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Superharmonic extension and harmonic approximation


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0. Introduction.

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$.

This paper solves the following problems. Find necessary and sufficient conditions on $(\Omega, E)$ so that :

(i) for each superharmonic function $u$ on $E$, there is a superharmonic function $\overline{u}$ on $\Omega$ such that $\overline{u} = u$ on $E$ (or on an open set which contains $E$);

(ii) for each harmonic (resp. superharmonic) function $u$ on $E$ and each positive number $\varepsilon$, there is a harmonic (resp. superharmonic) function $v$ on $\Omega$ such that $u - \varepsilon \leq v \leq u + \varepsilon$ on $E$;

(iii) for each function $h$ which is continuous on $E$ and harmonic on $E^\circ$, and for each positive number $\varepsilon$, there is a harmonic function $H$ on $\Omega$ such that $|H - h| < \varepsilon$ on $E$;

(iv) for each harmonic (resp. superharmonic) function $u$ on $E$, there is a harmonic (resp. superharmonic) function $v$ on $\Omega$ and a positive number $a$ such that $u - a \leq v \leq u + a$ on $E$.

Tangential harmonic and superharmonic approximation are also discussed.

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1. Superharmonic extension.

Let $\Omega$ be an open set in Euclidean space $\mathbb{R}^n$ ($n \geq 2$) and suppose that $E \subseteq \Omega$. A function $u$ will be called superharmonic (resp. harmonic) on $E$ if $u$ is defined and superharmonic (resp. harmonic) on an open set which contains $E$. We call $(\Omega, E)$ an extension pair for superharmonic functions if, for each superharmonic function $u$ on $E$, there is a superharmonic function $\bar{u}$ on $\Omega$ such that $\bar{u} = u$ on $E$. Further, $(\Omega, E)$ will be called a strong extension pair for superharmonic functions if it can be arranged that $\bar{u} = u$ on an open set which contains $E$. In the latter case we preserve not only the values of $u$ on $E$, but also the associated Riesz measure on an open set which contains $E$. It can be observed immediately that, for either of the above extension properties to hold, $E$ must be closed relative to $\Omega$. For, if $\eta$ is the fundamental subharmonic function with pole at some point $X_0$ of $(\bar{E}\setminus E) \cap \Omega$, then $u$ is harmonic on $E$, but any function $\bar{u}$ on $\Omega$ which satisfies $\bar{u} = u$ on $E$ is not bounded below near $X_0$, and so cannot be superharmonic.

We will use $\Omega^*$ to denote the Alexandroff one-point compactification of $\Omega$, and $A$ to denote the ideal point. However, $\bar{A}$, $A^\circ$ and $\partial A$ will always represent the Euclidean closure, interior and boundary (respectively) of a subset $A$ of $\mathbb{R}^n$. A subset $A$ of $\Omega$ will be called $\Omega$-bounded if $\bar{A}$ is a compact subset of $\Omega$. Recall that a topological space is called locally connected if, for each point $X$ in the space and each neighbourhood $\omega$ of $X$, there is a connected neighbourhood $\omega'$ of $X$ such that $\omega' \subseteq \omega$. In the following result the set $\Omega^*\setminus E$ can fail to satisfy this condition only in the case where $X = A$.

**Theorem 1.** — Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$. Then $(\Omega, E)$ is a strong extension pair for superharmonic functions if and only if $\Omega^*\setminus E$ is both connected and locally connected.

The condition that $\Omega^*\setminus E$ be connected is clearly equivalent to saying that $\Omega\setminus E$ has no $\Omega$-bounded (connected) components. The particular case of Theorem 1 where $E$ is compact (and so the local connectedness condition on $\Omega^*\setminus E$ is redundant) is closely related to several known results: see, for example, [8, Lemma 2.3 and §7], [2, Theorem 1] and [16, Theorem 2.4]. It appears that only Armitage [2] has previously considered the non-compact case (but see also the final note of this paper). Theorem 2 of [2] gives conditions on an open set $\omega$ which are sufficient to ensure that, for each
superharmonic function $u$ on $\omega$, there is a superharmonic function $\overline{u}$ on $\mathbb{R}^n$ satisfying $\overline{u} = u$ on the set $\{X : \text{dist}(X, \mathbb{R}^n \setminus \omega) > a\}$, where $a$ is a fixed positive number. A question raised by [2] (see the last two lines of p. 216) corresponds to asking if $(\mathbb{R}^2, E)$ is a strong extension pair for superharmonic functions, where

$$E = \{(x_1, x_2) : x_2 \geq 0\} \setminus \bigcup_{k=1}^{\infty} \{(x_1, x_2) : 2k < x_1 < 2k + 1 \text{ and } x_2 < 5k\}.$$ 

Theorem 1 supplies an affirmative answer. Below we give an example of a pair $(\Omega, E)$ such that $\Omega^* \setminus E$ is connected but not locally connected.

**Example 1.** — If

$$S = \bigcup_{k=1}^{\infty} \left\{ (x_1, x_2) : \frac{1}{2k+1} < x_1 < \frac{1}{2k} \text{ and } x_2 < k \right\},$$

then $(\mathbb{R}^2)^* \setminus \partial S$ is connected. However, $(\mathbb{R}^2)^* \setminus \partial S$ is not locally connected: the set $(\mathbb{R}^2)^* \setminus (\partial S \cup K)$, where $K = [0,1] \times \{0\}$, is a neighbourhood of $A$ in $(\mathbb{R}^2)^* \setminus \partial S$ which does not contain any connected neighbourhood of $A$.

It follows from Theorem 1 that $(\mathbb{R}^2, \partial S)$ is not a strong extension pair for superharmonic functions. (In fact, more can be said: see Example 3(b) below.)

The condition that $\Omega^* \setminus E$ be both connected and locally connected has arisen in the theory of holomorphic and harmonic approximation (see, for example, Arakeljan [1] and Theorem A below), but we do not make use of such results in proving Theorem 1. Our proof is based, in part, on ideas contained in [2].

If $E$ is a relatively closed subset of $\Omega$, then $\overline{E}$ will denote the union of $E$ with the $\Omega$-bounded components of $\Omega \setminus E$. In the case where $E$ is compact, we note that dist $(\overline{E}, \mathbb{R}^n \setminus \Omega) > 0$, and so $\mathbb{R}^n \setminus \overline{E}$ has finitely many components. If $V$ is an open set such that $\mathbb{R}^n \setminus V$ is not polar, then we use $\mu_{\overline{E},X}$ to denote harmonic measure for $V$ and a point $X$ in $V$. (For an account of the Dirichlet problem and related concepts, see Helms [13] or Doob [9].) The collection of all Borel subsets of $\mathbb{R}^n$ will be denoted by $\mathcal{B}$. Before presenting a complete characterization of extension pairs for superharmonic functions (see Theorem 3) we give below a special case of the solution which has a simpler formulation.
Theorem 2. — Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a compact subset of $\Omega$ such that each point of $\partial \widehat{E}$ is regular for the Dirichlet problem on $\mathbb{R}^n \setminus \widehat{E}$. Then $(\Omega, E)$ is an extension pair for superharmonic functions if and only if each $\Omega$-bounded component $V_0$ of $\Omega \setminus E$ satisfies the following conditions:

(i) $V_0$ is regular for the Dirichlet problem, and

(ii) given $X_k$ in $V_k$ ($k = 0, \ldots, m$), where $V_1, \ldots, V_m$ denote the components of $\mathbb{R}^n \setminus \widehat{E}$, there are positive constants $c_1, \ldots, c_m$ such that

\[
\mu_{V_0, X_0}(A) \leq \sum_{k=1}^{m} c_k \mu_{V_k, X_k}(A) \quad (A \in \mathcal{B}).
\]

It is clear from Harnack’s inequalities that, if there exist constants $c_1, \ldots, c_m$ such that (2) holds for a given choice of $X_0, \ldots, X_m$ then corresponding constants can be found for any other choice of $X_0, \ldots, X_m$. Also, conditions (i) and (ii) above together imply that $\partial V_0 \subseteq \partial \widehat{E}$. (To see this, we note that $\partial V_0 \setminus \widehat{E}$ is a relatively open subset of $\partial V_0$ which has zero harmonic measure, by (ii). Thus every point of $\partial V_0 \setminus \widehat{E}$ is irregular, and (i) now shows that $\partial V_0 \subseteq \partial \widehat{E}$.) Theorem 2 will be illustrated below by means of pairs $(\mathbb{R}^2, E)$, where $E$ is a union of finitely many line segments. It is straightforward to write down corresponding examples in higher dimensions. Our assertions are based on the elementary observation that, if

\[S_\alpha = \{re^{i\theta} : 0 < \theta < \alpha \text{ and } 0 < r < 2\}, \quad z_\alpha = e^{i\alpha/2} \quad (0 < \alpha \leq 2\pi)\]

(identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual manner), then the restriction of $\mu_{S_\alpha, z_\alpha}$ to the interval $(0,2)$ is absolutely continuous with respect to one-dimensional Lebesgue measure $\lambda$, and there are positive constants $k_1(\alpha)$, $k_2(\alpha)$ such that

\[k_1(\alpha)t^{\pi/\alpha-1} \leq (d\mu_{S_\alpha, z_\alpha}/d\lambda)(t) \leq k_2(\alpha)t^{\pi/\alpha-1} \quad (0 < t \leq 1)\]

Examples 2. — (a) Let $P$ denote an open polygon in $\mathbb{R}^2$. Then $(\mathbb{R}^2, \partial P)$ is an extension pair for superharmonic functions if and only if $P$ is convex.

