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ON LEIBNIZ HOMOLOGY

by Teimuraz PIRASHVILI

Introduction.

Let $\mathbf{g}$ be a Lie algebra over a field k and let M be a right $\mathbf{g}$-module. Leibniz homology $HL_*(\mathbf{g}, M)$ was defined by Jean-Louis Loday (see 10.6 in [CH]). Moreover, HL_* is well-defined on a much bigger category – the so called Leibniz algebras and their corepresentations (see [LP]). The purpose of this paper is to construct a few spectral sequences. These give a relation between Leibniz and ordinary homologies for Lie algebras and relate Leibniz homology of a given Leibniz algebra $\mathbf{h}$ with its Liezation $\mathbf{h}_{\text{Lie}}$. As a consequence we compute $HL_*(\mathbf{g}, k)$, where $\mathbf{g}$ is a semi-simple Lie algebra or has the dimension 2 or 3 (char $k = 0$), or is free Lie algebra. We also prove that in some cases (see Proposition 4.3) one has an isomorphism $HL_*(\mathbf{h}, k) \cong HL_*(\mathbf{h}_{\text{Lie}}, k)$.

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1. Relation between Chevalley-Eilenberg homology and Leibniz homology for Lie algebras.

Let M be a right module over the Lie algebra $\mathbf{g}$. We denote

$$C_*(\mathbf{g}, M) = (M \otimes \Lambda^* \mathbf{g}, d')$$
$$CL_*(\mathbf{g}, M) = (M \otimes T^* \mathbf{g}, d),$$

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where T^*g is the tensor algebra over $\mathbf{g}$, d' is the Chevalley-Eilenberg boundary map, and d is the Loday boundary map (see 10.6.2.1 in [CH]). The natural epimorphism

$$CL_*(\mathbf{g}, M) \longrightarrow C_*(\mathbf{g}, M)$$

is a morphism of chain complexes, which is an isomorphism in dimensions 0 and 1. We define $C_*^{\mathrm{rel}}(\mathbf{g}, M)$ such that $C_*^{\mathrm{rel}}(\mathbf{g}, M)[2]$ is the kernel of the above epimorphism. Let

$$H_*^{\mathrm{rel}}(\mathbf{g}, M) = H_*(C_*^{\mathrm{rel}}(\mathbf{g}, M)).$$

We recall that $HL_*(\mathbf{g}, M) = H_*(CL_*(\mathbf{g}, M))$ and $H_*(\mathbf{g}, M) = H_*(C_*(\mathbf{g}, M))$. Thus, by definition we have

PROPOSITION 1.1. — *One has an exact sequence*

$$0 \longleftarrow H_2(\mathbf{g}, M) \longleftarrow HL_2(\mathbf{g}, M) \longleftarrow H_0^{\mathrm{rel}}(\mathbf{g}, M) \longleftarrow H_3(\mathbf{g}, M) \longleftarrow HL_3(\mathbf{g}, M) \longleftarrow \dots$$

and isomorphisms

$$HL_i(\mathbf{g}, M) \cong H_i(\mathbf{g}, M), \quad for \quad i = 0, 1.$$

The product map $m : \mathbf{g} \otimes \Lambda^n\mathbf{g} \longrightarrow \Lambda^{n+1}\mathbf{g}$ in the exterior algebra yields an epimorphism of chain complexes

$$C_*(\mathbf{g}, \mathbf{g}) \longrightarrow \overline{C}_*(\mathbf{g}, k)[-1].$$

Here $\overline{C}_*(\mathbf{g}, k)$ is the reduced chain complex :

$$\overline{C}_0(\mathbf{g}, k) = 0, \overline{C}_i(\mathbf{g}, k) = C_i(\mathbf{g}, k), i > 0.$$

We define $CR_*(\mathbf{g})$ such that $CR_*(\mathbf{g})[1]$ is the kernel of $C_*(\mathbf{g}, \mathbf{g}) \longrightarrow \overline{C}_*(\mathbf{g}, k)[-1]$. We denote the homology of $CR_*(\mathbf{g})$ by $HR_*(\mathbf{g})$. Then, by definition we have

