1, j_2}^\gamma(1, x) = \min_{y_2 \in \mathcal{O}_{j_2}} g_{j_1}^\gamma(1, x + y_2)$$

so that we have

$$\tilde{g}_{j_1, j_2}^\gamma \text{ is } \mathcal{O}_{j_1} + \mathcal{O}_{j_2}-\text{locally constant, and}$$

$$1 \leq \tilde{g}_{j_1, j_2}^\gamma(1, x) \leq g_{j_1}^\gamma(1, x).$$

Hence if we put

$$\tilde{J}_{j_1, j_2}^{-\beta} = \mathcal{F} \tilde{g}_{j_1, j_2}^\gamma(1, x)^\beta \mathcal{F}$$

then $\tilde{J}_{j_1, j_2}^{-\beta}$ acting on $\mathcal{S}'(V)$, preserves by (2.1.14) $C^\infty_{\text{loc}}(\mathfrak{A}_{j_1})$, the distributions supported in $\mathfrak{A}_{j_1}$; moreover, it commutes with convolution, and by (2.1.15), (1.2.12), and Plancherel, we have for $\beta > 0$

$$\| \tilde{J}_{j_1, j_2}^{-\beta} f_1 \|_{B_{2,r}^\infty(\mathfrak{A}_{j_1})} \leq \| J_{g_{j_1}}^{-\beta} f_1 \|_{B_{2,r}^\infty(\mathfrak{A}_{j_1})} = \| f_1 \|_{B_{2,r}^{\infty+\beta}(\mathfrak{A}_{j_1})}.$$

Similarly we can define $\tilde{g}_{j_1, j_2}^\gamma$ and $\tilde{J}_{j_1, j_2}^{-\beta}$, and they satisfy the same properties.

Notice that we have in the distributional sense,

$$\mathcal{F}_{x'} \bar{\psi}(2(x' - x_1, x'' - x_2)) = \psi(2(x_1, x'' - x_2)) \cdot \delta_2(x'' - x_2)$$

and hence, letting $\tilde{J}_{j_1, j_2}^{-\beta}$ act through the variable $x'$, we have

$$\tilde{J}_{j_1, j_2}^{-\beta} \bar{\psi}(2(x' - x_1, x'' - x_2)) = \tilde{g}_{j_1, j_2}^\gamma(1, x'' - x_2)^\beta \bar{\psi}(2(x' - x_1, x'' - x_2)).$$
Putting,

\[ g^{\gamma}_{j_1}(x_2, \mathfrak{A}_{j_2}) = \min_{x'' \in \mathfrak{A}_{j_2}} g^{\gamma}_{j_1}(1, x'' - x_2) \]
\[ = \tilde{g}^{\gamma}_{j_1, j_2}(1, x'' - x_2), \quad \text{for any } x'' \in \mathfrak{A}_{j_2}, \]

we obtain from (2.1.12) and (2.1.19),

\[ \psi(Q(D)) f_1 \otimes f_2(x_1, x_2) \]
\[ = g^{\gamma}_{j_1}(x_2, \mathfrak{A}_{j_2})^{-\beta} \cdot \psi(Q(D)) [\tilde{J}_{j_1, j_2}^{-\beta} f_1 \otimes f_2](x_1, x_2). \]

Similarly, using \( \tilde{g}^{\gamma}_{j_2, j_1} \) and \( \tilde{J}_{j_2, j_1}^{-\beta} \), and the analogous formula to (2.1.19), we have

\[ \psi(Q(D)) f_1 \otimes f_2(x_1, x_2) \]
\[ = g^{\gamma}_{j_1}(x_2, \mathfrak{A}_{j_2})^{-\beta} g^{\gamma}_{j_2}(x_1, \mathfrak{A}_{j_1})^{-\beta} \cdot \psi(Q(D)) [\tilde{J}_{j_1, j_2}^{-\beta} f_1 \otimes \tilde{J}_{j_2, j_1}^{-\beta} f_2](x_1, x_2). \]

Now we can estimate, as in the short range-estimate (2.1.2),

\[
\begin{aligned}
\left| \psi(Q(D)) f_1 \otimes f_2(x_1, x_2) \right| \\
= g^{\gamma}_{j_1}(x_2, \mathfrak{A}_{j_2})^{-\beta} \cdot g^{\gamma}_{j_2}(x_1, \mathfrak{A}_{j_1})^{-\beta} \\
&\cdot \left| \psi(Q(D)) \tilde{J}_{j_1, j_2}^{-\beta} f_1 \otimes \tilde{J}_{j_2, j_1}^{-\beta} f_2(x_1, x_2) \right| \\
\leq g^{\gamma}_{j_1}(x_2, \mathfrak{A}_{j_2})^{-\beta} \cdot g^{\gamma}_{j_2}(x_1, \mathfrak{A}_{j_1})^{-\beta} \\
&\cdot \| \psi(Q(D)) \tilde{J}_{j_1, j_2}^{-\beta} f_1 \otimes \tilde{J}_{j_2, j_1}^{-\beta} f_2 \|_{B_{2,1}^{d}(V \oplus V, g_{j_1} \otimes g_{j_2})} \\
\leq g^{\gamma}_{j_1}(x_2, \mathfrak{A}_{j_2})^{-\beta} \cdot g^{\gamma}_{j_2}(x_1, \mathfrak{A}_{j_1})^{-\beta} \\
&\cdot \| f_1 \|_{B_{2,1}^{d+\beta}(\mathfrak{A}_{j_1})} \cdot \| f_2 \|_{B_{2,1}^{d+\beta}(\mathfrak{A}_{j_2})} \quad \text{by (2.1.17)}. \\
\end{aligned}
\]

Using the fact that by our normalization we have \( L_{\infty}(\mathfrak{A}_j, g_j) \rightarrow L_2(\mathfrak{A}_j, g_j) \) with norm 1, and the embedding (1.2.6), we obtain from (2.1.23) that for any \( \beta' > \beta + d \)

\[ \left| \psi(Q(D)) f_1 \otimes f_2(x_1, x_2) \right| \leq C \cdot g^{\gamma}_{j_1}(x_2, \mathfrak{A}_{j_2})^{-\beta} g^{\gamma}_{j_2}(x_1, \mathfrak{A}_{j_1})^{-\beta} \]
\[ \| f_1 \|_{B_{\infty,\infty}^{\alpha'}(\mathfrak{A}_{j_1})} \cdot \| f_2 \|_{B_{\infty,\infty}^{\alpha'}(\mathfrak{A}_{j_2})}, \]
This, of course gives an estimate for \( \|f_1 \# f_2\|_{L^\infty(\mathfrak{A}_j)} \), but to estimate the Besov norms of \( f_1 \# f_2 \) we need a further argument since we have different metrics \( g_j \) and \( g_{j_1} \). We remark that since we have, by (1.4.2), for any \( x_i \in \mathfrak{A}_j \)

\[
g_{j_1}(y) \leq g_j(y) \cdot p^{n_0} \cdot g_{j_1}(x_i, \mathfrak{A}_{j_1})^N
\]

an easy estimate gives

\[
sup_{g_j(y) \leq 1} g_j(y)^{-\alpha} \|g(\delta_0 - \delta_y) \ast f_1\|_{B_{\infty, \infty}^{\alpha}(V, g_{j_1})} \leq 6 \cdot p^{n_0} g_{j_1}(x_i, \mathfrak{A}_{j_1})^N \cdot \|f_1\|_{B_{\infty, \infty}^{\alpha + \beta}(V, g_{j_1})}.
\]