(b) Let

\[
\begin{align*}
F_1 &= [0, 2]^2 \cup ([2, 4] \times \{2\}), \quad F_2 = [0, 2]^2 \cup ([2, 4] \times \{1\}), \\
F_3 &= ([0, 1] \cup [2, 3])^2 \cup [1, 2]^2, \quad F_4 = ([0, 1) \cup (1, 2]) \times [0, 1], \\
F_5 &= [0, 2]^2 \setminus \{(1, 1)\}.
\end{align*}
\]
Then \((\mathbb{R}^2, \partial F_1)\) and \((\mathbb{R}^2, \partial F_3)\) are extension pairs for superharmonic functions. However \((\mathbb{R}^2, \partial F_2)\) and \((\mathbb{R}^2, \partial F_4)\) violate condition (ii) of Theorem 2, and \((\mathbb{R}^2, \partial F_5)\) violates condition (i), so these are not extension pairs.

We come now to the question of characterizing extension pairs \((\Omega, E)\) in the absence of any special conditions on \(E\). If \(W\) is an open set which satisfies \(\widetilde{E} \subseteq W \subseteq \Omega\), then we define a class of superharmonic functions on \(W\) by

\[
S_W = \{ v : v \text{ is positive and superharmonic on } W, v = 1 \text{ on } \widetilde{E} \}.
\]

Also, the Riesz measure associated with a superharmonic function \(v\) is denoted by \(\nu_v\). By a countable set we mean one which is either finite or countably infinite.

**Theorem 3.** — Let \(\Omega\) be an open set in \(\mathbb{R}^n\) and \(E\) be a relatively closed subset of \(\Omega\). Then \((\Omega, E)\) is an extension pair for superharmonic functions if and only if:

(i) each \(\Omega\)-bounded component of \(\Omega \setminus E\) is regular for the Dirichlet problem,

(ii) \(\Omega^* \setminus \widetilde{E}\) is locally connected, and

(iii) for each countable collection \(\{(X_k, c_k) : k \in I\}\) of pairs from \((\widetilde{E} \setminus E) \times (0, \infty)\) such that the points \(X_k\) are distinct and have no limit point in \(\Omega\), there exist an open set \(W\) satisfying \(\widetilde{E} \subseteq W \subseteq \Omega\) and a function \(v\) in \(S_W\) such that

\[
\sum_{k \in I} c_k \mu_{(\widetilde{E} \setminus E), X_k}(A) \leq \nu_v(A) \quad (A \in \mathcal{B}).
\]

As in the case of Theorem 2, we observe that conditions (i) and (iii) above together imply that \(\partial \Omega \subseteq \partial \widetilde{E}\) for each \(\Omega\)-bounded component \(V\) of \(\Omega \setminus E\). Condition (iii) is similar in nature to condition (ii) of Theorem 2, but it also implies that a given compact subset of \(\Omega\) cannot intersect “arbitrarily large” \(\Omega\)-bounded components of \(\Omega \setminus E\). This is made precise below.

**Lemma 1.** — Let \(\Omega\) be an open set in \(\mathbb{R}^n\), let \(E\) be a relatively closed subset of \(\Omega\), and suppose that condition (iii) of Theorem 3 holds. Then,
for each compact subset $K$ of $\Omega$, there is a compact subset $L$ of $\Omega$ that contains every $\Omega$-bounded component of $\Omega \setminus E$ which intersects $K$.

Examples 3. — (a) Let $(P_k)$ be a sequence of open polygons in $\mathbb{R}^2$ such that the closures $\overline{P}_k$ are pairwise disjoint and only a finite number of the polygons intersect any given compact set. Then $(\mathbb{R}^2, \bigcup_k \partial P_k)$ is an extension pair for superharmonic functions if and only if each of the polygons is convex. The “if” part of this assertion can be checked by choosing $W$ to be $\bigcup_k Q_k$ in condition (iii) of Theorem 3, where $(Q_k)$ is a suitable sequence of pairwise disjoint open sets such that $\overline{P}_k \subset Q_k$ for each $k$. The “only if” part follows from Example 2(a).

(b) Let $S$ be as in (1). Then $(\mathbb{R}^2, \partial S)$ is not an extension pair, because condition (ii) of Theorem 3 is violated. Also, $(\mathbb{R}^2, \partial S \cup ([0,1] \times \{0\}))$ is not an extension pair because (iii) fails, by Lemma 1.

Theorems 1-3 are established in §4-6, following some preparatory material in §3. Lemma 1 is proved in §3.3.

2. Harmonic approximation.

We call $(\Omega, E)$ a Runge pair for harmonic (resp. superharmonic) functions if, for each harmonic (resp. superharmonic) function $u$ on $E$ and each positive number $\varepsilon$, there is a harmonic (resp. superharmonic) function $v$ on $\Omega$ such that $u - \varepsilon \leq v \leq u + \varepsilon$ on $E$. Further, inspired by the main result of [1], we call $(\Omega, E)$ an Arakeljan pair for harmonic functions if, for each function $h$ which is continuous on $E$ and harmonic on $E^\circ$, and for each positive number $\varepsilon$, there is a harmonic function $H$ on $\Omega$ such that $|H - h| < \varepsilon$ on $E$. Reasoning as in the opening paragraph of §1, it is clear that these approximation properties also require $E$ to be closed relative to $\Omega$. The following important result is due to Gauthier, Goldstein and Ow (see [10, Theorem 3] when $n = 2$, and [11, Theorem 1] when $n \geq 3$).

**Theorem A.** — Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$. If $\Omega^* \setminus E$ is both connected and locally connected, then $(\Omega, E)$ is a Runge pair for harmonic functions.

The next result shows that the hypotheses of Theorem A can be considerably weakened.
THEOREM 4. — Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( E \) be a relatively closed subset of \( \Omega \). The following are equivalent:

(a) \((\Omega, E)\) is a Runge pair for superharmonic functions;

(b) \((\Omega, E)\) is a Runge pair for harmonic functions;

(c) \((\Omega, E)\) satisfies the conditions below:

(i) \( \Omega \setminus \widehat{E} \) and \( \Omega \setminus E \) are thin at the same points of \( E \), and

(ii) for each compact subset \( K \) of \( \Omega \), there is a compact subset \( L \) of \( \Omega \) which contains every \( \Omega \)-bounded component of \( \Omega \setminus (E \cup K) \) whose closure intersects \( K \).

Theorem 4 appears to be new even in the case where \( E \) is compact (and hence condition (c)(ii) is redundant). It is clear from this result and Examples 2 that every extension pair for superharmonic functions is a Runge pair for harmonic functions, but not conversely. Condition (c)(ii) of Theorem 4 implies that \( \Omega^* \setminus \widehat{E} \) is locally connected, and also that the conclusion of Lemma 1 holds. This condition is presented in [10, Theorem 2] as necessary for \((\Omega, E)\) to be a Runge pair for harmonic functions when \( n = 2 \), but the proof given there is defective: see [5], where it is shown that \( \Omega^* \setminus \widehat{E} \) must be locally connected. Condition (c)(i) implies that each \( \Omega \)-bounded component \( V \) of \( \Omega \setminus E \) is regular for the Dirichlet problem and satisfies \( \partial V \subseteq \partial \widehat{E} \) (see the second paragraph of §7.1). However, the converse of this statement is false when \( n \geq 3 \), as the following example shows. Let \( \phi_n : [0, +\infty) \to \mathbb{R} \cup \{ +\infty \} \) be the function defined by \( \phi_2(t) = \log(1/t) \) or \( \phi_n(t) = t^{2-n} \) if \( n \geq 3 \). (We interpret \( \phi_n(0) \) as \( +\infty \) in either case.)

Example 4. — Let \( n \geq 3 \), let \( \{ Y_k : k \in \mathbb{N} \} \) be a dense subset of \([0,1]^{n-1}\), and define

\[
u(X') = \sum_{k=1}^{\infty} 2^{-k} \phi_{n-1}(|X' - Y_k'|) \quad (X' \in \mathbb{R}^{n-1}).
\]

Further, let

\[
E = \partial([0,1]^{n-1} \times [-1,0]) \cup \{(X', x_n) : (0,1]^{n-1} \times (0,1) : u(X') \leq \phi_n(x_n) \}.
\]

Then the only bounded component of \( \mathbb{R}^n \setminus E \) is given by \( V = (0,1)^{n-1} \times (-1,0) \), which is regular for the Dirichlet problem and satisfies \( \partial V \subseteq \partial \widehat{E} \). However, \( \mathbb{R}^n \setminus \widehat{E} \) is thin at each point of \((0,1)^{n-1} \times \{0\} \), whereas \( \mathbb{R}^n \setminus E \) is
not. (See §12 for details.) Thus \((\mathbb{R}^n, E)\) is not a Runge pair for harmonic functions.

Theorem 4 can be combined with known results to obtain the following.

**Theorem 5.** — Let \(\Omega\) be an open set in \(\mathbb{R}^n\) and \(E\) be a relatively closed subset of \(\Omega\). Then \((\Omega, E)\) is an Arakeljan pair for harmonic functions if and only if:

(i) \(\Omega \setminus \hat{E}\) and \(\Omega \setminus E^o\) are thin at the same points of \(E\); and

(ii) for each compact subset \(K\) of \(\Omega\), there is a compact subset \(L\) of \(\Omega\) which contains every \(\Omega\)-bounded component of \(\Omega \setminus (E \cup K)\) whose closure intersects \(K\).

**Example 5.** — Let \(n \geq 3\), let \(E\) be as in Example 4, and let \(E_1 = \hat{E}\). Then \((\mathbb{R}^n, E_1)\) (trivially) satisfies conditions (c)(i)-(ii) of Theorem 4, but not condition (i) of Theorem 5 (see §12). Hence \((\mathbb{R}^n, E_1)\) is a Runge pair for harmonic functions, but not an Arakeljan pair for harmonic functions. (Another such example may be found in [5, p. 21].)