PROPOSITION 1.2. — *One has an exact sequence*

$$0 \longleftarrow H_2(\mathbf{g}, k) \longleftarrow H_1(\mathbf{g}, \mathbf{g}) \longleftarrow HR_0(\mathbf{g}) \longleftarrow H_3(\mathbf{g}, k) \longleftarrow H_2(\mathbf{g}, \mathbf{g}) \longleftarrow HR_1(\mathbf{g}) \longleftarrow \dots$$

As usual, we denote the kernel of $\otimes^2\mathbf{g} \longrightarrow \Lambda^2\mathbf{g}$ *by* $\Gamma\mathbf{g}$. *Thus* $\Gamma\mathbf{g}$ *is spanned by elements* $x \otimes x$. *Moreover,* $\Gamma\mathbf{g}$ *is a* $\mathbf{g}$*-submodule of* $\mathbf{g} \otimes \mathbf{g}$ *and the restriction of*

$$1 \otimes m : \mathbf{g} \otimes \mathbf{g} \otimes \Lambda^i\mathbf{g} \longrightarrow \mathbf{g} \otimes \Lambda^{i+1}\mathbf{g}$$

to $\Gamma\mathbf{g} \otimes \Lambda^i\mathbf{g}$ *gives an epimorphism of chain complexes*

$$C_*(\mathbf{g}, \Gamma\mathbf{g}) \longrightarrow CR_*(\mathbf{g}),$$

which is an isomorphism in dimension 0. Hence $HR_0(\mathbf{g}) \cong H_0(\mathbf{g}, \Gamma\mathbf{g})$ *and for the cohomological version we have*

$$HR^0(\mathbf{g}) \cong \{\text{invariant quadratic forms on } \mathbf{g}\}.$$

It follows from Theorem A that the same is also true for $H_0^{\mathrm{rel}}(\mathbf{g}, k)$ *and* $H^0_{\mathrm{rel}}(\mathbf{g}, k)$. *Moreover, the corresponding map*

$$\{\text{invariant quadratic forms on } \mathbf{g}\} \longrightarrow H^3(\mathbf{g}, k)$$

coincides with the classical homomorphism of Koszul (see section 11 of [K]).

THEOREM A. — *Let* $\mathbf{g}$ *be a Lie algebra and* M *be a right* $\mathbf{g}$*-module. Then there exists a spectral sequence*

$$E^2_{pq} = HR_p(\mathbf{g}) \otimes HL_q(\mathbf{g}, M) \Longrightarrow H^{\mathrm{rel}}_{p+q}(\mathbf{g}, M).$$

Proof. — We consider the following filtration on $C^{\mathrm{rel}}_*(\mathbf{g}, M)$. Let $\omega \in CL_n(\mathbf{g}, M)$ and $x \in \mathbf{g}$. Then it follows from 10.6.3.1 of [CH] that $d(\omega, x, x) = (d\omega, x, x)$. Here we write (α, y) instead of $\alpha \otimes y$. Thus the subspaces $\mathcal{F}^0_n = C_n(\mathbf{g}, M) \otimes \Gamma\mathbf{g} \subset C^{\mathrm{rel}}_n(\mathbf{g}, M)$ give us a subcomplex and

$$H_*(\mathcal{F}^0_*) \cong HL_*(\mathbf{g}, M) \otimes \Gamma\mathbf{g}.$$

More generally, for $\rho = (x_1, ..., x_i)$ and $\rho_j = (x_1, ..., \hat{x}_j, ...x_i)$, $1 \le j \le i$, one has

$$\begin{aligned} d(\omega, x, x, \rho) = (d\omega, x, x, \rho) \ &+ \ (-1)^{\deg\omega}(\omega, x, x, d\rho) \\ &+ \sum(-1)^{\deg\omega + j}([\omega, x_j], x, x, \rho_j) \\ &+ \sum(-1)^{\deg\omega + j}(\omega, (([x, x_j], x) + (x, [x, x_j])), \rho_j). \end{aligned}$$