Hence from (2.1.24) and (2.1.26) we deduce, since \( \psi(Q(D)) \) commutes with translations, that for \( x \in \mathfrak{A}_j \), \( \beta' > \beta + d \), we have

\[
sup_{g_j(y) \leq 1} g_j(y)^{-\alpha} \left| f_1 \# f_2(x) - f_1 \# f_2(x + y) \right| \leq \sup_{g_j(y) \leq 1} g_j(y)^{-\alpha} \left| \psi(Q(D)) f_1 \otimes (\delta_0 - \delta_y) \ast f_2(x, x) \right|
\]

\[
+ \sup_{g_j(y) \leq 1} g_j(y)^{-\alpha} \left| \psi(Q(D)) (\delta_0 - \delta_y) \ast f_1 \otimes f_2(x + y, x + y) \right|
\]

\[
\leq C \cdot g_{j_1}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}(x, \mathfrak{A}_{j_1})^{-\beta} \cdot \|f_1\|_{B_{\infty, \infty}^{\alpha'}(\mathfrak{A}_{j_1})} \cdot \sup_{g_j(y) \leq 1} g_j(y)^{-\alpha} \left| (\delta_0 - \delta_y) \ast f_2 \right|_{B_{\infty, \infty}^{\alpha'}(V, g_{j_2})}
\]

\[
\leq C' \cdot g_{j_1}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}(x, \mathfrak{A}_{j_1})^{-\beta} \cdot \|f_1\|_{B_{\infty, \infty}^{\alpha'}(\mathfrak{A}_{j_1})} \cdot \|f_2\|_{B_{\infty, \infty}^{\alpha'}(\mathfrak{A}_{j_2})}
\]

\[
\leq C'' \cdot g_{j_1}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}(x, \mathfrak{A}_{j_1})^{-\beta} \cdot \|f_1\|_{B_{\infty, \infty}^{\alpha'}(\mathfrak{A}_{j_1})} \cdot \|f_2\|_{B_{\infty, \infty}^{\alpha'}(\mathfrak{A}_{j_2})}
\]

\[
\leq C''' \cdot g_{j_1}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}(x, \mathfrak{A}_{j_1})^{-\beta} g_{j_1}(x, \mathfrak{A}_{j_1})^{-\beta} \cdot \|f_1\|_{B_{\infty, \infty}^{\alpha'}(\mathfrak{A}_{j_1})} \cdot \|f_2\|_{B_{\infty, \infty}^{\alpha'}(\mathfrak{A}_{j_2})}
\]

where the constant \( C''' \) depends only on \( \alpha, (\beta - \beta'), n_0 \) and \( d \).
2.2. The main theorem of symbolic calculus.

We now give a global estimate for \( f_1 \neq f_2 \), where \( f_i \in S(m_i, g) \) and the weights \( m_1, m_2 \) are temperate, i.e. they satisfy (1.4.8). We let \( f_{i,(j)} \) denote the restriction of \( f_i \) to \( A_j \), and we note that the partial sums \( \sum_{j \leq M} f_{i,(j)} \) are bounded in \( S(m_i, g) \), and they converge to \( f_i \) in \( C_\infty(V) \), since they are ultimately equal to \( f_i \) on any compact set. Letting \( \beta_0 \) denote the constant in (1.4.10), we deduce from (2.1.24) that for any \( \beta' > \beta + d + \beta_0 \),

\[
|f_{1,(j_1)} \# f_{2,(j_2)}(x)| 
\leq C \cdot g_{j_1}^\gamma(x, A_{j_2})^{-\beta} g_{j_2}^\gamma(x, A_{j_1})^{-\beta} \cdot m_1(x)m_2(x) 
\cdot \|m_1^{-1} f_{1,(j_1)}\|_{B_\infty,\infty'(A_{j_1})} \cdot \|m_2^{-1} f_{2,(j_2)}\|_{B_\infty,\infty'(A_{j_2})}.
\]

Taking \( \beta \) sufficiently large so that (1.4.7) applies, we obtain

\[
\sum_{j_1, j_2} |f_{1,(j_1)} \# f_{2,(j_2)}(x)| 
\leq C' \cdot m_1 \cdot m_2(x) \cdot \|f_1\|_{B_\infty,\infty'(m_1, g)} \cdot \|f_2\|_{B_\infty,\infty'(m_2, g)}.
\]

From the estimate (2.2.2), being valid for all \( f_i \) in a bounded subset of \( S(m_i, g) \) we deduce that the form \( f_1 \otimes f_2 \mapsto f_1 \neq f_2 \) defined on \( C_\infty,c(V) \otimes C_\infty,c(V) = C_\infty,c(V \oplus V) \) extends uniquely to a weakly continuous linear form on \( S(m_1, g) \otimes S(m_2, g) \), given by

\[
f_1 \neq f_2(x) = \sum_{j_1, j_2} f_{1,(j_1)} \neq f_{2,(j_2)}(x).
\]

Similarly, note that the functions

\[
\tilde{m}_i(y) = m_i(y) \cdot g_y^\gamma(x, A_y)^{-N \alpha}
\]

are again temperate, with the corresponding constants in (1.4.8) depending only on \( m_i, g \) and \( \alpha \) (and independent of \( x \)); letting \( \beta_\alpha \) denote the associated constant in (1.4.10) (with \( m_i \) replaced by \( \tilde{m}_i \)), and using (1.4.5), we deduce from (2.1.27) that for any \( \beta' > \beta + d + \beta_\alpha + (N + 1)^2 \cdot N \alpha \), we have

\[
\|f_{1,(j_1)} \neq f_{2,(j_2)}(x)\|_{B_\infty,\infty'(A_x)} 
\leq C'' \cdot g_{j_1}^\gamma(x, A_{j_2})^{-\beta} g_{j_2}^\gamma(x, A_{j_1})^{-\beta} m_1m_2(x) 
\cdot \|m_1^{-1} f_{1,(j_1)}\|_{B_\infty,\infty'(A_{j_1})} \cdot \|m_2^{-1} f_{2,(j_2)}\|_{B_\infty,\infty'(A_{j_2})}.
\]
Taking again a $\beta$ satisfying (1.4.7), we conclude that with some new constant $C$

\[(2.2.5) \sum_{j_1,j_2} \| f_{1,(j_1)} \# f_{2,(j_2)} \|_{B_{\infty,\infty}^0(a_x)} \leq C \cdot m_1 m_2(x) \cdot \| f_1 \|_{B_{\infty,\infty}^{\beta',\gamma}(m_1,g)} \cdot \| f_2 \|_{B_{\infty,\infty}^{\beta',\gamma}(m_2,g)}.
\]

Thus the map $f_1 \otimes f_2 \mapsto f_1 \# f_2$ is weakly continuous when viewed as a map

\[S(m_1,g) \otimes S(m_2,g) \to S(m_1 \cdot m_2,g).
\]

Moreover, since the function $x \mapsto h(x)$ of (1.4.9) is temperate, and since for $x \notin \mathfrak{A}_{j_i}$

\[(2.2.6) \quad 1 \leq \min_{y \in \mathfrak{A}_{j_i}} g_{j_i}(x - y) \leq h_{j_i} \cdot g_{j_i}^\gamma(x, \mathfrak{A}_{j_i}),
\]

we see that, using (1.4.5), we can bound for $x \notin \mathfrak{A}_{j_1} \cap \mathfrak{A}_{j_2}$,

\[(2.2.7) \quad g_{j_1}^\gamma(x, \mathfrak{A}_{j_2})^{-\beta_1 \cdot \gamma} g_{j_2}^\gamma(x, \mathfrak{A}_{j_1})^{-\beta_1 \cdot \gamma} \leq h(x)^\gamma
\]

for some $\beta_1 > 0$. Hence, by taking $\beta$ large, we can replace $m_1 \cdot m_2(x)$ in the right hand side of (2.2.4) by $h^\gamma \cdot m_1 \cdot m_2(x)$, for $x \notin \mathfrak{A}_{j_1} \cap \mathfrak{A}_{j_2}$, and a posteriori we can similarly replace $m_1 \cdot m_2$ by $h^\gamma \cdot m_1 \cdot m_2$ in (2.2.5) if we sum only over $j_1 \neq j_2$. When $j_1 = j_2 = j$, we can use the estimate (2.1.11). Thus we conclude that the map $f_1 \otimes f_2 \mapsto f_1 \# f_2 - f_1 \cdot f_2$ is weakly continuous when viewed as a map $S(m_1,g) \otimes S(m_2,g) \to S(h^\gamma \cdot m_1 \cdot m_2,g)$, for any $\gamma$. We summarize the above discussion in the following