The next result shows that the situation described in Example 5 cannot arise when \(n = 2\).

**Theorem 6.** — Let \(\Omega\) be an open set in \(\mathbb{R}^2\) and \(E\) be a relatively closed subset of \(\Omega\). The following are equivalent:

(a) \((\Omega, E)\) is a Runge pair for superharmonic functions;
(b) \((\Omega, E)\) is a Runge pair for harmonic functions;
(c) \((\Omega, E)\) is an Arakeljan pair for harmonic functions;
(d) (i) \(\partial E = \partial \hat{E}\), and

(ii) for each compact subset \(K\) of \(\Omega\), there is a compact subset \(L\) of \(\Omega\) which contains every \(\Omega\)-bounded component of \(\Omega \setminus (E \cup K)\) whose closure intersects \(K\).

Theorems 5 and 6 solve [6, Problem 9.10], posed by M. Goldstein. Necessary and sufficient conditions for \((\Omega, E)\) to be an Arakeljan pair for harmonic functions have recently been given also in [5, Theorem 3.3] and [12, Theorem 1], but the conditions given there are not as explicit as those above.
Now suppose that $\Omega$ has a Green function $G_\Omega(.,.)$, fix $X_0$ in $\Omega$, and define

$$g(X) = \min\{1, G_\Omega(X_0, X)\} \quad (X \in \Omega).$$

In this case we can add the following equivalent conditions to Theorem 4:

(d) (resp. (e)) for each harmonic (resp. superharmonic) function $u$ on $E$ and each positive number $\varepsilon$, there is a harmonic (resp. superharmonic) function $v$ on $\Omega$ such that $u - \varepsilon g \leq v \leq u + \varepsilon g$ on $E$.

Also, conditions (i)-(ii) of Theorem 5 are equivalent to the following:

for each function $h$ which is continuous on $E$ and harmonic on $E^c$, and for each positive number $\varepsilon$, there is a harmonic function $H$ on $\Omega$ such that $|H - h| < \varepsilon g$ on $E$. This leads to three additional equivalent conditions in Theorem 6. These assertions have essentially the same proofs as Theorems 4-6 except that, in place of Theorem A above and Theorem B of §8.1, we appeal to corresponding recent results of Armitage and Goldstein [3] concerning tangential harmonic approximation. Saginyan [17, Theorem 1] has announced a result for tangential harmonic approximation which is similar in nature to the modified form of Theorem 5 described above, but no proof has yet appeared.

We call $(\Omega, E)$ a weak Runge pair for harmonic (resp. superharmonic) functions if, for each harmonic (resp. superharmonic) function $u$ on $E$, there is a harmonic (resp. superharmonic) function $v$ on $\Omega$ and a positive number $a$ such that $u - a \leq v \leq u + a$ on $E$.

**THEOREM 7.** — Let $\Omega$ be an open set in $\mathbb{R}^n$ ($n \geq 3$) and $E$ be a relatively closed subset of $\Omega$. The following are equivalent:

(a) $(\Omega, E)$ is a weak Runge pair for superharmonic functions;

(b) $(\Omega, E)$ is a weak Runge pair for harmonic functions;

(c) $(\Omega, E)$ satisfies the conditions below:

(i) there is a compact subset $C$ of $\Omega$ such that $\Omega \setminus \tilde{E}$ and $\Omega \setminus E$ are thin at the same points of $E \setminus C$, and

(ii) for each compact subset $K$ of $\Omega$, there is a compact subset $L$ of $\Omega$ which contains every $\Omega$-bounded component of $\Omega \setminus (E \cup K)$ whose closure intersects $K$.

**THEOREM 8.** — Let $\Omega$ be an open set in $\mathbb{R}^2$ and $E$ be a relatively closed proper subset of $\Omega$. The following are equivalent:
(a) $(\Omega, E)$ is a weak Runge pair for superharmonic functions;
(b) $(\Omega, E)$ is a weak Runge pair for harmonic functions;
(c) (i) there is a compact subset $C$ of $\Omega$ such that $\partial E \setminus C = \partial \hat{E} \setminus C$,
    (ii) either $\hat{E} \neq \Omega$ or $\mathbb{R}^2 \setminus \Omega$ is non-polar, and
    (iii) for each compact subset $K$ of $\Omega$, there is a compact subset $L$ of $\Omega$ which contains every $\Omega$-bounded component of $\Omega \setminus (E \cup K)$ whose closure intersects $K$.

Theorems 4-8 are proved in §7-11. They rely on Theorems 1 and A.

3. Preparatory material.

3.1. The following is a variant of [8, Lemma 2.3 and §7] suitable for our present purposes.

**Lemma 2.** — Let $\omega_0$ be an open set in $\mathbb{R}^n$ and $E$ be a compact subset of $\omega_0$. Further, let $\omega_1, \ldots, \omega_l$ denote the bounded components of $\mathbb{R}^n \setminus E$ which are not subsets of $\omega_0$, and let $X_k \in \omega_k$ ($k = 1, \ldots, l$). If $u$ is a superharmonic function on $\omega_0$, then there exist a superharmonic function $v$ on $\mathbb{R}^n$ and a non-negative constant $c$ such that

$$u(X) = v(X) - c \sum_{k=1}^{l} \phi_n(|X - X_k|)$$

on some open set which contains $E$.

To prove this, let $F_0$ be a compact set such that

$$E \subset F_0^\circ \subset F_0 \subset \bigcup_{k=0}^{l} \omega_k$$

and $\mathbb{R}^n \setminus F_0$ is connected. Also, for each $k$ in $\{1, \ldots, l\}$, let $F_k$ be a compact set with connected interior such that

$$\{X_k\} \cup (\omega_k \setminus \omega_0) \subset F_k^\circ \subset F_k \subset \omega_k,$$

and define

$$F = F_0 \setminus \left( \bigcup_{k=1}^{l} F_k^\circ \right).$$
Now let $U$, $W$ be bounded open sets such that

$$E \subset U \subset \overline{U} \subset F^o \subset F \subset W \subset \overline{W} \subset \omega_0 \backslash \{X_1, \ldots, X_l\},$$

and define

$$w(X) = \begin{cases} H_u^{W \backslash \overline{U}}(X) & (X \in W \backslash U) \\ u(X) & \text{(elsewhere in } \omega_0) \end{cases},$$

where $H_\Omega$ denotes the PWB solution of the Dirichlet problem on $\Omega$ with boundary function $f$. The lower regularization $w^*$ of $w$, is superharmonic on $\omega_0$, and equals $u$ on $U$. Next we define $h$ on $\mathbb{R}^n \setminus F$ as follows. On $F_k^o$ ($k = 1, \ldots, l$) let $h$ be the Green function for $F_k^o$ with pole at $X_k$. On $\mathbb{R}^n \setminus F_0$ let $h$ be the Green function for $(\mathbb{R}^2)^* \setminus F_0$ with pole at $A$ if $n = 2$, or the solution to the Dirichlet problem for $\mathbb{R}^n \setminus F_0$ with boundary data 0 on $\partial F_0$ and 1 at the Alexandroff point for $\mathbb{R}^n$ if $n \geq 3$.

Now let

$$M > \sup \{w^*(X) : X \in \partial F\},$$

$$m < \inf \\{(w^*(X) - M)/h(X) : X \in \partial W\} \cup \{0\},$$

and

$$s(X) = \begin{cases} w^*(X) & (X \in F) \\ \min \{M + mh(X), w^*(X)\} & (X \in W \setminus F) \\ M + mh(X) & (X \in \mathbb{R}^n \setminus W). \end{cases}$$

It is straightforward to check that $s$ is superharmonic on $\mathbb{R}^n \setminus (\partial F \cup \{X_1, \ldots, X_l\})$, and also on an open set $T$ which contains the regular boundary points of $\mathbb{R}^n \setminus F$. Since $\partial F \setminus T$ is polar, $s^*$ is superharmonic on $\mathbb{R}^n \setminus \{X_1, \ldots, X_l\}$. Clearly $s^* = u$ on $U$ and the function $v$ defined by

$$v(X) = s^*(X) + (-m) \sum_{k=1}^l \phi_n(|X - X_k|) \quad (X \notin \{X_1, \ldots, X_l\})$$

has a superharmonic extension to all of $\mathbb{R}^n$. This completes the proof of the lemma.

3.2. Lemma 3. — Let $\Omega$ be an open set in $\mathbb{R}^n$, let $E$ be a relatively closed subset of $\Omega$, and suppose that, for each harmonic function $h$ on $E$, there exists a superharmonic function $u$ on $\Omega$ such that $u \geq h$ on $E$. Then, for each compact subset $K$ of $\Omega$, there is a compact subset $L$ of $\Omega$ which contains every $\Omega$-bounded component of $\Omega \setminus (E \cup K)$ whose closure intersects $K$. 
To prove this, suppose that the conclusion of the lemma fails to hold. Then there exist a compact subset $K$ of $\Omega$, a sequence $(V_k)$ of distinct $\Omega$-bounded components of $\Omega \setminus (E \cup K)$, and two sequences $(X_k), (Y_k)$ of points, such that $X_k, Y_k \in V_k$ for each $k$, and such that $X_k \to A$ and $(Y_k)$ converges to some point $Y_0$ in $K$. Now let $U$ be an $\Omega$-bounded open set which contains $K$ and let $U_0$ be the component of $U$ which contains $Y_0$. By deleting the first few members of the sequence $(V_k)$ we can arrange that $V_k \cap U_0 \neq \emptyset$ for each $k$. We define

\begin{equation}
(4) \quad a_k = \mu_{V_k, X_k} (U_0 \cap \partial V_k) \quad (k \in \mathbb{N}).
\end{equation}

If $a_k = 0$, then (see [9, I.VIII.5(b)]) there is a superharmonic function $v_1$ on $V_k$ with limit $+\infty$ at each point of $U_0 \cap \partial V_k$. Hence the function

$$v_2(X) = \begin{cases} v_1(X) & (X \in U_0 \cap V_k) \\ +\infty & (X \in U_0 \setminus V_k) \end{cases}$$

is lower semicontinuous and super-meanvalued on $U_0$. This is impossible, since $U_0 \cap V_{k+1}$ is a non-polar subset of $U_0 \setminus V_k$. Hence $a_k > 0$ for each $k$.