Since $(y, z) + (z, y) \in \Gamma\mathbf{g}$, $y, z \in \mathbf{g}$ we conclude that $\mathcal{F}^0_* \subset \mathcal{F}^1_* \subset ...$ are subcomplexes of $C^{\mathrm{rel}}_*(\mathbf{g}, M)$ with $\mathcal{F}^n_n = C^{\mathrm{rel}}_n(\mathbf{g}, M)$ and

$$\mathcal{F}^s_*/\mathcal{F}^{s-1}_* \cong CL_*(\mathbf{g}, M)[-s] \\ \otimes(\mathrm{Ker}(\otimes^{s+2}\mathbf{g} \longrightarrow \Lambda^{s+2}\mathbf{g})/(\mathbf{g} \otimes \mathrm{Ker}(\otimes^{s+1}\mathbf{g} \longrightarrow \Lambda^{s+1}\mathbf{g}))).$$

Here $\mathcal{F}^s_n = CL_{n-s}(\mathbf{g}, M) \otimes \mathrm{Ker}(\otimes^{s+2}\mathbf{g} \longrightarrow \Lambda^{s+2}\mathbf{g})$ is spanned by $(\omega, x, x, x_1, ..., x_i)$ with $i \leq s$. Thus the corresponding spectral sequence has the form :

$$E^1_{pq} = HL_q(\mathbf{g}, M) \otimes CR_p(\mathbf{g})$$

because there is a natural isomorphism

$$\ker(\otimes^{n+2}\mathbf{g} \longrightarrow \Lambda^{n+2}\mathbf{g})/\ker(\otimes^{n+2}\mathbf{g} \longrightarrow \mathbf{g} \otimes \Lambda^{n+1}\mathbf{g}) \\ \cong \ker(\mathbf{g} \otimes \Lambda^{n+1}\mathbf{g} \longrightarrow \Lambda^{n+2}\mathbf{g}).$$

Hence the theorem follows from Proposition 10.1.7 of [CH] and from the above formula for d.

Corollary 1.3. — *If $H_*(\mathbf{g}, M) = 0$, then $HL_*(\mathbf{g}, M) = 0$.*

Proof. — It follows from Proposition 1.1 that $HL_i(\mathbf{g}, M) = 0$ for $i = 0, 1$ and $HL_{n+2}(\mathbf{g}, M) \cong H^{\mathrm{rel}}_n(\mathbf{g}, M)$ for $n \geq 0$. We prove by induction that $H^{\mathrm{rel}}_n(\mathbf{g}, M) = 0$. It holds when $n = 0$ because $E^2_{pq} = 0$ for $q = 0, 1$ and Theorem A implies $H^{\mathrm{rel}}_n(\mathbf{g}, M) = 0$ for $n = 0, 1$. If we assume $H^{\mathrm{rel}}_i(\mathbf{g}, M) = 0$ for all $i \leq n$, then $HL_i(\mathbf{g}, M) = 0$ for $i \leq n + 2$. Thus $E^2_{pq} = 0$ for $q \leq n + 2$ and $H^{\mathrm{rel}}(\mathbf{g}, M) = 0$ for $i \leq n + 2$.

The cohomological versions of Theorem A and Corollary 3.3 are still true with the same proofs (of course now one will consider $C^*(\mathbf{g}, \mathbf{g}^*)$ instead of $C_*(\mathbf{g}, \mathbf{g})$, where $\mathbf{g}^*$ is the dual of $\mathbf{g}$). There are well-known examples which satisfy the assumption of Corollary 1.3 (see for example Theorem 1 of [D] and the main result of [T]).

In the next section we consider the case when $\mathbf{g}$ is a finite dimensional semi-simple Lie algebra over a characteristic 0 field. Now as a sample application of Theorem A we compute

$$HL_*\mathbf{g} := HL_*(\mathbf{g}, k)$$

where $\mathbf{g}$ is two or three dimensional Lie algebra defined over the complex numbers.