**Main theorem (2.2.8).** — Given a temperate, certain, metric-covering $g$ in the symplectic space $V$, and two temperate weights $m_1, m_2$, then

\[\# : S(m_1,g) \otimes S(m_2,g) \to S(m_1 m_2,g)
\]

is weakly continuous. If $h(x) = \sup_y g_x(y)/g_x^\gamma(y)$, then for any $\gamma$ the map $f_1 \otimes f_2 \mapsto f_1 \# f_2 - f_1 \cdot f_2$ is weakly continuous

\[S(m_1,g) \otimes S(m_2,g) \to S(h^\gamma \cdot m_1 \cdot m_2,g).
\]
2.3. Bounds for operators.

In this section we study the continuity of the operators \( \hat{\rho}(f) \) on the spaces \( \mathcal{S}(V_0) \hookrightarrow L_2(V_0) \hookrightarrow \mathcal{S}'(V_0) \), for \( f \in S(m, g) \), \( g, m \) temperate. We first remark that since by (0.1.13) \( \hat{\rho} \) takes complex conjugates to adjoints, the continuity in \( \mathcal{S}'(V_0) \) follows from the continuity in \( \mathcal{S}(V_0) \). Secondly, we note that the Bargman isomorphism \( B^{-1} : L_2(V_0) \rightarrow L_2(V_0) \psi \) induces an isomorphism \( B^{-1} : \mathcal{S}(V_0) \rightarrow \mathcal{S}(V_0) \psi = \mathcal{S}(V) \cap L_2(V) \psi \) onto the subspace of \( \mathcal{S}(V) \) consisting of \( \psi \)-holomorphic functions. By the description of \( \mathcal{S}(V) \) given in (1.4.11), and the description of \( B^{-1} \hat{\rho}(f)B \) given in (0.2.12), we deduce from our main theorem (2.2.8) the continuity in \( \mathcal{S}(V_0) \). Thus we have:

**Theorem (2.3.1).** — For a temperate certain-metric-covering \( g \) and a temperate weight \( m \), the operators \( \hat{\rho}(f) \) are continuous in \( \mathcal{S}(V_0) \), and in \( \mathcal{S}'(V_0) \), for every \( f \in S(m, g) \). Moreover, the map \( f \mapsto \hat{\rho}(f) \) induces weakly continuous embeddings

\[
\begin{align*}
S(m, g) \hookrightarrow & \text{Hom}(\mathcal{S}(V_0), \mathcal{S}(V_0)) = \mathcal{S}(V_0) \otimes \mathcal{S}'(V_0), \\
S(m, g) \hookrightarrow & \text{Hom}(\mathcal{S}'(V_0), \mathcal{S}'(V_0)) = \mathcal{S}'(V_0) \otimes \mathcal{S}(V_0).
\end{align*}
\]

(2.3.2)

For the continuity on \( L_2(V_0) \) we take \( f \in S(1, g) \) and write \( f = \sum_i f_i \) for the corresponding decomposition of \( f \) with respect to the metric covering \( g \). We shall use the lemma of Cotlar-Knapp-Stein-Calderón-Vaillancourt, so that we need to estimate the operator norms

\[
\|\hat{\rho}(f_i) \hat{\rho}(f_j)\|_{op} = \|\hat{\rho}(\hat{f}_i \# f_j)\|_{op} = \|\rho(\mathcal{F}(\hat{f}_i \# f_j))\|_{op}
\]

\[
\leq \|\mathcal{F}(\hat{f}_i \# f_j)\|_{L_1(V)} \leq C_\alpha \|\hat{f}_i \# f_j\|_{B_{\infty, \infty}^\alpha(1, g)}
\]

(2.3.3)

for \( \alpha \) sufficiently large. From (2.2.4) we obtain for \( \beta' \gg \beta, \alpha \)

\[
\|\hat{f}_i \# f_j\|_{B_{\infty, \infty}^\beta(\mathfrak{A}_i, g_x)} \leq C \cdot g_i^\gamma(x, \mathfrak{A}_j)^{-\beta} g_j^\gamma(x, \mathfrak{A}_i)^{-\beta} \cdot \|f\|_{B_{\infty, \infty}^{\beta'}(1, g)}^2.
\]

(2.3.4)

By using repeatedly (1.4.1), and the inequality of (1.1.26), one easily obtains that for some constants \( C_0, M \), and all \( x \in V \)

\[
g_i^\gamma(\mathfrak{A}_i, \mathfrak{A}_j) \overset{\text{def}}{=} \min_{y' \in \mathfrak{A}_i, y'' \in \mathfrak{A}_j} g_i^\gamma(1, y' - y'') \leq C_0 \cdot g_i^\gamma(x, \mathfrak{A}_j)^M \cdot g_j^\gamma(x, \mathfrak{A}_i)^M.
\]

(2.3.5)
Putting (2.3.3), (2.3.4), (2.3.5) together we obtain, with some new constant $C$, for $\gamma \gg \beta \gg 0$,

\begin{equation}
\|\hat{\rho}(f)\|_{\text{op}} \leq C \cdot g^{\gamma}(\mathfrak{A}_i, \mathfrak{A}_j)^{-\beta} \cdot \|f\|_{B_{2,\infty}^\gamma(1,g)}.
\end{equation}

By an analogous proof to the proof of Lemma (1.4.6) we have, for $\beta \gg 0$

\begin{equation}
\sum_i g^{\gamma}(\mathfrak{A}_i, \mathfrak{A}_j)^{-\beta} < \infty, \quad \sum_j g^{\gamma}(\mathfrak{A}_i, \mathfrak{A}_j)^{-\beta} < \infty.
\end{equation}

From (2.3.6) and (2.3.7), we obtain by applying the lemma of Cotlar et al.,

\begin{equation}
\|\hat{\rho}(f)\|_{\text{op}} \leq C \cdot \|f\|_{B_{2,\infty}^\gamma(1,g)}, \quad \text{for } \gamma \text{ sufficiently large.}
\end{equation}

**Theorem (2.3.9).** — For a temperate certain metric-covering $g$, and a temperate bounded weight $m$, the operators $\hat{\rho}(f)$ are bounded on $L^2(V_0)$ for every $f \in S(m,g)$. Moreover, for some $C, \gamma$ we have

\begin{equation}
\|\hat{\rho}(f)\|_{\text{op}} \leq C \cdot \|f\|_{B_{2,\infty}^\gamma(m,g)}.
\end{equation}

If $m$ vanishes at infinity, the operators $\hat{\rho}(f)$ with $f \in S(m,g)$ are all compact.