Now let $h$ be a harmonic function on the set $\Omega_1 = \Omega \setminus \{X_k : k \in \mathbb{N}\}$, such that, for each $k$, the function

$$h(X) + a_k^{-1} \phi_n(|X - X_k|)$$

has a harmonic extension to $\Omega_1 \cup \{X_k\}$. (Such a function exists by [11, Lemma 2], for example.) By hypothesis there exists a superharmonic function $u$ on $\Omega$ such that $u \geq h$ on $E$. Also, since $u - h$ is superharmonic on $\Omega$, we can define $b$ to be a negative lower bound for $u - h$ on $\overline{U}$, and then define the open set

$$W = \{X \in \Omega : u(X) - h(X) > b - 1\}.$$

It follows from the minimum principle, and the fact that $K \subset U$, that $u - h \geq b$ on each $\overline{V}_k$, and so $\bigcup_k \overline{V}_k \subseteq W$. Also, $\overline{U} \subset W$. Clearly the function $v$ defined by $v(X) = u(X) - h(X) - b + 1$ is positive and superharmonic on $W$ and satisfies $\nu_v(\{X_k\}) \geq a_k^{-1}$ for each $k$. It follows from the Riesz decomposition theorem that a potential on $W$ is defined by

$$w(X) = \sum_k a_k^{-1} G_W(X_k, X) \quad (X \in W),$$

where $G_W(.,.)$ denotes the Green function for $W$. Let $T = \bigcup_k V_k$. Then the restriction of $w$ to $\partial T \cap W$ is $\mu_{T, X}$-integrable when $X \in T$. However, (4)
yields
\[ \sum_{k} a_k^{-1} \mu_{T, x_k}(U_0 \cap \partial T) = \sum_{k} a_k^{-1} \mu_{V_k, x_k}(U_0 \cap \partial V_k) = +\infty. \]

Also, by monotone convergence, we have
\[
\int_{\partial T \cap \Omega} w(Y) d\mu_{T, X}(Y) = \sum_{k} a_k^{-1} \int_{\partial T \cap \Omega} G_{W}(X_k, Y) d\mu_{T, X}(Y)
\]
\[= \sum_{k} a_k^{-1} \int_{\partial T \cap \Omega} G_{W}(X, Y) d\mu_{T, x_k}(Y). \]

Hence the Riesz measure associated with the superharmonic function 
\[ X \mapsto \int_{\partial T \cap \Omega} w d\mu_{T, X} \] is infinite on the compact set \( \overline{U}_0 \cap \partial T \), a contradiction. Therefore the conclusion of the lemma must hold.

3.3. Lemma 1 is now straightforward to prove. Suppose that condition (iii) of Theorem 3 holds, and that the conclusion of the lemma fails. Then there exist a compact subset \( K \) of \( \Omega \), a sequence \( (V_k) \) of distinct \( \Omega \)-bounded components of \( \Omega \setminus E \), and sequences \( (X_k), (Y_k) \) of points, such that \( X_k, Y_k \in V_k \) for each \( k \), and such that \( X_k \to A \) and \( (Y_k) \) converges to some point \( Y_0 \) in \( K \). Now let \( U \) be an \( \Omega \)-bounded open set which contains \( K \) and let \( U_0 \) be the component of \( U \) which contains \( Y_0 \), define \( a_k \) as in (4), and let \( c_k = a_k^{-1} \). (We know from §3.2 that \( a_k > 0 \).) Inequality (3) now implies that \( \nu_b(\overline{U} \cap E) = +\infty \). This is impossible, since \( \overline{U} \cap E \) is a compact subset of \( \Omega \). Hence Lemma 1 is proved.

4. Proof of Theorem 1.

4.1. We begin with the "if" part of the proof. Let \( u \) be a superharmonic function on \( E \), fix \( X_0 \) in \( E \), let \( A_1 = \{X_0\} \), and let \( (A_k) \) be a sequence of compact subsets of \( \Omega \) such that \( A_k \subset A_{k+1}^\circ \) for each \( k \) and also \( \bigcup_k A_k = \Omega \). A subset \( A \) of \( \Omega \) will be called \( \Omega \)-solid if \( \Omega^* \setminus A \) is connected.

We will now inductively define a new sequence \( (C_k) \) of compact subsets of \( \Omega \) which satisfy \( C_k \subset C_{k+1}^\circ \) for each \( k \) and also

(I) \( A_k \subset C_k \), \hspace{1cm} (II) \( C_k \) is \( \Omega \)-solid, \hspace{1cm} (III) \( C_k \cup E \) is \( \Omega \)-solid.

Let \( C_1 = A_1 \). Then (I)-(III) hold when \( k = 1 \). Given \( C_k \), we choose a compact subset \( F_1 \) of \( \Omega \) which satisfies \( A_{k+1} \cup C_k \subset F_1^\circ \). Since \( \Omega^* \setminus E \) is
locally connected, there is a compact set $F_2$ such that $F_1 \subseteq F_2 \subseteq \Omega$ and $
abla \Omega \setminus (F_2 \cup E)$ is connected; that is, $F_2 \cup E$ is $\Omega$-solid. We now define $C_{k+1}$ to be the union of $F_2$ with all the $\Omega$-bounded components of $\Omega \setminus F_2$, and observe that $C_{k+1} \cup E = F_2 \cup E$. It is clear that $C_k \subseteq C_{k+1}$ and that (I)-(III) hold when $k$ is replaced by $k + 1$.

Secondly, we inductively define a sequence $(u_k)$ of functions such that

(a) $u_k$ is superharmonic on $C_k \cup E$;

(b) $u_k = u$ on an open set $U_k$ which contains $E$,

and such that $u_{k+1} = u_k$ on $C_k$ for each $k$. If we define $u_1 = u$, then (a) and (b) hold when $k = 1$. Given $u_k$, we construct $u_{k+1}$ as follows. We know that $u_k$ is superharmonic on an open set $\omega$ (where $\omega \subseteq \Omega$) which contains $C_k \cup E$, and so also contains the compact set $E_1$ defined by $E_1 = C_{k+2} \cap (C_k \cup E)$. Since $C_{k+2}$ and $C_k \cup E$ are $\Omega$-solid by (II) and (III) above, it follows that $E_1$ is $\Omega$-solid. Thus $\mathbb{R}^n \setminus E_1$ has finitely many bounded components $\omega_1, \ldots, \omega_l$, and we can choose $X_j$ in $\omega_j \setminus \Omega$ for each $j$ in $\{1, \ldots, l\}$.

Lemma 2 can now be applied (with $\omega_0, E, u$ replaced by $\omega, E_1, u_k$ respectively) to obtain a superharmonic function $\overline{u}_k$ on $\mathbb{R}^n \setminus \{X_1, \ldots, X_l\}$, and hence on $\Omega$, such that $\overline{u}_k = u_k$ on an open set $\omega'$ which contains $E_1$. We define $V = (\omega \setminus C_{k+2}) \cup \omega'$ and

$$u_{k+1}(X) = \begin{cases} \overline{u}_k(X) & (X \in C_{k+2}^o) \\ u_k(X) & (X \in V). \end{cases}$$

This function is well-defined, and hence superharmonic, on the open set $C_{k+2}^o \cup V$, because the two parts of the definition agree on the region of overlap, namely $C_{k+2}^o \cap \omega'$. We know that

$$E \setminus C_{k+2} \subseteq \omega \setminus C_{k+2} \quad \text{and} \quad E \cap C_{k+2} \subseteq E_1 \subseteq \omega',$$

so $E \subseteq V$ and $C_{k+1} \cup E \subseteq C_{k+2}^o \cup V$.

It follows that $u_{k+1}$ is superharmonic on $C_{k+1} \cup E$, that $u_{k+1} = u_k = u$ on the open set $U_{k+1} = U_k \cap V$ which contains $E$, and that $u_{k+1} = u_k$ on $C_k$ (since $C_k \subseteq E_1 \subseteq \omega' \subseteq V$).

