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EXTENDING TAMM’S THEOREM

by L. van den DRIES & C. MILLER

Introduction.

The theorem of M. Tamm [T] referred to in the title of this paper can be stated as follows:

Given a finitely subanalytic function $f : U \to \mathbb{R}$ on an open set $U \subseteq \mathbb{R}^n$, there is a natural number $N$ such that for all open $U' \subseteq U$, if $f \upharpoonright U'$ is $C^N$, then $f \upharpoonright U'$ is analytic.

(Here and throughout this paper, “analytic” means “real analytic”.)

“Finitely subanalytic” [D2] is the same as “globally subanalytic” [KR], and is a better behaved notion than “subanalytic”. We give several definitions of “finitely subanalytic” below. Here we just mention that bounded subanalytic sets in $\mathbb{R}^n$ as well as their complements are finitely subanalytic. (A map $f : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^m$ is finitely subanalytic if its graph is a finitely subanalytic subset of $\mathbb{R}^{m+n}$.)

In this paper we extend Tamm’s theorem simultaneously in two ways:

(1) We allow $U$ and $f$ to depend on parameters, with an $N$ independent of the parameters.

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(2) We allow \( f \) to be definable, not just in terms of addition, multiplication, and analytic functions on sets \([-1,1]^m\) for \( m \in \mathbb{N} \) — this would give us just the finitely subanalytic functions — but also in terms of the power functions \( x \mapsto x^r : (0,\infty) \to \mathbb{R}, \) which are not subanalytic at 0 for irrational \( r. \)

In (2) above, "definable" is a certain technical notion arising from logic; we introduce it without referring explicitly to logical concepts.

**DEFINITION.** — A structure \( S \) on \( \mathbb{R} \) consists of a collection \( S_n \) of subsets of \( \mathbb{R}^n, \) for each \( n \in \mathbb{N}, \) such that

1. \( S_n \) is a boolean algebra of subsets of \( \mathbb{R}^n, \) in particular \( \mathbb{R}^n \in S_n; \)
2. \( S_n \) contains the diagonals \( \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_i = x_j\} \) for \( 1 \leq i < j \leq n; \)
3. if \( A \in S_n, \) then \( A \times \mathbb{R} \) and \( \mathbb{R} \times A \) belong to \( S_{n+1}; \)
4. if \( A \in S_{n+1}, \) then \( \pi(A) \in S_n, \) where \( \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n \) is the projection on the first \( n \) coordinates.

We say that a set \( A \subseteq \mathbb{R}^n \) belongs to \( S \) if \( A \in S_n, \) and that a map \( f : A \to \mathbb{R}^k \) with \( A \subseteq \mathbb{R}^n \) belongs to \( S \) if its graph \( \Gamma(f) := \{(x,f(x)) \in \mathbb{R}^{n+k} : x \in A\} \) belongs to \( S. \) Instead of "\( A \) belongs to \( S\)" we also say "\( S \) contains \( A\);" (similarly with maps).

Given structures \( S = (S_n) \) and \( S' = (S'_n) \) on \( \mathbb{R} \) we put \( S \subseteq S' \) if \( S_n \subseteq S'_n \) for all \( n \in \mathbb{N}; \) this defines a partial order on the set of all structures on \( \mathbb{R}. \) Given sets \( A_i \subseteq \mathbb{R}^{m(i)} \) (i in some index set \( I), \) and functions \( f_j : B_j \to \mathbb{R} \) with \( B_j \subseteq \mathbb{R}^{n(j)} \) (j in some index set \( J), \) there is clearly a smallest structure on \( \mathbb{R} \) containing all sets \( A_i \) and all functions \( f_j; \) we call this the structure on \( \mathbb{R} \) generated by the \( A_i \)'s and the \( f_j \)'s. (A function \( f : \mathbb{R}^0 = \{0\} \to \mathbb{R} \) is identified with the corresponding real constant \( f(0).\)

A set \( A \subseteq \mathbb{R}^n \) is said to be definable in terms of the \( A_i \)'s and the \( f_j \)'s, or to be definable in \( (\mathbb{R}, (A_i)_{i \in I}, (f_j)_{j \in J}), \) if \( A \) belongs to the structure on \( \mathbb{R} \) generated by the \( A_i \)'s and the \( f_j \)'s; (similarly with maps). For example, by Tarski-Seidenberg, a set \( X \subseteq \mathbb{R}^n \) is definable in \( (\mathbb{R}, +, \cdot, (r)_{r \in \mathbb{R}}) \) if and only if \( X \) is semialgebraic.