**Proof.** — The inequality in (2.3.10) follows immediately from (2.3.8), since if $m \leq C$, then $\|f\|_{B_{2,\infty}^\gamma(1,g)} \leq C \cdot \|f\|_{B_{2,\infty}^\gamma(m,g)}$. The compactness assertion follows since the operators $\hat{\rho}(f_i)$ are all compact, and by (2.3.8)

\begin{equation}
\|\hat{\rho}(f) - \sum_{i \leq j} \hat{\rho}(f_i)\|_{\text{op}} \leq C \cdot \|\sum_{i > j} f_i\|_{B_{2,\infty}^\gamma(1,g)} \leq C \cdot \|f\|_{B_{2,\infty}^\gamma(m,g)} \cdot \sup_{i > j} m_i
\end{equation}

which converges to 0 as $j \to \infty$ by assumption. \hfill \Box

**Remark.** — We note that if $f_j$ is a bounded sequence in $S(\mathfrak{I}, g)$ converging to $f$ in $C_\infty(V)$ then for any $\varphi \in L^2(V_0)$, $\hat{\rho}(f_j) \varphi \to \hat{\rho}(f) \varphi$ in $L^2(V_0)$. Indeed, we may assume $f = 0$, and in view of the uniform bound (2.3.8) we can take $\varphi \in S(V_0)$, then by the weak-continuity assertion of (2.3.1), $\hat{\rho}(f_j) \varphi$ is bounded in $S(V_0)$, and converges to zero in $S'(V_0)$, hence a posteriori in $S(V_0)$ and in $L^2(V_0)$.

The inequality (2.3.8) can be sharpened in various ways, one of which we describe next. For a self-dual lattice $\mathcal{O} = \mathcal{O}^\vee$, denoting by $\Phi$
its characteristic function, we know from (0.4.15), (0.4.17) that there is
a unit vector \( \varphi_0 \in L_2(V_0) \) such that \( W \varphi_0 = \Phi = \Phi_0 \). If \( f \) is such that
supp \( \hat{f} \subseteq \Omega \), then by (0.4.19), we have \( \hat{\rho}(f) = \hat{\rho}(f \ast \Phi) = C_{\varphi_0} M_f \varphi_0 \), and
hence \( \| \hat{\rho}(f) \|_{\text{op}} \leq \| f \|_{L_\infty} \). This also holds for \( f \) such that supp \( \hat{f} \subseteq x + \Omega \),
since the function \( t_x \hat{f}(y) = \hat{f}(y + x) = \mathcal{F}(\psi_x \cdot f)(y) \) is supported in \( \Omega \),
so
\[
\| \hat{\rho}(f) \|_{\text{op}} = \| \rho(-\frac{1}{2}x) \hat{\rho}(f) \rho(-\frac{1}{2}x) \|_{\text{op}} = \| \hat{\rho}(\psi_x \cdot f) \|_{\text{op}} \leq \| \psi_x \cdot f \|_{L_\infty} = \| f \|_{L_\infty}.
\]

Thus for a self-dual metric covering \( g_x = g'_x \), corresponding to
\( V = \bigsqcup_j \mathfrak{A}_j, \mathfrak{A}_j = x_j + \Omega_j, \Omega_j = \Omega'_j \), letting \( \Phi_j \) denote the characteristic
function of \( \mathfrak{A}_j \), we have for any function \( f \)
\[
\| \hat{\rho}(f) \|_{\text{op}} = \| \hat{\rho} \left( \sum_j \Phi_j \ast f \right) \|_{\text{op}} \leq \sum_j \| \hat{\rho}(\Phi_j \ast f) \|_{\text{op}} \leq \sum_j \| \Phi_j \ast f \|_{L_\infty} \overset{\text{def}}{=} \| f \|_{B_{\infty,1}^\frac{\alpha}{\alpha-1}(g)}.
\]

Here the right hand defines a norm for a Banach space \( B_{\infty,1}^\alpha(g) \)
analogous to a Besov space. If for example \( g_x = g \) is independent of \( x \),
i.e., \( g = g' \) is a self-dual metric corresponding to a lattice \( \Omega \subseteq V \) and
\( V = \bigsqcup_{x \in V / \Omega} (x + \Omega) \) is the associated covering, then an easy calculation gives
\[
\| \Phi_x \ast f \|_{L_\infty} \leq g(1,x)^{-\alpha} \cdot \| J_g^{-\alpha} f \|_{L_\infty}
\]
and since \( \sum_{x \in V / \Omega} g(1,x)^{-\alpha} = \zeta(\alpha - 2d)/\zeta(\alpha) \) is finite for \( \alpha > 2d = \dim V \),
we obtain
\[
(2.3.12) \quad \| \hat{\rho}(f) \|_{\text{op}} \leq \| f \|_{B_{\infty,1}^\alpha(g)} \leq \frac{\zeta(\alpha - 2d)}{\zeta(\alpha)} \cdot \| f \|_{L_\infty^\alpha(V,g)}.
\]
From this we deduce the following

**Theorem (2.3.13).** — For a self-dual metric \( g \), the operators \( \hat{\rho}(f) \)
are bounded on \( L_2(V_0) \) for every \( f \in L_\infty^\alpha(V,g), \alpha > \dim V \). If such an \( f \)
vanishes at infinity, then \( \hat{\rho}(f) \) is compact.

For a \( g \)-locally constant function \( m \), we define the operator \( J_g^m = \hat{\rho}(m^{-1}) \); if \( m \geq 1 \), it is bounded on \( L_2(V_0) \) by (2.3.8). We define the
\( m \)-th Soboleff space by
\[
(2.3.14) \quad L_2^{(m)}(V_0) = J_g^m \left( L_2(V_0) \right), \quad \| \varphi \|_{L_2^{(m)}(V_0)} = \| \hat{\rho}(m) \varphi \|_{L_2(V_0)}.
\]
Since for $g$-locally constant functions $m_1, m_2$ we have $m_1 \# m_2 = m_1 \cdot m_2$, we deduce that $J_g^{m_1} J_g^{m_2} = J_g^{m_1 \cdot m_2}$, and in particular, we see that $J_g^{m} = J_g^{m^{1/2}} \cdot (J_g^{m^{1/2}})^* \text{ is positive, and } J_g^m J_g^{m^{-1}} = \text{id}$, so that $J_g^m$ is invertible. For example, taking $g$ and $m$ as in (1.4.12) (or as in (1.4.18)) we get essentially the operator $K^\alpha$ of (1.1.20) (respectively, the usual $J^\alpha$), and the space $L_2^{(\alpha)}(V_0)$ of (1.1.21) (respectively, $L_2^\alpha(V_0)$). For a temperate weight $m_0$ if $f \in S(m_0, g)$, we have $f \# m_0^{-1} \in S(1, g)$, hence by (2.3.8) $\hat{\rho}(f \# m_0^{-1}) = \hat{\rho}(f) J_g^{m_0}$ is bounded.

Similarly replacing $f$ by $m_1^{-1} \# f \# m_1 \in S(m_0, g)$, we obtain the

**Corollary (2.3.15).** — For temperate weights $m_0, m_1$, every $f$ in $S(m_0, g)$ defines a bounded operator $\hat{\rho}(f) : L_2^{(m_0 m_1)}(V_0) \to L_2^{(m_1)}(V_0)$.

### 3. Elliptic Operators

#### 3.1. The Gårding inequalities.

Fix a temperate certain metric covering $g$ in $V$ corresponding to $V = \coprod_j \mathfrak{A}_j$, $\mathfrak{A}_j = x_j + \mathcal{O}_j$, $\mathcal{O}_j \supset \mathcal{O}_j'$, and write $h_j = p^{-a_j}$ for the quantity defined in (1.4.9) for the metric $g_j$. For each $j$ choose a self-dual lattice $\mathcal{O}_j^0$ such that $\mathcal{O}_j \supset \mathcal{O}_j^0 \supset \mathcal{O}_j'$, and moreover such that we have

$$\tag{3.1.1} p^b_j \mathcal{O}_j \supset \mathcal{O}_j^0,$$

with $b_j = [\frac{1}{2} a_j] = (\text{the greatest integer} < \frac{1}{2} a_j)$. To see that such a choice for $\mathcal{O}_j^0$ is possible, write in some symplectic basis

$$\mathcal{O}_j = p^{-e_1}\mathbb{Z}_p \oplus \cdots \oplus p^{-e_d}\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p, \quad 0 \leq e_1 \leq \cdots \leq e_d, $$

$$\mathcal{O}_j' = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p \oplus p^{e_1}\mathbb{Z}_p \oplus \cdots \oplus p^{e_d}\mathbb{Z}_p, \quad \text{so } e_1 = a_j, \text{ and take}$$

$$\mathcal{O}_j^0 = p^{-c_1}\mathbb{Z}_p \oplus \cdots \oplus p^{-c_d}\mathbb{Z}_p \oplus p^{c_1} \oplus \cdots \oplus p^{c_d}\mathbb{Z}_p, \quad \text{with } c_i = [\frac{1}{2} e_i].$$

We denote by $\Phi_j$ the characteristic function of $\mathcal{O}_j^0$, which is by (0.4.15), (0.4.17) the Wigner transform of a unit vector $\phi_j \in L_2(V_0)$, $\Phi_j = W\phi_j$. For a function $f$ in $V$, we denote, as usual, by $f_j$ the restriction of $f$ to $\mathfrak{A}_j$, and we let $\tilde{f}(x) = \Phi_j * f_j(x)$ for $x \in \mathfrak{A}_j$, $\tilde{f}$ is the average of $f$ over the $x + \mathcal{O}_j^0$'s.