The final step of the argument is to define $\overline{u}(X) = \lim_{k \to \infty} u_k(X)$ for each $X$ in $\Omega$. Given $Y_0$ in $\Omega$, there exists $k_0$ such that $Y_0 \in A_{k_0}^o \subseteq C_{k_0}^o$, and $u_k = u_{k_0}$ on $C_{k_0}$ when $k \geq k_0$. It follows that $\overline{u}$ is superharmonic on a neighbourhood of $Y_0$. Thus $\overline{u}$ is superharmonic on $\Omega$. From property (b) above, and the fact that $\overline{u} = u_k$ on $C_k$, it is clear that $\overline{u} = u$ on the open

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locally connected, there is a compact set $F_2$ such that $F_1 \subseteq F_2 \subseteq \Omega$ and $
abla \Omega \setminus (F_2 \cup E)$ is connected; that is, $F_2 \cup E$ is $\Omega$-solid. We now define $C_{k+1}$

Secondly, we inductively define a sequence $(u_k)$ of functions such that

(a) $u_k$ is superharmonic on $C_k \cup E$;

(b) $u_k = u$ on an open set $U_k$ which contains $E$,

and such that $u_{k+1} = u_k$ on $C_k$ for each $k$. If we define $u_1 = u$, then (a) and

(b) hold when $k = 1$. Given $u_k$, we construct $u_{k+1}$ as follows. We know that

$u_k$ is superharmonic on an open set $\omega$ (where $\omega \subseteq \Omega$) which contains $C_k \cup E$, and so also contains the compact set $E_1$ defined by $E_1 = C_{k+2} \cap (C_k \cup E)$. Since $C_{k+2}$ and $C_k \cup E$ are $\Omega$-solid by (II) and (III) above, it follows that $E_1$ is $\Omega$-solid. Thus $\mathbb{R}^n \setminus E_1$ has finitely many bounded components $\omega_1, \ldots, \omega_l$, and we can choose $X_j$ in $\omega_j \setminus \Omega$ for each $j$ in $\{1, \ldots, l\}$. Lemma 2 can now be applied (with $\omega_0, E, u$ replaced by $\omega, E_1, u_k$ respectively) to obtain a superharmonic function $\overline{u}_k$ on $\mathbb{R}^n \setminus \{X_1, \ldots, X_l\}$, and hence on $\Omega$, such that $\overline{u}_k = u_k$ on an open set $\omega'$ which contains $E_1$. We define $V = (\omega \setminus C_{k+2}) \cup \omega'$ and

$$u_{k+1}(X) = \begin{cases} \overline{u}_k(X) & (X \in C_{k+2}^o) \\ u_k(X) & (X \in V). \end{cases}$$

This function is well-defined, and hence superharmonic, on the open set $C_{k+2}^o \cup V$, because the two parts of the definition agree on the region of overlap, namely $C_{k+2}^o \cap \omega'$. We know that

$$E \setminus C_{k+2} \subseteq \omega \setminus C_{k+2} \quad \text{and} \quad E \cap C_{k+2} \subseteq E_1 \subseteq \omega',$$

so $E \subseteq V$ and $C_{k+1} \cup E \subseteq C_{k+2}^o \cup V$.

It follows that $u_{k+1}$ is superharmonic on $C_{k+1} \cup E$, that $u_{k+1} = u_k = u$ on the open set $U_{k+1} = U_k \cap V$ which contains $E$, and that $u_{k+1} = u_k$ on $C_k$ (since $C_k \subseteq E_1 \subseteq \omega' \subseteq V$).

The final step of the argument is to define $\overline{u}(X) = \lim_{k \to \infty} u_k(X)$ for each $X$ in $\Omega$. Given $Y_0$ in $\Omega$, there exists $k_0$ such that $Y_0 \in A_{k_0}^o \subseteq C_{k_0}^o$, and $u_k = u_{k_0}$ on $C_{k_0}$ when $k \geq k_0$. It follows that $\overline{u}$ is superharmonic on a neighbourhood of $Y_0$. Thus $\overline{u}$ is superharmonic on $\Omega$. From property (b) above, and the fact that $\overline{u} = u_k$ on $C_k$, it is clear that $\overline{u} = u$ on the open...
set \( \bigcup_k (U_k \cap C_F^c) \) which contains \( E \). Hence \((\Omega, E)\) is a strong extension pair for superharmonic functions.

4.2. Conversely, suppose that \((\Omega, E)\) is a strong extension pair for superharmonic functions. If \( \Omega^* \setminus E \) is not connected, then there is an \( \Omega \)-bounded component \( V \) of \( \Omega \setminus E \). We fix \( X_0 \in V \), define \( u(X) = -\phi_n(|X - X_0|) \) and conclude, by hypothesis, that there is a superharmonic function \( \bar{u} \) on \( \Omega \) such that \( \bar{u} = u \) on an open set \( \omega \) which contains \( E \). Now let \( W \) be an \( \Omega \)-bounded connected open set such that \( \bar{V} \subset W \) and \( \bar{W} \subset \omega \cup V \).

Since \( u \) is subharmonic on \( \Omega \), we know that \( H_u^V \leq H_u^W \) on \( V \). Since \( \bar{u} \) is superharmonic on \( \Omega \), it is also true that \( H_{\bar{u}}^V \geq H_{\bar{u}}^W \) on \( V \). Observing that \( \bar{u} = u \) on \( \partial V \) and \( \partial W \), it follows that \( H_{\bar{u}}^V = H_u^W \) on \( V \). Hence \( H_u^W - u \), which is a positive superharmonic function on \( W \), takes the value 0 at every regular boundary point of \( \partial V \) : a contradiction. Thus \( \Omega^* \setminus E \) must be connected.

Since any strong extension pair satisfies the hypotheses of Lemma 3, we deduce that \( \Omega^* \setminus \bar{E} \) is locally connected. The connectedness of \( \Omega^* \setminus E \), shown above, means that \( E = \bar{E} \), so \( \Omega^* \setminus E \) is locally connected. The proof of Theorem 1 is now complete.

5. Proof of Theorem 2.

5.1. Let \((\Omega, E)\) be as in the first sentence of Theorem 2, suppose that each \( \Omega \)-bounded component \( V_0 \) of \( \Omega \setminus E \) satisfies conditions (i) and (ii) of the theorem, and let \( u \) be a superharmonic function on some open set \( \omega \) (where \( \omega \subset \subseteq \Omega \)) which contains \( E \). Further, let \( U_1, \ldots, U_l \) be the bounded components of \( \mathbb{R}^n \setminus E \) which are not subsets of \( \Omega \), and let \( W_1, \ldots, W_p \) be the remaining bounded components of \( \mathbb{R}^n \setminus E \) which are not subsets of \( \omega \).

We choose \( Y_k \) in \( U_k \setminus \Omega \) for each \( k \in \{1, \ldots, l\} \), and \( Z_k \) in \( W_k \) for each \( k \in \{1, \ldots, p\} \). It follows from Lemma 2 that there is a non-negative constant \( c \) and a superharmonic function \( v \) on \( \mathbb{R}^n \) such that

\[
    u(X) = v(X) - c \left\{ \sum_{k=1}^l \phi_n(|X - Y_k|) + \sum_{k=1}^p \phi_n(|X - Z_k|) \right\} \quad (X \in E).
\]

In particular, there is a superharmonic function \( v_1 \) on \( \Omega \) such that

\[
    u(X) = v_1(X) - c \sum_{k=1}^p \phi_n(|X - Z_k|) \quad (X \in E).
\]
Now let \( V_0 = W_1 \) and \( X_0 = Z_1 \). Further, let \( V_1, \ldots, V_m \) denote the components of \( \mathbb{R}^n \setminus \hat{E} \) and let \( X_k \in V_k \) \((k = 1, \ldots, m)\). By hypothesis (ii) there are non-negative constants \( c_1, \ldots, c_m \) such that (2) holds. We now define

\[
(5) \quad s(X) = \phi_n(|X - X_0|) - \int \phi_n(|X - Y|)d\mu_{V_0, X_0}(Y) + \sum_{k=1}^m c_k \left\{ \int \phi_n(|X - Y|)d\mu_{V_k, X_k}(Y) - \phi_n(|X - X_k|) \right\}.
\]

Inequality (2) shows that the function \( s(X) - \phi_n(|X - Z_1|) \) is superharmonic on \( \mathbb{R}^n \setminus \{X_1, \ldots, X_m\} \), in view of the fact that \( X_0 = Z_1 \). Further, the regularity of \( V_0 \) (hypothesis (i)) and of the (finite) boundary points of \( \mathbb{R}^n \setminus \hat{E} \) ensure that \( s = 0 \) on \( E \). To see this when \( n \geq 3 \) we note that, if \( X \in E \), then \( \Omega \setminus V_k \) is not thin at \( X \), and so

\[
\phi_n(|X - X_k|) = R^{\mathbb{R}^n \setminus V_k}_{\phi_n(|X - |)}(X_k) = \int \phi_n(|X - Y|)d\mu_{V_k, X_k}(Y) \quad (k = 0, \ldots, m),
\]

where \( R^F_f \) denotes the reduced function (réduite) of \( f \) relative to a set \( F \) in \( \mathbb{R}^n \). A modified form of this argument applies also when \( n = 2 \). Hence, if we define

\[
v_2(X) = v_1(X) + c \{ s(X) - \phi_n(|X - Z_1|) \} \quad (X \in \Omega),
\]

we obtain a superharmonic function \( v_2 \) on \( \Omega \setminus \{X_1, \ldots, X_m\} \) such that

\[
u(X) = v_2(X) - c \sum_{k=2}^p \phi_n(|X - Z_k|) \quad (X \in E).
\]

If we repeat the argument of the previous paragraph with \( V_0 = W_k \) \((k = 2, \ldots, p)\), it follows that there is a superharmonic function \( v_{p+1} \) on \( \Omega \setminus \{X_1, \ldots, X_m\} \) such that \( u = v_{p+1} \) on \( E \). Since \( \Omega \setminus \hat{E} \) is connected, we can apply Theorem 1 to the pair \( (\Omega, \hat{E}) \) to conclude that there is a superharmonic function \( \overline{u} \) on \( \Omega \) such that \( \overline{u} = v_{p+1} \) on \( \hat{E} \), and hence \( \overline{u} = u \) on \( E \). It follows that \( (\Omega, E) \) is an extension pair for superharmonic functions.

5.2. Conversely, suppose that \( (\Omega, E) \) is an extension pair for superharmonic functions, let \( V_0 \) be an \( \Omega \)-bounded component of \( \Omega \setminus E \), let \( X_0 \in V_0 \), and define \( u(X) = -\phi_n(|X - X_0|) \). By hypothesis there is a superharmonic
function \( \overline{u} \) on \( \Omega \) such that \( \overline{u} = u \) on \( E \). Thus the function \( v = \overline{u} - u \) is superharmonic on \( \Omega \) and vanishes on \( \partial \Omega \). In particular, \( v \) is a positive superharmonic function on \( \Omega \) which vanishes on \( \partial \Omega \), so \( \Omega \) is regular for the Dirichlet problem.