These notions all make sense with \( \mathbb{R} \) replaced by any set. However, of special interest for analysis and topology are the "\( o \)-minimal" structures on \( \mathbb{R}, \) which are the simplest structures on \( \mathbb{R} \) compatible with the ordering of the real line.
DEFINITION. — A structure $S$ on $\mathbb{R}$ is o-minimal ("order-minimal") if

(S1) $\{(x, y) : x < y\} \in S_2$, and $\{a\} \in S_1$ for each $a \in \mathbb{R}$;

(S2) each set in $S_1$ is a finite union of intervals $(a, b)$, $-\infty \leq a < b \leq +\infty$, and points $\{a\}$.

(We think of (S2) as a minimality requirement, since each structure on $\mathbb{R}$ satisfying (S1) must contain at least all finite unions of intervals and points.) If an o-minimal structure $S$ is generated by sets $A_i \subseteq \mathbb{R}^{m(i)}$ ($i$ in some index set $I$) and functions $f_j : B_j \to \mathbb{R}$ with $B_j \subseteq \mathbb{R}^{n(j)}$ ($j$ in some index set $J$), then we also say that $(\mathbb{R}, (A_i)_{i \in I}, (f_j)_{j \in J})$ is o-minimal.

Each subset of $\mathbb{R}^n$ belonging to an o-minimal structure $S$ on $\mathbb{R}$ has only finitely many connected components, and each component also belongs to $S$. The class of semialgebraic sets is an o-minimal structure on $\mathbb{R}$, as is the larger class of finitely subanalytic sets: $B \subseteq \mathbb{R}^n$ is finitely subanalytic if and only if $B = f(A)$ for some bounded semianalytic set $A \subseteq \mathbb{R}^m$ and some semialgebraic map $f : \mathbb{R}^m \to \mathbb{R}^n$. (A map from a subset of $\mathbb{R}^m$ into $\mathbb{R}^n$ is semialgebraic if its graph is a semialgebraic subset of $\mathbb{R}^{m+n}$; unlike some authors, we do not require semialgebraic maps to be continuous.)

DEFINITION. — A structure on $(\mathbb{R}, +, \cdot)$ is a structure on $\mathbb{R}$ containing the graphs of both addition and multiplication.

Let $S$ be a structure on $(\mathbb{R}, +, \cdot)$. Then the usual order relation $<$ necessarily belongs to $S$; the set $\{(x, y) \in \mathbb{R}^2 : x \leq y\}$ is the projection of

$\{(x, y, z) \in \mathbb{R}^3 : y = x + z^2\}$,

and $\{(x, y) \in \mathbb{R}^2 : x < y\} = \{(x, y) \in \mathbb{R}^2 : x \leq y\} - \{(x, y) \in \mathbb{R}^2 : x = y\}$.

Given a set $X \in S_n$, its closure and interior are also in $S_n$. Given a function $f : U \to \mathbb{R}$ belonging to $S$ with $U$ open in $\mathbb{R}^n$, the set of points in $U$ where $f$ is differentiable belongs to $S$, and if $f$ is differentiable on $U$, then each partial derivative also belongs to $S$. Throughout this paper, we use many such basic facts (familiar to logicians); proofs are left as exercises.

An o-minimal structure on $(\mathbb{R}, +, \cdot)$ shares many of the nice properties of the class of semialgebraic sets; the sets in such a structure can be triangulated by means of homeomorphisms in the structure, and Hardt's semialgebraic triviality theorem [H] extends to such o-minimal structures on $\mathbb{R}$. The theory of o-minimal structures is a wide-ranging generalization of semialgebraic and subanalytic geometry; one can view the subject as a
realization of Grothendieck's idea of topologie modéré, (outlined in the unpublished notes Esquisse d'un programme, 1984). The first papers on o-minimality are [D1], [PS] and [KPS]; for an extensive and systematic account, see [D3].

We now return to the subject of this paper.

Notation. — Given a subfield $K$ of $\mathbb{R}$, $\mathbb{R}^K$ denotes the set $\mathbb{R}$ equipped with

1. addition and multiplication (functions on $\mathbb{R}^2$),
2. all analytic functions $f : [-1,1]^m \rightarrow \mathbb{R}$, for all $m \in \mathbb{N}$,
3. the power functions $x \mapsto x^r : (0,\infty) \rightarrow \mathbb{R}$ for all $r \in K$.

Convention. — In this paper we say that $f : A \rightarrow B$ with $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ is analytic if $f$ is the restriction to $A$ of an analytic map $g : U \rightarrow \mathbb{R}^n$ with $U$ an open neighborhood of $A$ in $\mathbb{R}^m$ and $g(A) \subseteq B$. We also say that such a map $f$ is analytic at a point $a \in A$ if there is an open set $U \subseteq \mathbb{R}^n$ with $a \in U \subseteq A$ such that $f \upharpoonright U$ is analytic; (note then that $a \in \text{int}(A)$). We also work similarly with "analytic" replaced by "$C^p$", $1 \leq p \leq \infty$.