We have the following:
LEMMA (3.1.2). — For \( f \in S(m,g) \), we have \( f - \tilde{f} \in S(mh^\gamma,g) \), for any \( \gamma \).

Proof. — It is enough to prove that for any \( \alpha, \gamma > 0 \), we have for all \( j \)

\[
(3.1.3) \quad h_j^{-\gamma}\|f_j - \tilde{f}_j\|_{B_{\infty,\infty}(\mathbb{A}_j)} \leq 2 \cdot \|f_j\|_{B_{\infty,\infty}^{\alpha+2\gamma}(\mathbb{A}_j)}.
\]

With the notations of (1.2.3), using (3.1.1), we have

\[
(3.1.4) \quad \phi_{g_j,k} * f = \phi_{g_j,k} * \tilde{f}, \quad \text{and hence} \quad \varphi_{g_j,k} * f = \varphi_{g_j,k} * \tilde{f},
\]

for \( k = 0, 1, \ldots, b_j \). From (3.1.4) we obtain

\[
(3.1.5) \quad \hat{p}(\hat{f}) = \sum_j \hat{p}(f_j * \Phi_j) = \sum_j \hat{p}(f_j * W \phi_j) = \sum_j C_{\phi_j}^*, M_{f_j} C_{\phi_j}.
\]

In particular, if \( f \geq 0 \) is positive, \( \hat{p}(\hat{f}) \) is positive in the sense that

\[
(3.1.6) \quad (\hat{p}(\hat{f}) \varphi, \varphi)_{L_2(V_0)} \geq 0 \quad \text{for all} \quad \varphi \in S(V_0).
\]

Thus we have obtained the following

COROLLARY (3.1.7). — If \( f \in S(m,g) \) is positive, \( f \geq 0 \), then \( f = \tilde{f} + f_0 \), where the operator \( \hat{p}(\hat{f}) \), is positive, and \( f_0 \in S(mh^\gamma,g) \) for all \( \gamma \).
COROLLARY (3.1.8). — If \( f \in S(m, g) \) is positive, then for any \( \gamma \)
\[
(\hat{\rho}(f)\varphi, \varphi)_{L^2(V_0)} \geq -C \cdot \|\varphi\|^2_{L^2(m^{1/2}h^{-\gamma})(V_0)}
\]
for all \( \varphi \in S(V_0) \).

Proof. — Writing \( f = \tilde{f} + f_0 \) as in (3.1.7), we have
\[
(\hat{\rho}(f)\varphi, \varphi)_{L^2(V_0)} \\
\geq (\hat{\rho}(f_0)\varphi, \varphi)_{L^2(V_0)} = \left(\hat{\rho}(m^{-1/2}h^{-\gamma})\hat{\rho}(f_0)\varphi, \hat{\rho}(m^{1/2}h^{-\gamma})\varphi\right)_{L^2(V_0)} \\
\geq -\|\hat{\rho}(m^{-1/2}h^{-\gamma})\hat{\rho}(f_0)\varphi\|_{L^2(V_0)} \cdot \|\hat{\rho}(m^{1/2}h^{-\gamma})\varphi\|_{L^2(V_0)} \\
\geq -C \cdot \|\varphi\|^2_{L^2(m^{1/2}h^{-\gamma})(V_0)}
\]
where \( C \) denotes the norm of \( \hat{\rho}(f_0) : L^2(m^{1/2}h^{-\gamma})(V_0) \to L^2(m^{-1/2}h^{-\gamma}) \), which is bounded by (2.3.15).

As a special case of (3.1.8) we note that we have

COROLLARY (3.1.9). — If \( f \in S(h^{-\gamma}, g) \) is positive, then, for all \( \varphi \in S(V_0) \),
\[
(\hat{\rho}(f)\varphi, \varphi)_{L^2(V_0)} \geq -C \cdot \|\varphi\|^2_{L^2(V_0)}.
\]

We say that \( f \in S(m, g) \) is strongly-elliptic if for some \( A > 0 \), we have
\[
\text{Re} f(x) \geq A \cdot m(x) \quad \text{for } x \text{ outside a compact subset of } V.
\]

SHARP GÅRDING INEQUALITY (3.1.11). — For a strongly-elliptic \( f \) in \( S(m, g) \), we have for any \( \gamma \) :
\[
\text{Re}(\hat{\rho}(f)\varphi, \varphi)_{L^2(V_0)} \geq A \cdot \|\varphi\|_{L^2(m^{1/2})}(V_0) - C \cdot \|\varphi\|_{L^2(m^{1/2}h^{-\gamma})(V_0)}.
\]

Proof. — Set \( f_1 = \text{Re} f - A \cdot m + \Phi \), where \( \Phi \in C_{\infty,c}(V) \) is large enough to make \( f_1 \geq 0 \). Applying (3.1.8) to \( f_1 \), and noting that by (0.1.13) \( \text{Re}(\hat{\rho}(\text{Re} f)\varphi, \varphi)_{L^2(V_0)} = (\hat{\rho}(\text{Re} f)\varphi, \varphi)_{L^2(V_0)} \), we get the desired result.

We can write \( \mathfrak{A}_j = \bigcup_i \mathfrak{A}_{ji} \), with \( \mathfrak{A}_{ji} = x_{ji} + \mathcal{O}_{ji}^0, i = 1, \ldots, [\mathcal{O}_j, \mathcal{O}_j^0] \), so that the decomposition \( V = \coprod_{ji} \mathfrak{A}_{ji} \) corresponds to a self-dual metric covering \( g_x \) with \( g_x \leq \tilde{g}_x \leq g_x' \). For any \( f \in S(m, g) \), denoting by \( f_{ji} = \tilde{f}(x_{ji}) \) the average of \( f \) over \( \mathfrak{A}_{ji} \), the operator \( \hat{\rho}(\tilde{f}) \) is diagonalizable with respect to the orthonormal basis \( \varphi_{ji} \in L^2(V_0) \), \( W_{\varphi_{ji}} = \) characteristic function of \( \mathfrak{A}_{ji} \), cf. (0.4.20) : \( \hat{\rho}(\tilde{f})\varphi_{ji} = f_{ji} \cdot \varphi_{ji} \). Thus we have «almost diagonalized» \( \hat{\rho}(f) \).
3.2. Fredholm operators.

By the Stone-von Neumann theorem, \( \hat{\rho} : L_2(V) \to K_2(L_2(V_0)) \) is an isometry onto the ideal of Hilbert-Schmidt operators. Since an operator is trace-class if and only if it is a product of two Hilbert-Schmidt operators, we conclude that

\[
(3.2.1) \quad \hat{\rho}(f) \text{ is trace-class, if and only if } f \in L_2(V) \# L_2(V),
\]

and note that \( L_2(V) \# L_2(V) \subseteq L_2(V) \cap C_0(V) \).