Examples 1.4. — i) Let $\mathbf{g} = \mathbf{r}_2, \mathbf{r}_3$ or $\mathbf{r}_{3,\lambda}$, where $\lambda \neq -1, \frac{1}{2}$. Here $\mathbf{r}_2$ is the nonabelian two dimensional Lie algebra, $\mathbf{r}_3$ and $\mathbf{r}_{3,\lambda}$ are three dimensional Lie algebras with

$$\mathbf{r}_3: \quad [x,y]=y, \quad [x,z]=y+z$$

$$\mathbf{r}_{3,\lambda}: \quad [x,y]=y, \quad [x,z]=\lambda z, \qquad 0<\mid \lambda \mid\leq 1.$$

Then $H_i(\mathbf{g})=0$ for $i>1$. Thus $HL_{n+2}(\mathbf{g}) \cong H_n^{\mathrm{rel}}(\mathbf{g}), n \geq 0$. By the direct calculations we obtain : $HR_i(\mathbf{g})=0$ for $i>0$ and dim $HR_0(\mathbf{g})=1$. It follows from Theorem A that $H_n^{\mathrm{rel}}(\mathbf{g}) \cong HL_n(\mathbf{g})$. Hence dim $HL_n(\mathbf{g})=1,\ n\geq 0$.

ii) Let $\mathbf{g}=\mathbf{r}_{3,\lambda}$ with $\lambda=-1$. In this case we still have $HR_i(\mathbf{g})=0$ for $i>0$ and $\dim HR_0(\mathbf{g})=1$, but now $\dim H_i(\mathbf{g})=1$, for $0\leq i\leq 3$, and $H_i(\mathbf{g})=0, i>3$. Easy calculation shows that the connected homomorphism $H_3(\mathbf{g}) \longrightarrow H_0^{\mathrm{rel}}(\mathbf{g})=HR_0(\mathbf{g})$ is zero. Thus for $x_n:=\dim HL_n(\mathbf{g})$ one has $x_0=1, x_1=1, x_n=2, n\geq 2$.

iii) Let $\mathbf{g}=\mathbf{r}_{3,\lambda}$ with $\lambda=\dfrac{1}{2}$. In this case $H_i(\mathbf{g})=0$ for $i>1$, dim $HR_i(\mathbf{g})=1$ for $i=0,1,2$ and $HR_i(\mathbf{g})=0$ for $i>2$. Thus $HL_{n+2}(\mathbf{g}) \cong H_n^{\mathrm{rel}}(\mathbf{g})$, $E^2_{pq}=0$ for $p>2$ and $E^2_{pq}=HL_q(\mathbf{g})$ for $p=0,1,2$. We claim that d_2 and hence all differentials are zero. Indeed, this follows from the fact that the class of $z\otimes(z\wedge x\wedge y)$ in $HR_2(\mathbf{g})$ is a nonzero element and if ω is some element of $CL_*(\mathbf{g})$ with $d\omega=0$ then $d\rho=0$. Here $\rho=(\omega,z,z,x,y)-(\omega,x,z,z,y)+(\omega,y,z,z,x)-(\omega,y,x,z,z)$. Thus $x_0=1, x_1=1, x_2=1, x_3=2, x_n=x_{n-2}+x_{n-3}+x_{n-4}, n\geq 4$, where $x_n=\dim HL_n(\mathbf{g})$.