The sets definable in $\mathbb{R}^K_{an}$ form an o-minimal structure on $(\mathbb{R},+,-)$, and some basic properties of this structure are established in [M2].

For $K = \mathbb{Q}$ the sets definable in $\mathbb{R}^K_{an}$ are exactly the finitely subanalytic sets (see [DD], [D2]), and in fact the power functions $x^q$ for $q \in \mathbb{Q}$ are superfluous here, since they are definable in terms of just multiplication.

We can now give a precise formulation of our extension of Tamm's theorem:

**Main Theorem.** Let $f : A \rightarrow \mathbb{R}$ be definable in $\mathbb{R}^K_{an}$, $A \subseteq \mathbb{R}^{m+n}$. Then there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$ and all open sets $U \subseteq \mathbb{R}^n$ with $U \subseteq A_x := \{y \in \mathbb{R}^n : (x,y) \in A\}$, if $f(x,-)$ is $C^N$ on $U$, then $f(x,-)$ is analytic on $U$.

(We let $f(x,-)$ denote the function $y \mapsto f(x,y) : A_x \rightarrow \mathbb{R}$.)

**Corollary.** Let $A \subseteq \mathbb{R}^n$ be definable in $\mathbb{R}^K_{an}$. Then $\text{Sing}(A)$, the set of singular points of $A$, is definable in $\mathbb{R}^K_{an}$.

(See §5 for a definition of $\text{Sing}(A)$).
We cannot follow here Tamm's original proof [T], nor the proof by Bierstone and Milman [BM], since these depend on properties of subanalytic sets not shared by all sets definable in $\mathbb{R}^K_{\text{an}}$ if $K \neq \mathbb{Q}$. Instead we adapt (and simplify in some places) the proof of Tamm's theorem given by Kurdyka [K]. One important tool used in [K] is Pawlucki's "Puiseux expansion with parameters for subanalytic functions" from [P]. Much of the technical work in this paper goes into establishing the Expansion Theorem of §4, which for $K = \mathbb{Q}$ is a somewhat stronger version of Pawlucki's result.

Here then is a brief outline of the contents of this paper. In §1 we review some basic properties of o-minimal structures needed for our purpose. In §2, we discuss Gateaux differentiability and its relation to analyticity and o-minimality. In §3, some results about $\mathbb{R}^K_{\text{an}}$ are given. The statement and proof of the aforementioned Expansion Theorem constitutes §4. Finally, in §5, we prove the Main Theorem and some corollaries.

1. o-minimal structures on $\mathbb{R}$.

Throughout this section, $S$ denotes some fixed, but arbitrary, o-minimal structure on $\mathbb{R}$. "Definable" means "belonging to $S$".

1.1. MONOTONICITY THEOREM. — Let $f : \mathbb{R} \to \mathbb{R}$ be definable. Then there exist (extended) real numbers $-\infty = a_0 < a_1 < \ldots < a_N < a_{N+1} = +\infty$ such that $f \upharpoonright (a_n, a_{n+1})$ is either constant, or strictly monotone and continuous, for $n = 0, \ldots, N$.

(See [D1] for a proof.)

Remarks.

(1) The statement holds with "differentiable" instead of "continuous" if $S$ is an o-minimal structure on $\langle \mathbb{R}, +, \cdot \rangle$; (see [D1]). Consequently, the ring of germs at $+\infty$ of all definable functions $f : \mathbb{R} \to \mathbb{R}$ is a Hardy field. The converse is also true: a structure $\mathcal{R}$ on $\langle \mathbb{R}, +, \cdot \rangle$ containing all singletons $\{r\}$ for $r \in \mathbb{R}$ is o-minimal if every function $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{R}$ is of constant sign ($-1$, 0 or 1) for all sufficiently large (depending on $f$) positive real arguments; (see [DMM]).

(2) For every presently-known o-minimal structure on $\langle \mathbb{R}, +, \cdot \rangle$, the statement holds true with "analytic" in place of "continuous".
Cells and cell decomposition.

We define the cells in $\mathbb{R}^n$ as certain kinds of definable subsets of $\mathbb{R}^n$; the definition is by induction on $n$:

1. The cells in $\mathbb{R}$ ($= \mathbb{R}^1$) are just the points $\{r\}$ and the open intervals $(a, b)$, $-\infty < a < b < +\infty$;

2. Let $C \subseteq \mathbb{R}^n$ be a cell and let $f, g : C \to \mathbb{R}$ be definable continuous functions such that $f < g$ on $C$, then $((f, g)) := \{(x, r) \in C \times \mathbb{R} : f(x) < r < g(x)\}$ is a cell in $\mathbb{R}^{n+1}$; also, given definable continuous $f : C \to \mathbb{R}$ on a cell $C$ in $\mathbb{R}^n$, the graph $\Gamma(f) \subseteq C \times \mathbb{R}$ and the sets $\{(x, r) \in C \times \mathbb{R} : r < f(x)\}$, $\{(x, r) \in C \times \mathbb{R} : f(x) < r\}$ and $C \times \mathbb{R}$ are cells in $\mathbb{R}^{n+1}$.