Given an invertible trace-class operator \( \mathcal{K} \), if \( \mathcal{K}^{-1} \hat{\rho}(f) \) is bounded then \( \hat{\rho}(f) \) is trace-class, so we can use the criteria for boundedness of paragraph 2.3 to get criteria for \( \hat{\rho}(f) \) to be trace-class. In particular, using (1.1.23), (1.1.24) we get:

**Lemma (3.2.2).** The operator \( \hat{\rho}(f) \) will be of trace-class provided either \( \hat{\rho}(f \# g(1, \xi, x)^\alpha) \) is a bounded operator for some metric \( g \) in \( V \), and some \( \alpha > 2d = \dim V \), or if \( \hat{\rho}(f \# g_0(1, \xi)^{\alpha_1} \cdot g_0(1, x)^{\alpha_2}) \) is a bounded operator for some metric \( g_0 \) in \( V_0 \), and some \( \alpha_1, \alpha_2 > d \).

For example, using the first criteria of the lemma, we see that the Toeplitz symbols \( \Sigma^\alpha \) correspond to trace-class operators if \( \alpha < -2d \).

A symbol \( f \in S(m, g) \) will be called **elliptic** if

\[
(3.2.3) \quad |f(x)| \geq A \cdot m(x) \text{ for } x \text{ outside a compact set } K \text{ of } V.
\]

For such an \( f \) we define

\[
(3.2.4) \quad f_K^{-1}(x) = \begin{cases} f(x)^{-1} & \text{if } x \in V \setminus K, \\ 0 & \text{if } x \in K,
\end{cases}
\]

where we can take \( K \) to be a finite union of the sets \( \mathfrak{A}_j \) corresponding to the metric-covering \( g \). Since we have for \( x \in V \setminus K \), and for \( y \) such that

\[
g_x(y) \leq 1, \\
|f_K^{-1}(x)| \leq A^{-1} \cdot m(x)^{-1},
\]

\[
g_x(y)^{-\alpha} |f_K^{-1}(x) - f_K^{-1}(x + y)| = \frac{g_x(y)^{-\alpha} |f(x + y) - f(x)|}{|f(x)f(x + y)|} \leq A^{-2} \cdot C_{\alpha} \cdot m(x)^{-1},
\]

we have
we see that $f_K^{-1} \in S(m^{-1}, g)$. By the main theorem (2.2.8) we get,

\[(3.2.5)\quad 1 - f \# f_K^{-1}, \quad 1 - f_K^{-1} \# f \in S(h^\gamma, g) \quad \text{for all } \gamma.\]

If $h$ vanishes at infinity we conclude from (2.3.9), (2.3.15), that $\hat{\rho}(f)$ defines a Fredholm operator $: L_2^m(V_0) \rightarrow L_2(V_0)$. Assume now that $g$ is *global*, in the sense that we have

\[(3.2.6)\quad h(x) \leq C \cdot g_0(1, x)^{-\delta} \quad \text{for some } \delta > 0.\]

We first remark that from (3.2.6) we obtain for any temperate weight $m$,

\begin{align*}
\begin{cases}
    m(x) \leq C_1 \cdot m(0) \cdot g_0^\gamma(1, x)^{\beta_1} & \text{by (1.4.8)} \\
    \leq C_2 \cdot g_0^\gamma(1, x)^{\beta_2} & \text{by (1.4.3)} \\
    \leq C_3 \cdot h(x)^{-\beta_3} & \text{by (3.2.6)}
\end{cases}
\end{align*}

so that $\bigcap_{\gamma > 0} S(mh^{\gamma}, g) = \bigcap_{\gamma > 0} S(h^{\gamma}, g) = S(V)$ by (1.4.11).

To calculate the index of $\hat{\rho}(f)$, $f \in S(m, g)$ elliptic, we may assume that $m = 1$ by the logarithmic law for the index. So let $f \in S(1, g)$, $|f(x)| \geq A$ for $x \notin K$, and assume that (3.2.6) holds, so that

\[1 - f \# f_K^{-1}, \quad 1 - f_K^{-1} \# f \in S(V),\]

and they correspond by (3.2.2) to trace-class operators. Hence we have,

\begin{align*}
\begin{cases}
\text{Index}(\hat{\rho}(f)) = \text{tr}(1 - \hat{\rho}(f_K^{-1} \# f)) - \text{tr}(1 - \hat{\rho}(f \# f_K^{-1})) \\
\quad = \text{tr}(\hat{\rho}(f \# f_K^{-1}) - \hat{\rho}(f_K^{-1} \# f)) \\
\quad = \int_V dx(f \# f_K^{-1}(x) - f_K^{-1} \# f(x)).
\end{cases}
\end{align*}

Writing $f = \check{f} + f_0$, $f_K^{-1} = \check{f}_K^{-1} + f_1$ as in Lemma (3.1.2), with $f_0, f_1 \in S(V)$ by (3.2.7), so that $\hat{\rho}(f_i)$ are trace-class, we obtain from (3.2.8)

\[\text{Index}(\hat{\rho}(f)) = \int_V dx(\check{f} \# \check{f}_K^{-1}(x) - \check{f}_K^{-1} \# \check{f}(x)).\]

But since $\check{f}$ and $\check{f}_K^{-1}$ are $\tilde{g}$-locally-constant, we have $\check{f} \# \check{f}_K^{-1} = \check{f}_K^{-1} \# \check{f} = f \cdot f_K^{-1}$, and we have proved the following

**Theorem (3.2.9).** — For global $g$, every elliptic $f \in S(m, g)$ defines a Fredholm operator of index zero

\[\hat{\rho}(f) : L_2^m(V_0) \rightarrow L_2(V_0).\]
3.3. Zeta functions and spectral asymptotics.

In this section we fix a global metric-covering $g$, temperate and certain, so that we have from (3.2.7) : \( \bigcap \nabla \gamma S(\gamma h, g) = S(V) \), for any temperate weight $m$. Let $f \in S(m, g)$ be a real valued elliptic symbol. By (0.1.13), \( \hat{\rho}(f) \) is formally self-adjoint. From (2.3.15) we know that \( \hat{\rho}(f) \) is bounded as an operator from $L^2_{(m)}(V_0)$ to $L_2(V_0)$. Let us denote by $\mathcal{A}$ the unbounded operator on $L_2(V_0)$ defined by \( \hat{\rho}(f), \) with domain $L^2_{(m)}(V_0)$.

Note that we can write $f^{-1}_{K} \# f = 1 - R_0$, with $R_0 \in S(V)$ as in (3.2.5), hence if $\varphi \in L_2(V_0)$ is any element such that $\hat{\rho}(f)\varphi \in L_2(V_0)$, then

\[
\hat{\rho}(m)\varphi = \hat{\rho}(m \# f^{-1}_{K}) \hat{\rho}(f) \varphi + \hat{\rho}(m \# R_0) \varphi,
\]

showing that $\hat{\rho}(m)\varphi \in L_2(V_0)$, and so $\varphi \in L^2_{(m)}(V_0)$. From the continuity of $\hat{\rho}(f)$ in $S'(V_0)$, we deduce that the operator $\mathcal{A}$ is closed. We claim that $\mathcal{A}$ is self-adjoint, and to prove it we shall show that $\mathcal{A}$ equals the closure of its restriction to $S(V_0)$. Let $\Phi_k$ denote the partial sums of the characteristic functions of the sets $\mathfrak{A}_j$ corresponding to the metric-covering $g$. The $\Phi_k$ are a bounded sequence in $S(1, g)$ covering to 1 in $C^\infty(V)$, and by (2.3.11) we have for any $\varphi \in L_2(V_0)$, $\hat{\rho}(\Phi_k)\varphi \to \varphi$ in $L_2(V_0)$; moreover, since $\Phi_k \in S(V)$ the operators $\hat{\rho}(\Phi_k)$ are smoothening, and so $\hat{\rho}(\Phi_k)\varphi \in S(V_0)$. For $\varphi \in L^2_{(m)}(V_0)$ we have

\[
\mathcal{A}\hat{\rho}(\Phi_k)\varphi = \hat{\rho}(f \# \Phi_k \# f^{-1}_{K}) \hat{\rho}(f) \varphi + \hat{\rho}(f \# \Phi_k \# R_0) \varphi.
\]