iv) Let $\mathbf{g}=\mathbf{n}_3$ be the three dimensional nilpotent Lie algebra : $[x,y]=z$. Then dim $H_1(\mathbf{g})=2$, dim $H_2(\mathbf{g})=2$, dim $H_3(\mathbf{g})=1$ and dim $HR_0(\mathbf{g})=3$, dim $HR_1(\mathbf{g})=3$, dim $HR_2(\mathbf{g})=1$. The connected homomorphism $H_3(\mathbf{g}) \longrightarrow HR_0(\mathbf{g})$ as well as all differentials of the spectral sequence are still zero. To verify the last fact we remark that $z\otimes(z\wedge x\wedge y)$ gives a nonzero element in $HR_2(\mathbf{g})$ and if $\omega\in CL_*(\mathbf{g})$ is an element with $d\omega=0$, then $d\rho=0$, where $\rho=(\omega,z,z,x,y)-(\omega,x,z,z,y)+(\omega,y,z,z,x)+(\omega,x,y,z,z)$. Hence for $x_n:=\dim HL_n(\mathbf{g})$ we have $x_0=1, x_1=2, x_2=5, x_3=10$ and $x_n=3x_{n-2}+3x_{n-3}+x_{n-4},\ n\geq 4$.

2. Semi-simple Lie algebras as Leibniz algebras.

In this section we assume that char $k = 0$ and $\mathbf{g}$ is a finite dimensional semi-simple Lie algebra. Proposition 2.1 was proved independently by Ntolo [N]. She used Casimir element to construct the explicit homotopy.

PROPOSITION 2.1. — *Let M be a finite dimensional right $\mathbf{g}$-module and A be a finite dimensional corepresentation of $\mathbf{g}$ (see 1.5 and 1.11 of [LP]). Then*

$$HL_i(\mathbf{g}, M) = 0, \quad \text{if} \quad i > 0$$
$$HL_i(\mathbf{g}, A) = 0, \quad \text{if} \quad i > 1.$$

Proof. — It follows from the dual of Proposition 5.6 of [HS] and Corollary 1.3 that $HL_*(\mathbf{g}, M) = 0$ if M is a nontrivial simple $\mathbf{g}$-module. Thus, for arbitrary M, one has

$$HL_*(\mathbf{g}, M) = HL_*(\mathbf{g}, k) \otimes M_{\mathbf{g}}.$$

But $HL_{i+1}(\mathbf{g}, k) \cong HL_i(\mathbf{g}, \mathbf{g})$ (see Exercise E.10.6.1 of [CH] or take $C = k$ in Lemma 2.2) and $\mathbf{g}_{\mathbf{g}} = 0$. Thus $HL_i(\mathbf{g}, k) = 0$ for $i \geq 1$. This completes the proof in the case of right modules. Let A be a corepresentation. Then there exists an extension (compare with the corresponding extension for representations in 1.10 of [LP])

$$0 \longrightarrow B^s \longrightarrow A \longrightarrow C^a \longrightarrow 0$$

such that B^s (resp. C^a) is an (anti)symmetric corepresentation and the proposition follows from Lemma 2.2.

LEMMA 2.2. — *Let $\mathbf{h}$ be a Leibniz algebra. Let B^s (resp. C^a) be an (anti)symmetric corepresentation of $\mathbf{h}$. Let B and C be the underlying right $\mathbf{h}$-modules. Then $HL_*(\mathbf{h}, B^s) = HL_*(\mathbf{h}, B)$, $HL_0(\mathbf{h}, C^a) = C$ and*

$$HL_{i+1}(\mathbf{h}, C^a) \cong HL_i(\mathbf{h}, C \otimes \mathbf{h}),$$

where $C \otimes \mathbf{h}$ is considered as right $\mathbf{h}$-module by the action :

$$[c \otimes h, x] \; = \; -[x, c] \otimes h \; + \; c \otimes [h, x], \;\; h, x \in \mathbf{h}, c \in C.$$

Proof. — One checks that the isomorphisms exist on the level of chain complexes.