Now let \( V_1, \ldots, V_m \) be the components of \( \mathbb{R}^n \setminus \overline{E} \), and let \( G_k(\cdot, \cdot) \) be the Green function for \( V_k \) for each \( k \) in \( \{0, \ldots, m\} \). Further, let \( \mathcal{W} \) be an \( \Omega \)-bounded open set which contains \( \overline{E} \) and let \( X_k \in V_k \setminus \overline{W} \) for each \( k \) in \( \{1, \ldots, m\} \). For each \( k \) in the latter set we can find a positive constant \( c_k \) such that

\[
-c_k G_k(X_k, X) < v(X) \quad (X \in \partial \mathcal{W} \cap V_k).
\]

It follows from the minimum principle that inequality (6) remains true for all \( X \) in \( \mathcal{W} \cap V_k \). It is also clear that \( v \geq G_0(X_0, \cdot) \) on \( \Omega \) and that \( v \geq 0 \) on \( \overline{E} \). Hence the function \( s \) defined on \( \mathbb{R}^n \) by

\[
s(X) = \begin{cases} 
G_0(X_0, X) & (X \in \Omega) \\
-c_k G_k(X_k, X) & (X \in V_k; k \in \{1, \ldots, m\}) \\
0 & (X \in \overline{E} \setminus \Omega)
\end{cases}
\]

is superharmonic on \( \mathbb{R}^n \setminus \{X_1, \ldots, X_m\} \). The function \( s \) can be written as in (5). Since \( \Delta s \leq 0 \) on \( \mathbb{R}^n \setminus \{X_1, \ldots, X_m\} \) in the sense of distributions, we conclude that (2) holds. Thus Theorem 2 is established.

6. Proof of Theorem 3.

6.1. Suppose that conditions (i)-(iii) of the theorem hold and let \( u \) be a superharmonic function on some open set \( \omega \) (where \( \omega \subseteq \Omega \)) which contains \( E \). We denote by \( \{V_k : k \in I\} \) the collection of \( \Omega \)-bounded components of \( \Omega \setminus E \) which are not subsets of \( \omega \), choose \( X_k \) in \( V_k \) for each \( k \) in \( I \), and let

\[
\omega_1 = \omega \cup \left( \bigcup_{k \in I} (V_k \setminus \{X_k\}) \right).
\]

Let \( K \) be a compact subset of \( \Omega \), and define

\[
S = \bigcup_{k \in J} V_k, \text{ where } J = \{k \in I : \overline{V}_k \cap K \neq \emptyset\}.
\]

It follows from (iii) and Lemma 1 that \( S \) is \( \Omega \)-bounded. Since \( \text{dist}(S \cap E, \mathbb{R}^n \setminus \omega) > 0 \), it is clear that \( J \) is a finite set. Thus \( \{X_k : k \in I\} \) has no
limit point in \( \Omega \). Next, for each \( k \) in \( I \), we apply Lemma 2 with \( \partial V_k \) in place of \( E \). This allows us to construct a superharmonic function \( s \) on \( \omega_1 \) such that \( s = u \) on an open set which contains \( E \), and such that the function

\[
s(X) + c_k \phi_n(|X - X_k|)
\]

has a superharmonic extension to \( \omega_1 \cup \{X_k\} \) for a suitable choice of non-negative constant \( c_k \).

By condition (iii) there exist an open set \( W \) satisfying \( \widehat{E} \subseteq W \subseteq \Omega \) and a function \( v \) in \( S_W \) such that (3) holds. Let

\[
w(X) = \sum_{k \in I} c_k \left\{ \phi_n(|X - X_k|) - \int \phi_n(|X - Y|) d\mu_{V_k, X_k}(Y) \right\} + s(X) + v(X) - 1 \quad (X \in W \cap \omega_1).
\]

Inequality (3) ensures that \( w \) is superharmonic on \( W \cap \omega_1 \), and we have arranged \( s \) in such a way that \( w \) has a superharmonic extension to \( W \cap (\omega \cup (\bigcup V_k)) \), which contains \( \widehat{E} \). Since \( \Omega^* \setminus \widehat{E} \) is connected (by the definition of \( \widehat{E} \)) and locally connected (by (ii)), we can apply Theorem 1 to the pair \( (\Omega, \widehat{E}) \) to obtain a superharmonic function \( \overline{u} \) on \( \Omega \) such that \( \overline{u} = w \) on \( \widehat{E} \). Also, \( w = s = u \) on \( E \) by condition (i) and the definition of \( S_W \). Hence \( \overline{u} = u \) on \( E \). It follows that \( (\Omega, E) \) is an extension pair for superharmonic functions.

**6.2.** Conversely, suppose that \( (\Omega, E) \) is an extension pair for superharmonic functions. It follows as in §5.2 that (i) holds, and Lemma 3 shows that (ii) also holds.

It remains to establish (iii). Let \( \{(X_k, c_k) : k \in I\} \) be a countable collection of pairs from \( (\widehat{E} \setminus E) \times (0, \infty) \) such that the points \( X_k \) are distinct and have no limit point in \( \Omega \). As in the proof of Lemma 3 we can choose \( u \) to be a harmonic function on \( \Omega_1 = \Omega \setminus \{X_k : k \in I\} \) such that \( u(X) + c_k \phi_n(|X - X_k|) \) has a harmonic extension to \( \Omega_1 \cup \{X_k\} \) for each \( k \) in \( I \). By hypothesis there is a superharmonic function \( \overline{u} \) on \( \Omega \) such that \( \overline{u} = u \) on \( E \). Since \( \overline{u} - u \) is superharmonic on \( \Omega \), it follows from the minimum principle that \( \overline{u} - u \geq 0 \) on \( \widehat{E} \). For each \( k \) in \( I \) let \( V_k \) be the component of \( \widehat{E} \setminus E \) to which \( X_k \) belongs. We know from Lemma 3 that any given compact subset of \( \Omega \) intersects only finitely many of the sets \( V_k \). Also, let \( G_k(\cdot, \cdot) \) be the Green function for \( V_k \), and define \( G_k(\cdot, \cdot) = 0 \) outside \( V_k \times V_k \). Clearly

\[
\overline{u} - u \geq \sum_k c_k G_k(X_k, \cdot) \text{ on } \bigcup V_k.
\]
Let $W = \{X \in \Omega : \bar{u}(X) - u(X) > -1\}$ and
\[ v(X) = 1 + \min \{\bar{u}(X) - u(X), 0\} \quad (X \in W). \]
Clearly $W$ is an open set satisfying $\hat{E} \subseteq W \subseteq \Omega$, and also $v \in S_W$. Further, the function $s$ defined by
\[ s(X) = v(X) - 1 + \sum_{k \in I} c_k G_k(X_k, X) \quad (X \in W) \]
is also superharmonic on $W$. We can rewrite $s$ as
\[ s(X) = v(X) - 1 + \sum_{k \in I} c_k \left\{ \phi_n(|X - X_k|) - \int \phi_n(|X - Y|)d\mu_{\omega_k,X_k}(Y) \right\} \quad (X \in W), \]
and so (3) must hold. This completes the proof of Theorem 3.


7.1. Suppose that $(\Omega, E)$ satisfies conditions (c)(i)-(ii) of the theorem, let $\varepsilon > 0$, and let $u$ be a superharmonic function on an open set $\omega$ (where $\omega \subseteq \Omega$) which contains $E$. Further, let $V_k$, $X_k$, and $\omega_1$ be as in §6.1. Following the reasoning given there we can construct a superharmonic function $s$ on $\omega_1$ such that $s = u$ on an open set which contains $E$, and such that $s(X) + c_k \phi_n(|X - X_k|)$ has a superharmonic extension to $\omega_1 \cup \{X_k\}$ for a suitable choice of positive constant $c_k$. Using (c)(ii) in place of Lemma 1, we can also choose $\{U_k : k \in I\}$ to be a collection of $\Omega$-bounded open sets such that $V_k \subseteq U_k$ for each $k$, and such that any given compact subset of $\Omega$ intersects only finitely many of the sets $\bar{U}_k$.

Next we observe two consequences of condition (c)(i). Let $V$ be any $\Omega$-bounded component of $\Omega \setminus E$. If $Y_0$ is an irregular boundary point of $V$, then $\Omega \setminus V$, and hence $\Omega \setminus \hat{E}$, is thin at $Y_0$. However, $\Omega \setminus E$ contains $V$, and so is non-thin at $Y_0$. This contradicts (c)(i), and so $V$ must be regular for the Dirichlet problem. Secondly, we note that $\partial V \subseteq \partial \hat{E}$. To see this, let $A = \partial V \setminus \partial \hat{E}$. Then $A \subseteq (\hat{E})^0$, so $\Omega \setminus \hat{E}$ is certainly thin at each point of $A$. It follows from (c)(i) that $\Omega \setminus E$, and hence $V$, is thin at each point of $A$. Hence $A$ is a relatively open subset of $\partial V$ which has zero harmonic measure for $V$ (see [9, 1.XI.13]). Each point of $A$ is therefore irregular for the Dirichlet problem on $V$, and so $A = \emptyset$, as claimed.
Now fix $k$ temporarily. For each $m$ in $\mathbb{N}$ let

$$A_{k,m} = \{ X \in \overline{U}_k : \text{dist} (X, \widehat{E}) \geq 1/m \},$$