The $f \# \Phi_k \# f^{-1}_{K}$, and $f \# \Phi_k \# R_0$, are bounded in $S(1, g)$, and converge to $f \# f^{-1}_{K}$, and to $f \# R_0$, in $C^\infty(V)$, respectively, hence by (2.3.11) we have

\[
\mathcal{A}\hat{\rho}(\Phi_k)\varphi \longrightarrow \hat{\rho}(f \# f^{-1}_{K}) \hat{\rho}(f) \varphi + \hat{\rho}(f \# R_0) \varphi = \mathcal{A}\varphi
\]
in $L_2(V_0)$. Thus $\mathcal{A}$ is indeed self-adjoint. We note that if $f \geq 0$ is positive, so that by Gårding inequality (3.1.11) $\mathcal{A}$ is bounded from below, and hence for some real number $\lambda_0$, $(\mathcal{A} - \lambda_0)$ is invertible. Its inverse $(\mathcal{A} - \lambda_0)^{-1}$ is a self-adjoint injective operator with image $L^2_{(m)}(V_0)$. Assume now that $m^{-1} \in C^0(V)$ vanishes at infinity. From Theorem (2.3.9) we see that $(\mathcal{A} - \lambda_0)^{-1}$ is a compact operator. From the spectral theorem for compact, self-adjoint, injective operators, we deduce that there is an orthonormal basis $\{\phi_j\}$ for $L_2(V_0)$ consisting of eigenfunctions for $(\mathcal{A} - \lambda_0)^{-1}$, with real non-zero eigenvalues $\mu_j$ converging to 0. Hence the $\phi_j$'s are eigenfunctions
of \( A \) with the real eigenvalues \( \lambda_j = \lambda_0 + \mu_j^{-1} \), converging to \( +\infty \). Moreover, writing

\[(f - \lambda)_{R_j}^{-1} \# (f - \lambda) = 1 - R_{\lambda},\]

with \( R_{\lambda_j} \in S(V) \), we have \( \phi_j = \rho(R_{\lambda_j})\phi_j \) is in \( S(V_0) \). We summarize the above discussion by the following

**Theorem (3.3.1).** — For a global certain temperate metric-covering \( g \), and a temperate weight \( m \), every real-valued elliptic \( f \in S(m, g) \) defines an essentially self-adjoint operator \( \hat{\rho}(f) \). Denoting by \( A \) the unique self-adjoint extension of \( \hat{\rho}(f) \), \( \text{Domain}(A) = L_2^{(m)}(V_0) \), and if \( f \geq 0 \) is positive, \( A \) is bounded from below. If moreover, \( m^{-1} \in C_0(V) \) vanishes at infinity, then \( A \) has a discrete point spectrum, and there is an orthonormal basis for \( L_2(V_0) \) consisting of eigenfunctions \( \phi_j \in S(V_0) \), \( A\phi_j = \lambda_j \cdot \phi_j \), with eigenvalues \( \lambda_j \) converging to \( +\infty \).

To study the asymptotic behaviour of the eigenvalues \( \lambda_j \), we may assume that \( f(x) > c \cdot m(x) > \varepsilon > 0 \), and moreover that \( \min \lambda_j > \varepsilon \), since we can always achieve this by adding to \( f \) a large constant. For \( \lambda \in \mathbb{C} \setminus [\varepsilon, \infty] \) we can write

\[(3.3.2) \quad \begin{cases} (f - \lambda)^{-1} \# (f - \lambda) = 1 - R_{\lambda} \\ (f - \lambda)^{-1} \# (f - \lambda)^{-1} = 1 - R'_{\lambda} \end{cases} \]

with \( R_{\lambda}, R'_{\lambda} \in S(V) \). From (3.3.2) we obtain

\[(3.3.3) \quad (A - \lambda)^{-1} - \hat{\rho}(f(f - \lambda)^{-1}) = \hat{\rho}(R_{\lambda} \# (f - \lambda)^{-1}) + \hat{\rho}(R_{\lambda})(A - \lambda)^{-1} \hat{\rho}(R'_{\lambda}). \]

For \( \lambda \notin \{\lambda_j\} \) the operator \( (A - \lambda)^{-1} \) is bounded, so that the right hand side of (3.3.3) is a smoothening operator and can be written as \( \hat{\rho}(r_{\{\lambda\}}) \) for some \( r_{\{\lambda\}} \in S(V) \); thus we conclude that for \( \lambda \notin \{\lambda_j\} \),

\[(3.3.4) \quad (A - \lambda)^{-1} = \hat{\rho}(f_{\{\lambda\}}) \]

for some \( f_{\{\lambda\}} \in S(m^{-1}, g) \) with \( f_{\{\lambda\}} - (f - \lambda)^{-1} = r_{\{\lambda\}} \in S(V) \).

Note that for \( \text{Re}(\lambda) \leq \varepsilon_1 < \varepsilon \) we have the inequalities

\[(3.3.5) \quad \begin{cases} \sup_{x \in \mathbb{R}_j} |(f - \lambda)^{-1}(x)| \leq |c \cdot m_j - \lambda|^{-1} \leq |\varepsilon - \lambda|^{-1}, \\
\sup_{x, x_2 \in \mathbb{R}_j} \quad g_j(x_1 - x_2)^{-\alpha}|(f - \lambda)^{-1}(x_1) - (f - \lambda)^{-1}(x_2)| = \sup_{x, x_2 \in \mathbb{R}_j} \quad g_j(x_1 - x_2)^{-\alpha}|f(x_1) - f(x_2)| \leq c_\alpha \cdot |c \cdot m_j - \lambda|^{-2} \leq c_\alpha |\varepsilon - \lambda|^{-2}. \end{cases} \]
Hence in particular,

\[(3.3.6) \quad \| (f - \lambda)^{-1} \|_{B_{2, \infty}^\infty(1,g)} = O(|\lambda|^{-1}) \quad \text{for } |\lambda| \to \infty, \ \text{Re}(\lambda) \leq \epsilon_1.\]

From (3.3.5) and the error estimates of the main theorem (2.1.11), (2.2.4), (2.2.7), a straightforward calculation gives for

\[R_\lambda = (f - \lambda)^{-1} \cdot f - (f - \lambda)^{-1} \# f, \quad \text{Re}(\lambda) \leq \epsilon_1 < \epsilon,
\]

the inequalities

\[(3.3.7) \quad \begin{cases} 
\| R_\lambda \|_{B_{2, \infty}^\infty(h^\gamma, g)} \leq C_{\alpha, \gamma}, \quad \text{and similarly} \\
\| R'_\lambda \|_{B_{2, \infty}^\infty(h^\gamma, g)} \leq C_{\alpha, \gamma}.
\end{cases}
\]

From (3.3.6) and (3.3.7), combined with (2.3.8), and with (2.3.15), we have as \(|\lambda| \to \infty, \ \text{Re}(\lambda) \leq \epsilon_1,
\]

\[(3.3.8) \begin{cases} 
\| \hat{\rho}(f - \lambda)^{-1} \|_{\text{op}} = O(|\lambda|^{-1}), \\
\| \hat{\rho}(R_\lambda) \|_{\text{op}(h^\gamma)} = O(1), \\
\| \hat{\rho}(R'_\lambda) \|_{\text{op}(h^\gamma)} = O(1),
\end{cases}
\]

where \( \| R \|_{\text{op}(h^\gamma)} \) denotes the norm of \( R : L^2(h^\gamma)(V_0) \to L^2(h^{-\gamma})(V_0). \)

From (3.3.8) we have for the right hand side of (3.3.3) the estimate

\[(3.3.9) \quad \| \hat{\rho}(r_{\{\lambda\}}) \|_{\text{op}(h^\gamma)} = \| (A - \lambda)^{-1} - \hat{\rho}((f - \lambda)^{-1}) \|_{\text{op}(h^\gamma)} = O(|\lambda|^{-1})
\]
as \(|\lambda| \to \infty, \ \text{Re}(\lambda) \leq \epsilon_1.