The cohomological version of Proposition 2.1 is still true. As for Lie algebras one verifies that for an arbitrary Leibniz algebra $\mathbf{h}$ the space $H^2(\mathbf{h},\mathbf{h})$ in the sense of 1.8 of [LP] is a space of obstructions for deformations in the category of Leibniz algebras. Thus we deduce from Proposition 2.1, Corollary 1.3 and [T] the following

PROPOSITION 2.3. — *Let $\mathbf{p}$ be a parabolic subalgebra of $\mathbf{g}$. Then $\mathbf{p}$ as well as $\mathbf{g}$ is rigid in the category of all Leibniz algebras.*

PROPOSITION 2.4. — *Let $f : \mathbf{h} \longrightarrow \mathbf{g}$ be an epimorphism from an arbitrary finite dimensional Leibniz algebra $\mathbf{h}$ to the semi-simple Lie algebra $\mathbf{g}$. Then f has a section.*

Proof. — Let $\mathbf{h}_{\mathrm{Lie}} = \mathbf{h}/([x,x] = 0)$ be the Liezation of $\mathbf{h}$. Let $\mathrm{ch}(\mathbf{h}) \in H^2(\mathbf{h}_{\mathrm{Lie}}, \mathbf{h}^{\mathrm{ann}})$ be the characteristic element of $\mathbf{h}$ (see 1.10 in [LP]), corresponding to the extension

$$0 \longrightarrow \mathbf{h}^{\mathrm{ann}} \longrightarrow \mathbf{h} \longrightarrow \mathbf{h}_{\mathrm{Lie}} \longrightarrow 0.$$

By definition f factors through $f_1 : \mathbf{h}_{\mathrm{Lie}} \longrightarrow \mathbf{g}$. It follows from Levi theorem that f_1 has a section s. By Proposition 2.1 we have

$$s^*(\mathrm{ch}(\mathbf{h})) \in HL^2(\mathbf{g}, \mathbf{h}^{\mathrm{ann}}) = 0.$$

Thus s has a lifting $\mathbf{g} \longrightarrow \mathbf{h}$.

3. Free Lie algebra approach.

Let $\mathbf{f}$ be a free Lie algebra and M be a right $\mathbf{f}$-module. Since $H_i(\mathbf{f}, -) = 0$, if $i > 1$, we deduce from Propositions 1.1 and 1.2 that $HL_{n+2}(\mathbf{f}, M) \cong H_n^{\mathrm{rel}}(\mathbf{f}, M), n \geq 0$; $HR_i(\mathbf{f}) = 0, i > 0$ and $HR_0(\mathbf{f}) \cong H_1(\mathbf{f}, \mathbf{f})$. Then Theorem A gives

$$H_n^{\mathrm{rel}}(\mathbf{f}, M) \cong H_1(\mathbf{f}, \mathbf{f}) \otimes HL_n(\mathbf{f}, M).$$

Thus we prove the following proposition.

Proposition 3.1. — *Let* $\mathbf{f}$ *be a free Lie algebra and* M *be a right* $\mathbf{f}$*-module. Then*

$$\begin{aligned} HL_n(\mathbf{f}, M) &\cong H_1(\mathbf{f}, \mathbf{f})^{\otimes k} \otimes H_0(\mathbf{f}, M), \quad if \quad n = 2k \\ &\cong H_1(\mathbf{f}, \mathbf{f})^{\otimes k} \otimes H_1(\mathbf{f}, M), \quad if \quad n = 2k+1. \end{aligned}$$

Theorem B. — *Let* $\mathbf{g}$ *be a Lie algebra and* M *be a right* $\mathbf{g}$*-module. Then there exists a spectral sequence* $E^r_{pq} \Longrightarrow HL_{p+q}(\mathbf{g}, M)$ *with*

$$\begin{aligned} E^2_{p,2k} &= \bigoplus HR_{i_1}(\mathbf{g}) \otimes \ldots \otimes HR_{i_k}(\mathbf{g}) \otimes H_0(\mathbf{g}, M) \\ E^2_{p,2k+1} &= \bigoplus HR_{i_1}(\mathbf{g}) \otimes \ldots \otimes HR_{i_k}(\mathbf{g}) \otimes H_{i_{k+1}+1}(\mathbf{g}, M) \end{aligned}$$

where the sum in the first (resp. second) formula is taken over all $i_1 + \ldots + i_k = p$ *(resp.* $i_1 + \ldots + i_{k+1} = p$*).*

Proof. — Let $\mathbf{f}_*$ be a componentwise free simplicial Lie algebra, such that $\pi_0 \mathbf{f}_* = \mathbf{g}$ and $\pi_i \mathbf{f}_* = 0, i > 0$. Such an object exists because of [Q] and we obtain a spectral sequence

$$E^1_{pq} = HL_q(\mathbf{f}_p, M) \Longrightarrow HL_{p+q}(\mathbf{g}, M)$$

because $H_*(\text{Tot } CL_*(\mathbf{f}_*, M)) \cong HL_*(\mathbf{g}, M)$. Thus Theorem B is a consequence of Proposition 3.1, Künneth relation and Lemma 3.2.