and let $g_{k,m}$ be the Green function for $\Omega \setminus A_{k,m}$ with pole at $X_k$. (This must exist for all sufficiently large $m$, even if $n = 2$, because $\partial V_k \subseteq \partial \widehat{E}$.) If we define $g_{k,m}(X) = 0$ when $X \in A_{k,m}$, then the upper regularization $g_{k,m}^{**}$ is subharmonic on $\Omega \setminus \{ X_k \}$. Thus the function $g_k = \lim_{m \to \infty} g_{k,m}^{**}$, being the limit of a decreasing sequence, is subharmonic on $\Omega \setminus \{ X_k \}$ and harmonic on $V_k \setminus \{ X_k \}$. Further, $g_k$ vanishes on $U_k \setminus \widehat{E}$, and so vanishes at each point of $\partial V_k$ where $\Omega \setminus \widehat{E}$ is non-thin. If $X$ is a point of $\partial V_k$ at which $\Omega \setminus \widehat{E}$ is thin, then condition (c)(i) shows that $\Omega \setminus E$, and hence $V_k$, are also thin at $X$. The set of all such points $X$ therefore has $\mu_{V_k, X_k}$-measure zero. It follows easily that $g_k$ coincides (on $V_k$) with the Green function for the regular set $V_k$ with pole at $X_k$. Thus, given a positive number $\varepsilon$, there is a compact subset $K_k$ of $V_k$ such that $X_k \in K_k$ and $g_k < 2^{-k-1}c_k^{-1}\varepsilon$ on $V_k \setminus K_k$. Hence, by the monotonicity of the sequence $(g_{k,m}^{**})_{m \geq 1}$ and Dini’s theorem, there exists $m_k$ such that

$$g_{k,m_k}^{**}(X) \leq g_k(X) + 2^{-k-1}c_k^{-1}\varepsilon < 2^{-k}c_k^{-1}\varepsilon \quad (X \in \partial K_k).$$

It follows that $g_{k,m_k}^{**}(X) \leq 2^{-k}c_k^{-1}\varepsilon$ on $E$.

Now let

$$W = \Omega \setminus \left( \bigcup_{k \in I} A_{k,m_k} \right).$$

This is an open set because only finitely many of the sets $A_{k,m_k}$ intersect a given compact subset of $\Omega$. Also, $\widehat{E} \subseteq W$. Let $G_W(\cdot, \cdot)$ denote the Green function for $W$. Then

$$(7) \quad G_W(X_k, X) \leq g_{k,m_k}^{**}(X) \leq 2^{-k}c_k^{-1}\varepsilon \quad (X \in E).$$

We define

$$v_1(X) = \sum_{k \in I} c_k G_W(X_k, X) \quad (X \in W).$$

It follows from (7) that $v_1$ defines a potential on $W$, and that $v_1 \leq \varepsilon$ on $E$. The function $v_2 = s + v_1$, suitably redefined on the set where it is the difference of two infinite values, is superharmonic on the open set $W \cap (\omega \cup (\cup_k V_k))$, which contains $\widehat{E}$. Also, $s \leq v_2 \leq s + \varepsilon$ on $E$. Since $\Omega^* \setminus \widehat{E}$ is connected and locally connected (by (c)(ii)) we can apply Theorem 1
to obtain a superharmonic function \( v \) on \( \Omega \) such that \( v = v_2 \) on \( \hat{E} \). Hence \( u \leq v \leq u + \varepsilon \) on \( E \). It follows that \((\Omega, E)\) is a Runge pair for superharmonic functions.

7.2. Suppose that \((\Omega, E)\) is a Runge pair for superharmonic functions, let \( h \) be harmonic on \( E \), and let \( \varepsilon > 0 \). We know that there exist superharmonic functions \( u, v \) on \( \Omega \) such that \( |u - h| < \varepsilon/4 \) and \( |v + h| < \varepsilon/4 \) on \( E \). Now let \( W \) be the open set defined by

\[
W = \{ X \in \Omega : u(X) + v(X) > -\varepsilon/2 \}.
\]

Clearly \( E \subseteq W \), and the minimum principle implies that \( \hat{E} \subseteq W \). Since 
\[
-v(X) - \varepsilon/4
\]

is a subharmonic minorant of \( u(X) + \varepsilon/4 \) on \( W \), there is a greatest harmonic minorant, \( h_1 \) say, of \( u(X) + \varepsilon/4 \) on \( W \). Hence

\[
-v(X) - \varepsilon/4 \leq h_1(X) \leq u(X) + \varepsilon/4 \quad (X \in W),
\]

and so \( |h_1 - h| < \varepsilon/2 \) on \( E \).

The function \( h_1 \) is harmonic on \( \hat{E} \). Further, \( \Omega^* \setminus \hat{E} \) is connected and also locally connected by our hypothesis and Lemma 3. We can thus apply Theorem A to obtain a harmonic function \( H \) on \( \Omega \) such that \( |H - h_1| \leq \varepsilon/2 \) on \( \hat{E} \). Combining this with the conclusion of the previous paragraph, it follows that \( |H - h| \leq \varepsilon \) on \( E \). Hence \((\Omega, E)\) is a Runge pair for harmonic functions.

7.3. Finally, suppose that \((\Omega, E)\) is a Runge pair for harmonic functions, let \( V \) be an \( \Omega \)-bounded component of \( \Omega \setminus E \), let \( X_0 \in V \) and let \( h(X) = \phi_n(|X - X_0|) \). For each positive number \( \varepsilon \) there is a harmonic function \( H_\varepsilon \) on \( \Omega \) such that \( |H_\varepsilon - h| < \varepsilon/2 \) on \( E \). We define the open set

\[
W_\varepsilon = \{ X \in \Omega : h(X) - H_\varepsilon(X) + \varepsilon/2 > 0 \}.
\]

It follows from the minimum principle that \( \hat{E} \subseteq W_\varepsilon \), and clearly

\[
h(X) - H_\varepsilon(X) + \varepsilon/2 \geq G_{W_\varepsilon}(X_0, X) \quad (X \in W_\varepsilon).
\]

Hence \( G_{W_\varepsilon}(X_0, .) < \varepsilon \) on \( E \). It follows from the arbitrary nature of \( \varepsilon \) that the Green function for \((\hat{E})^\circ\), with pole at \( X_0 \), vanishes continuously on \( \partial V \). This implies that \( \partial V \subseteq \partial \hat{E} \) (and that \( V \) is regular for the Dirichlet problem).

In this paragraph we assume that \( \Omega \) has a Green function \( G_\Omega(., .) \). If this is not the case, we could work instead with \((\Omega_1, E)\), where \( \Omega_1 \) is
obtained from \( \Omega \) by deleting a closed ball contained in \( \Omega \setminus \hat{E} \). (Note that, if \( \hat{E} = \Omega \), then \( E = \Omega \) by the above paragraph, and so there is nothing to prove). Let \( V, X_0 \) and \( W_\varepsilon \) be as in the previous paragraph. Then

\[
(8) \quad G_\Omega(X_0, X) - R_{G_\Omega(X_0, \cdot)}^{\Omega \setminus \hat{E}}(X) \leq G_\Omega(X_0, X) - R_{G_\Omega(X_0, \cdot)}^{\Omega \setminus W_\varepsilon}(X) = G_{W_\varepsilon}(X_0, X) < \varepsilon \quad (X \in \partial V).
\]

Since \( \varepsilon \) can be arbitrarily small, it follows that

\[
G_\Omega(X_0, X) = R_{G_\Omega(X_0, \cdot)}^{\Omega \setminus \hat{E}}(X) = R_{G_\Omega(X_0, \cdot)}^{\Omega \setminus \hat{E}}(X_0) \quad (X \in \Omega \setminus V).
\]

Hence

\[
G_\Omega(Y, X) = R_{G_\Omega(X_0, \cdot)}^{\Omega \setminus \hat{E}}(Y) \quad (Y \in V)
\]

for each \( X \) in \( \Omega \setminus V \). This holds for all such components \( V \), so

\[
(9) \quad R_{G_\Omega(X_0, \cdot)}^{\Omega \setminus \hat{E}} = R_{G_\Omega(X_0, \cdot)}^{\Omega \setminus \hat{E}} \quad (X \in E),
\]

and this proves (c)(i).

Finally, Lemma 3 shows that (c)(ii) holds.

8. Proof of Theorem 5.

8.1. We require the following result:

**Theorem B.** — Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( E \) be a relatively closed subset of \( \Omega \). The following are equivalent:

(a) for each function \( h \) which is continuous on \( E \) and harmonic on \( E^0 \), and for each positive number \( \varepsilon \), there is a function \( H \) harmonic on \( E \) such that \( |H - h| < \varepsilon \) on \( E \);

(b) \( \Omega \setminus E \) and \( \Omega \setminus E^0 \) are thin at the same points of \( E \).

Theorem B is due to Keldyš [14] and Deny [7] under the additional assumption that \( E \) is compact. For the case of general closed sets \( E \), see either [15, Theorem 3.10] or [4, Section 8].

8.2. Suppose that conditions (i) and (ii) of Theorem 5 hold, let \( h \) be continuous on \( E \) and harmonic on \( E^0 \), and let \( \varepsilon > 0 \). Condition (i) implies that the three sets \( \Omega \setminus \hat{E}, \Omega \setminus E \) and \( \Omega \setminus E^0 \) are thin at the same points of \( E \).
It follows from Theorem B that there is a harmonic function $h_1$ on $E$ such that $|h_1 - h| < \varepsilon/2$ on $E$. By Theorem 4 there is a harmonic function $H$ on $\Omega$ such that $|H - h_1| < \varepsilon/2$ on $E$, and so $|H - h| < \varepsilon$ on $E$. It follows that $(\Omega, E)$ is an Arakeljan pair for harmonic functions.

Conversely, if $(\Omega, E)$ is an Arakeljan pair, then Theorems 4 and B immediately show that conditions (i) and (ii) hold. Thus Theorem 5 is established.