For \( \text{Re}(s) < 0 \) we have

\[(3.3.10) \quad A^s = \frac{1}{2\pi i} \int_{\epsilon_1 - i\infty}^{\epsilon_1 + i\infty} \lambda^s (A - \lambda)^{-1} \ d\lambda
\]

which converges in the operator norm since \( \| A - (\epsilon_1 + it) \|_{\text{op}} = O(|t|^{-1}). \)

From (3.3.5) we see that for \( \text{Re}(s) < 0 \) we have

\[(3.3.11) \quad f^s = \frac{1}{2\pi i} \int_{\epsilon_1 - i\infty}^{\epsilon_1 + i\infty} \lambda^s (f - \lambda)^{-1} \ d\lambda \in S(m^{\text{Re}(s)}, g).\]
From (3.3.9) we see that for $\Re(s) < 0$

\[ r^{(s)} = \frac{1}{2\pi i} \int_{\epsilon_1+i\infty}^{\epsilon_1+i\infty} \lambda^{s} r_{(\lambda)} \, d\lambda \in S(V) \]  

since the norms $\| \|_{\text{op}(h^\lambda)}$, $\lambda > 0$, define the space $S(V)$ by the remarks after (1.3.3).

Combining (3.3.10), (3.3.11), (3.3.12) with (3.3.4) we conclude that for $\Re(s) < 0$

\[ f_A^{(s)} = \hat{\rho}(f \#^s) \quad \text{with} \quad f \#^s = f^s + r^{(s)} \in S(m^{\Re(s)}, g), \]  

(3.3.13)  

\[ r^{(s)} \in S(V). \]

Since for $s$ an integer (3.3.13) holds with $f \#^s = f \# \cdots \# f$, $s$ times, we see that (3.3.13) actually holds for all $s \in \mathbb{C}$. Thus we have proved the following

**Theorem (3.3.14).** — For a global certain temperate metric covering $g$, a temperate weight $m$ with $m^{-1} \in C_0(V)$, and a positive elliptic symbol $f \in S(m, g)$, denoting by $A$ the unique self-adjoint extension of $\hat{\rho}(f)$, and assuming that $A$ is positive, $A$ generates a holomorphic group of operators $A^s$ (bounded only for $\Re(s) < 0$) and given by $A^s = \hat{\rho}(f \#^s)$, $f \#^s = f^s + r^{(s)} \in S(m^{\Re(s)}, g)$, $r^{(s)} \in S(V)$.

Note in particular that $s \mapsto r^{(s)}$ is an entire holomorphic function of $s$ with values in $S(V)$; for $\Re(s) \leq 0$ this follows from (3.3.9) (3.3.12), and we have a «step by step» holomorphic continuation by means of the recursion formula

\[ r^{(s+1)} = f \# r^{(s)} + (f \# f^s - f \cdot f^s). \]

Assume now that we have

\[ m^{-1}(x) \leq C \cdot g_0(1, x)^{-\alpha_0} \quad \text{for some} \quad \alpha_0 > 0. \]  

(3.3.15)

It follows from Lemma (3.2.2), and from Theorem (2.3.9), that the symbols in $S(m^{-s}, g)$ correspond to trace-class operators for $s > 2d/\alpha_0$. Thus we can define the *zeta function,*

\[ \zeta_A(s) = \text{tr}(A^{-s}) = \sum_j \lambda_j^{-s} = \int_V dx f \#(-s)(x), \]  

(3.3.16)
which is a holomorphic function of $s$ in the right half plane $\text{Re}(s) > 2d/\alpha_0$. Similarly we can define

\begin{equation}
\zeta_f(s) = \int_V dx \ f(x)^{-s}
\end{equation}

which is holomorphic in the same right half plane $\text{Re}(s) > 2d/\alpha_0$. Setting

\[
N_A(\lambda) = \# \{ \lambda_j \leq \lambda \},
\]

\[
N_f(\lambda) = \text{vol} \{ x \mid f(x) \leq \lambda \},
\]

we have the Mellin transform expressions for our zetas

\begin{equation}
\zeta_A(s) = \int \lambda^{-s} \ dN_A(\lambda),
\end{equation}

\begin{equation}
\zeta_f(s) = \int \lambda^{-s} \ dN_f(\lambda).
\end{equation}

Their difference is given via (3.3.14) by

\begin{equation}
\zeta_A(s) - \zeta_f(s) = \int \lambda^{-s} \ d(N_A(\lambda) - N_f(\lambda)) = \int_V dx \ r^{(-s)}(x),
\end{equation}

and is an entire function of $s$. Applying the inverse Mellin transform we obtain the following

**Theorem (3.3.21).** — For a global certain temperate metric-covering $g$, a temperate weight $m$ such that

\[
m(x) \geq C \cdot g_0(1, x)^{\alpha_0}, \quad \alpha_0 > 0,
\]

and a positive elliptic $f \in S(m, g)$, denoting by $\lambda_j$ the eigenvalues of the self-adjoint extension $A$ of $\rho(f)$, we have for every $\varepsilon > 0$

\[
N_A(\lambda) = \# \{ \lambda_j \leq \lambda \} = O(\lambda^{2d/\alpha_0 + \varepsilon}), \quad \text{as} \ \lambda \to +\infty.
\]

Moreover, we have for every $\varepsilon > 0$

\[
N_A(\lambda) = \text{vol} \{ x \mid f(x) \leq \lambda \} + O(\lambda^\varepsilon), \quad \text{as} \ \lambda \to +\infty.
\]

This theorem should be compared with the remark at the end of paragraph 3.1. Using the notation of the same paragraph, $\tilde{f}$ is $\tilde{g}$-locally
constant and the operator $\tilde{A} = \tilde{\rho}(\hat{f})$ is explicitly diagonalizable; denoting by $\tilde{\lambda}_j$ its eigenvalues, we have $\{\tilde{\lambda}_j\} = \hat{f}(V)$, so that for any positive real $\lambda$ we have an equality $\# \{\tilde{\lambda}_j \mid |\tilde{\lambda}_j| \leq \lambda\} = \text{vol}\{x \mid |\hat{f}(x)| \leq \lambda\}$ if finite, and more precisely we have an equality of measures

$$\sum_j \delta_{\tilde{\lambda}_j} = \hat{f}_*(dx).$$

Here there are no ellipticity or positivity assumptions on $f$. If however the assumptions of Theorem (3.3.21) are satisfied, then $\hat{f}$ is also positive elliptic, and so for any $\varepsilon > 0$, combining (3.3.21) with (3.1.2) we have as $\lambda \to +\infty$,

$$\begin{align*}
N_A(\lambda) &= \text{vol}\{x \mid f(x) \leq \lambda\} + O(\lambda^\varepsilon) \\
&= \text{vol}\{x \mid \hat{f}(x) \leq \lambda\} + O(\lambda^\varepsilon) \\
&= N_{\hat{A}}(\lambda) + O(\lambda^\varepsilon).
\end{align*}$$

For particular examples of Theorem (3.3.21) one can take the Toeplitz symbols defined in (1.4.12).

**BIBLIOGRAPHY**


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