Lemma 3.2. — *One has*

$$\begin{aligned} \pi_i H_0(\mathbf{f}_*, M) &\cong H_0(\mathbf{g}, M), \quad if \quad i = 0 \quad and \quad = 0, \quad if \quad i > 0 \\ \pi_* H_1(\mathbf{f}_*, M) &\cong H_{*+1}(\mathbf{g}, M) \\ \pi_* H_1(\mathbf{f}_*, \mathbf{f}_*) &\cong HR_*(\mathbf{g}). \end{aligned}$$

Proof. — The first part follows from the fact that $H_0(\mathbf{f}_*, M)$ is a constant simplicial object. The second one is well-known (see for example [Q]). Since $H_*(\text{Tot } CR_*(\mathbf{f}_*)) \cong H_* CR_*(\mathbf{g})$, the last part is a consequence of Proposition 3.1.

4. The relation between $HL_*\mathbf{h}$ and $HL_*\mathbf{h}_{\mathrm{Lie}}$.

Let $f : \mathbf{h} \longrightarrow \mathbf{b}$ be an epimorphism of Leibniz algebras and let M be a right $\mathbf{b}$-module. We may consider it also as an $\mathbf{h}$-module through f. Let $C_*(\mathbf{h}; \mathbf{b}, M)$ be a chain complex, such that

$$CL_*(\mathbf{h}; \mathbf{b}, M)[1] = \mathrm{Ker}(CL_*(\mathbf{h}, M) \longrightarrow CL_*(\mathbf{b}, M))$$

and $HL_*(\mathbf{h}; \mathbf{b}, M) := H_*(CL_*(\mathbf{h}; \mathbf{b}, M))$. Then, by definition, we have

PROPOSITION 4.1. — *There exists an exact sequence*

$$0 \longleftarrow HL_1(\mathbf{b}, M) \longleftarrow HL_1(\mathbf{h}, M) \longleftarrow HL_0(\mathbf{h}; \mathbf{b}, M)$$
$$\longleftarrow HL_2(\mathbf{b}, M) \longleftarrow HL_2(\mathbf{h}, M) \longleftarrow ...$$

THEOREM C. — *Let*

$$0 \longrightarrow \mathbf{a} \longrightarrow \mathbf{h} \longrightarrow \mathbf{b} \longrightarrow 0$$

be a short exact sequence of Leibniz algebras. Assume $[h, a] = 0,\ h \in \mathbf{h}$, $a \in \mathbf{a}$. *Then there exists a spectral sequence*

$$E^2_{pq} = HL_p(\mathbf{b}, \mathbf{a}) \otimes HL_q(\mathbf{h}, M) \Longrightarrow HL_{p+q}(\mathbf{h}; \mathbf{b}, M)$$

where a right $\mathbf{b}$*-module structure on* $\mathbf{a}$ *is given by* $[a, x] := [a, f^{-1}(x)]$, $x \in \mathbf{b}, a \in \mathbf{a}$.