Clearly (c) implies (b). Further, Theorem 4 shows that (b) is equivalent to (a), and that (a) implies (d). (It was observed in §7.1 that, if condition (c)(i) of Theorem 4 holds, then each $\Omega$-bounded component $V$ of $\Omega \setminus E$ satisfies $\partial V \subseteq \partial \widehat{E}$.) Now suppose that conditions (d)(i)-(ii) hold, and let $X_1$ be a point of $E$ at which $\Omega \setminus \widehat{E}$ is thin. Then (because $n = 2$) there are arbitrarily small circles, centred at $X_1$, which are contained in $\widehat{E}$ (see [13, Theorem 10.14]). Hence $X_1 \in (\widehat{E})^o$. It follows from (d)(i) that $X_1 \notin \partial E$, so $X_1 \in E^o$, and so $\Omega \setminus E^o$ is certainly thin at $X_1$. Hence $\Omega \setminus \widehat{E}$ and $\Omega \setminus E^o$ are thin at the same points of $E$. Applying Theorem 5, it follows that $(\Omega, E)$ is an Arakeljan pair for harmonic functions, i.e. (c) holds. Theorem 6 is now proved.


10.1. Let $n \geq 3$, suppose that $(\Omega, E)$ satisfies conditions (c)(i)-(ii) of the theorem, let $C_1$ be a compact subset of $\Omega$ such that $C \subset C_1^o$, and let $u$ be a superharmonic function on an open set $\omega$ (where $\omega \subset \Omega$) which contains $E$. It follows that there are only finitely many $\Omega$-bounded components $V_1, \ldots, V_m$ of $\Omega \setminus E$ which satisfy both $\overline{V_k} \cap C_1 \neq \emptyset$ and $V_k \setminus \omega \neq \emptyset$.

For each $k$ in $\{1, \ldots, m\}$ choose $X_k$ in $V_k \setminus \omega$. Next define $\Omega_1 = \Omega \setminus \{X_1, \ldots, X_m\}$, and define $E_1$ to be the union of $E$ with the $\Omega$-bounded components of $\Omega_1 \setminus E$ which are contained in $\omega$. The pair $(\Omega_1, E_1)$ satisfies conditions (c)(i)-(ii) of Theorem 4, so there exists a superharmonic function $v_1$ on $\Omega_1$ such that $u - 1 \leq v_1 \leq u + 1$ on $E_1$, and hence on $E$. Further, in view of Lemma 2, it can be arranged that there are non-negative constants $c_1, \ldots, c_m$ such that $v_1(X) + c_k \phi_n(|X - X_k|)$ has a superharmonic extension to $\Omega_1 \cup V_k$. Thus,
if we define

\[ v(X) = v_1(X) + \sum_{k=1}^{m} c_k G_{\Omega}(X_k, X) \quad (X \in \Omega), \]

we obtain a superharmonic function \( v \) on \( \Omega \) such that \( u - a \leq v \leq u + a \) on \( E \), where

\[ a = 1 + \sup_{X \in E} \sum_{k=1}^{m} c_k G_{\Omega}(X_k, X) < \infty. \]

Thus \((\Omega, E)\) is a weak Runge pair for superharmonic functions, i.e. (a) holds.

10.2. The proof that (a) implies (b) is directly analogous to the argument given in §7.2.

10.3. Suppose now that \((\Omega, E)\) is a weak Runge pair for harmonic functions. It follows from Lemma 3 that condition (c)(ii) must hold.

Next let \((V_k)\) be a sequence of \( \Omega \)-bounded components of \( \Omega \setminus E \) such that only a finite number of the sets \( V_k \) intersect any given compact subset of \( \Omega \). (If no such sequence exists, then (c)(i) clearly holds.) Also, let \((c_k)\) be a sequence of positive numbers, and let \( X_k \in V_k \) for each \( k \). We now define \( h \) to be a harmonic function on the set \( \Omega_2 = \Omega \setminus \{X_k : k \in \mathbb{N}\} \) such that, for each \( k \), the function \( h(X) + c_k \phi_n(|X - X_k|) \) has a harmonic extension to \( \Omega_2 \cup \{X_k\} \). By hypothesis there is a harmonic function \( H \) on \( \Omega \) and a positive number \( a \) such that \( |H - h| < a \) on \( E \). We define the open set

\[ W = \{X \in \Omega : H(X) - h(X) + a > 0\}. \]

It follows from the minimum principle that \( \tilde{E} \subseteq W \), and clearly

\[ H(X) - h(X) + a \geq c_k G_W(X_k, X) \quad (X \in W; k \in \mathbb{N}). \]

Hence \( c_k G_W(X_k, .) < 2a \) on \( E \), and we can argue as in (8) to deduce that

\[ c_k \left\{ G_\Omega(X_k, X) - R_{\Omega \setminus \tilde{E}}(X_k, .) \right\} < 2a \quad (X \in \partial V_k; k \in \mathbb{N}). \]

It follows from the arbitrary nature of the sequence \((c_k)\), and the reasoning given in §7.3 that, for all but a finite number of the components \( V_k \),

\[ G_\Omega(X_k, X) = R_{\Omega \setminus \tilde{E}}(X_k, X) = R_{\Omega \setminus \tilde{E}}(X_k) \quad (X \in \Omega \setminus V_k). \]
Arguing as in (9) we obtain condition (c)(i), and the proof of Theorem 7 is complete.


11.1. Let \( n = 2 \), let \( E \) be a relatively closed proper subset of \( \Omega \), and suppose that \( (\Omega, E) \) satisfies conditions (c)(i)-(iii) of the theorem. If \( \mathbb{R}^2 \setminus \Omega \) is non-polar, then \( \Omega \) possesses a Green function, and the reasoning of §9 and §10.1 establishes (a). If \( \mathbb{R}^2 \setminus \Omega \) is polar, then (c)(ii) implies that \( \hat{E} \neq \Omega \). In this case, let \( B \) be a closed ball in \( \Omega \setminus \hat{E} \), and let \( \Omega_0 = \Omega \setminus B \). Thus \( \Omega_0 \) possesses a Green function. The arguments in §9 and §10.1 show that \( (\Omega_0, E) \) is a weak Runge pair for superharmonic functions. It follows, by applying Theorem 1 to the pair \( (\Omega, \hat{E}) \), that \( (\Omega, E) \) is also a weak Runge pair for superharmonic functions.

11.2. The proof that (a) implies (b) is directly analogous to the argument given in §7.2.

11.3. Now suppose that (b) holds. As before, condition (c)(iii) follows from Lemma 3.

Next, suppose that \( \hat{E} = \Omega \). Let \( V_0 \) be an \( \Omega \)-bounded component of \( \Omega \setminus E \), let \( X_0 \in V_0 \), and let \( u(X) = \phi_2(|X - X_0|) \). It follows, by hypothesis, that there exist a harmonic function \( v \) on \( \Omega \) and a positive constant \( a \) such that \(|u - v| \leq a\) on \( E \). Hence, by the maximum principle, \(|u - v| \leq a\) on \( \Omega \setminus V_0 \). Thus \( u - v + 2a \) is a non-constant positive superharmonic function on \( \Omega \), and this implies that \( \mathbb{R}^2 \setminus \Omega \) is non-polar. It follows that (c)(ii) holds.

If \( \Omega \) has a Green function, then we can argue as in §9 and §10.3 that (c)(i) must hold. If \( \Omega \) does not possess a Green function, then we know from the previous paragraph that \( \hat{E} \neq \Omega \). We can thus reason as before with \( (\Omega_0, E) \) in place of \( (\Omega, E) \), where \( \Omega_0 \) is obtained from \( \Omega \) by deleting a closed ball contained in \( \Omega \setminus \hat{E} \). This completes the proof of Theorem 8.

12. Details of Examples 4 and 5.

12.1. Let \( n \geq 3 \), and let \( \{Y_k^r : k \in \mathbb{N}\} \), \( u \) and \( E \) be as in Example 4. The lower semicontinuity of \( u \) ensures that \( E \) is closed, and hence compact. Also, if we define \( E_y = \{X' \in \mathbb{R}^{n-1} : (X', y) \notin E\} \), then \( E_y \subseteq E_z \) whenever
0 < y < z. It follows that $\mathbb{R}^n \setminus E$ has only one bounded component, namely $V = (0,1)^{n-1} \times (-1,0)$. Clearly $V$ is regular for the Dirichlet problem and satisfies $\partial V \subseteq \partial E$.

Define $v(X', x_n) = u(X')$ on $\mathbb{R}^{n-1} \times \mathbb{R}$. Then $v$ is superharmonic. If $Y' \in (0,1)^{n-1}$, then

$$v(X', x_n) = u(X') > \phi_n(x_n) \geq \phi_n(||(X', x_n) - (Y', 0)||) \quad ((X', x_n) \in (0,1)^n \setminus E),$$

so $\mathbb{R}^n \setminus \hat{E}$ is thin at $(Y', 0)$, whereas $\mathbb{R}^n \setminus E$ is not. This establishes Example 4.

12.2. Let $E$ be as above, let $E_1 = \hat{E}$ and let $Y' \in (0,1)^{n-1}$. Then §12.1 shows that $\mathbb{R}^n \setminus E_1$ (which equals $\mathbb{R}^n \setminus \hat{E}_1$) is thin at $(Y', 0)$. However, $\mathbb{R}^n \setminus E_0$ is not, because it contains $(0,1)^{n-1} \times [0, +\infty)$. This establishes Example 5.

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Note. — Professor Paul Gauthier has independently obtained an extension theorem for subharmonic functions in a preprint entitled “Subharmonic extensions and approximations”, which is to appear in Canad. Math. Bull. His main result is distinct from, but related to, our Theorem 1. Connections with the theory of harmonic approximation are also discussed, as are a number of open problems. Problem 1 (concerning the characterization of extension pairs and Runge pairs for subharmonic functions) is solved by Theorems 2-4 of the present paper.

BIBLIOGRAPHY


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