Proof. — Let $a \in \mathbf{a}$ and $\omega \in CL_*(\mathbf{h}, M)$. Then $d(\omega, a) = (d\omega, a)$, because $[\omega, a] = 0$. Thus

$$\mathcal{F}^0_* = CL_*(\mathbf{h}, M) \otimes \mathbf{a} \subset CL_*(\mathbf{h}; \mathbf{b}, M)$$

is a subcomplex and $H_*(\mathcal{F}^0_*) = H_*(\mathbf{h}, M) \otimes \mathbf{a}$. More generally : for $\rho = (x_1, ..., x_i)$ and $\rho_j = (x_1, ..., \hat{x}_j, ..., x_i),\ 1 \le j \le i$, one has

$$d(\omega, a, \rho) = (d\omega, a, \rho) + \sum(-1)^{\deg \omega + j}([\omega, x_j], a, \rho_j)$$
$$+ \sum(-1)^{\deg \omega + j}(\omega, [a, x_j], \rho_j) + (-1)^{\deg \omega}(\omega, a, d\rho).$$

Since $[a, x] \in \mathbf{a}$ we conclude that $\mathcal{F}^0_* \subset \mathcal{F}^1_* \subset ...$ are subcomplexes of $CL_*(\mathbf{h}; \mathbf{b}, M)$ with $\mathcal{F}^n_n = CL_n(\mathbf{h}; \mathbf{b}, M)$. Here $\mathcal{F}^s_*$ is spanned by (ω, a, ρ) with $\deg \rho \le s$. One checks that there exists an isomorphism

$$\mathcal{F}^s_* / \mathcal{F}^{s-1}_* \cong CL_*(\mathbf{h}; \mathbf{b}, M)[-s] \otimes \mathbf{a} \otimes \mathbf{b}^{\otimes s}$$

and the theorem follows from 10.1.7. of [CH] and the above formula for d.

Remark 4.2. — The assumptions of Theorem C hold for

$$0 \longrightarrow \mathbf{h}^{\mathrm{ann}} \longrightarrow \mathbf{h} \longrightarrow \mathbf{h}_{\mathrm{Lie}} \longrightarrow 0.$$

Hence Theorem C gives a relation between $HL_*\mathbf{h}$ and $HL_*\mathbf{h}_{\mathrm{Lie}}$. Since $HL_1(\mathbf{h},k) = HL_1(\mathbf{h}_{\mathrm{Lie}},k)$, we obtain an exact sequence

$$0 \longleftarrow (\mathbf{h}^{\mathrm{ann}})_{\mathbf{h}_{\mathrm{Lie}}} \longleftarrow HL_2(\mathbf{h}_{\mathrm{Lie}},k) \longleftarrow HL_2(\mathbf{h},k).$$

PROPOSITION 4.3. — *Let* $\mathbf{h}$ *be a finite dimensional Leibniz algebra, such that* $\mathbf{g} = \mathbf{h}_{\mathrm{Lie}}$ *is a semi-simple Lie algebra over characteristic zero field or is a nilpotent Lie algebra with* $\mathrm{ch}(\mathbf{h}) = 0$. *Then* $HL_*(\mathbf{h},k) = HL_*(\mathbf{g},k)$.

Proof. — By Proposition 2.1 $\mathrm{ch}(\mathbf{h}) = 0$ in both cases. Hence the epimorphism $\mathbf{h} \longrightarrow \mathbf{g}$ has a section. It follows from Proposition 4.1 and Remark 4.2 that

$$HL_*(\mathbf{h},k) = HL_*(\mathbf{g},k) \oplus HL_{*-1}(\mathbf{h};\mathbf{g},k), \quad \text{and} \quad (\mathbf{h}^{\mathrm{ann}})_{\mathbf{g}} = 0.$$

Theorem C gives the spectral sequence

$$E^2_{pq} = HL_p(\mathbf{g},\mathbf{h}^{\mathrm{ann}}) \otimes HL_q(\mathbf{h},k) \Longrightarrow HL_{p+q}(\mathbf{h};\mathbf{g},k).$$

By Theorem 1 of [D], Corollary 1.3 and Proposition 2.1 we have $HL_*(\mathbf{g},\mathbf{h}^{\mathrm{ann}}) = 0$. Thus $HL_*(\mathbf{h};\mathbf{g},k) = 0$, because $E^2_{*q} = 0$ and $HL_*(\mathbf{h},k) = HL_*(\mathbf{g},k)$.

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