(We also consider $\mathbb{R}^0 = \{0\}$ as a cell in $\mathbb{R}^0$; so (2) even holds for $n = 0$.)

The dimension of a cell $C$ in $\mathbb{R}^n$, denoted $\text{dim}(C)$, is defined by induction on $n$:

1. For $n = 1$, put $\text{dim}(C) := 0$ if $C$ is a singleton, and put $\text{dim}(C) := 1$ if $C$ is an open interval.

2. Let $C$ be a cell in $\mathbb{R}^{n+1}$. Then $\pi(C)$ is a cell in $\mathbb{R}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ is the projection on the first $n$ coordinates. Put $\text{dim}(C) := \text{dim}(\pi(C))$ if $C$ is of the form $\Gamma(f)$ for some definable continuous $f : \pi(C) \to \mathbb{R}$, and put $\text{dim}(C) := 1 + \text{dim}(\pi(C))$ otherwise.

(We also put $\text{dim}(\mathbb{R}^0) := 0$.)

Note. — Clearly, if $C$ is a cell in $\mathbb{R}^n$ and $C$ is open, then $\text{dim}(C) = n$.

1.2. Given $i = (i_1, \ldots, i_m)$ with $1 \leq i_1 < \cdots < i_m \leq n$, define $\pi_i : \mathbb{R}^n \to \mathbb{R}^m$ by $\pi_i(x_1, \ldots, x_n) := (x_{i_1}, \ldots, x_{i_m})$. It is easy to check that if $C$ is a cell in $\mathbb{R}^n$ of dimension $m$, then there is some $i = (i_1, \ldots, i_m)$ as above such that $\pi_i$ maps $C$ homeomorphically onto an open cell in $\mathbb{R}^m$. Note also that $\pi_i \upharpoonright C$ is definable.

A decomposition of $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into finitely many cells. Definition is by induction on $n$:

1. A decomposition of $\mathbb{R}^1$ ($= \mathbb{R}$) is a collection of intervals and points of the form $$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\},$$ with $a_1 < \ldots < a_k$ real numbers. (For $k = 0$ this is just $\{(-\infty, \infty)\}$.)

2. A decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into cells $A$ such that the set of projections $\pi(A)$ is a decomposition of $\mathbb{R}^n$, where
π : \mathbb{R}^{n+1} \to \mathbb{R}^n is the projection on the first n coordinates. (Note that different cells can have the same image under π.)

In a similar manner, one can define \( C^p \) cells and \( C^p \) decompositions, by requiring that the functions occurring in part (2) of the definition of cells be \( C^p \), for \( p \) a positive integer or \( p = \infty \); similarly for analytic cells and analytic decompositions. Each \( C^p \) cell in \( \mathbb{R}^n \) is a connected \( C^p \) submanifold of \( \mathbb{R}^n \), \( C^p \) diffeomorphic via some coordinate projection \( π_i \mid C \) to an open \( C^p \) cell in \( \mathbb{R}^m \), for some \( m \leq n \); similarly with \( "C^p" \) replaced by \("analytic"\).

**Note.** — Cells and decompositions are always relative to some particular structure; (the structure \( S \) throughout this section).

The projection \( πC \) of a decomposition \( C \) of \( \mathbb{R}^{m+n} \) onto \( \mathbb{R}^m \) is the collection \( \{π(C) : C \in C\} \), where \( π : \mathbb{R}^{m+n} \to \mathbb{R} \) is the projection map onto the first \( m \) coordinates. (Note that \( πC \) is then a decomposition of \( \mathbb{R}^m \).) A decomposition of \( \mathbb{R}^n \) is said to **partition** a set \( A \subseteq \mathbb{R}^n \) if \( A \) is a union of cells in the decomposition.

**THEOREM.** — The structure \( S \) admits cell decomposition; i.e.,

(I) given definable sets \( A_1, \ldots, A_k \subseteq \mathbb{R}^n \), there is a decomposition of \( \mathbb{R}^n \) into cells partitioning \( A_1, \ldots, A_k \),

(II) for every definable function \( f : A \to \mathbb{R} \), \( A \subseteq \mathbb{R}^n \), there is a decomposition of \( \mathbb{R}^n \) into cells partitioning \( A \) such that each restriction \( f \mid C : C \to \mathbb{R} \) is continuous for each cell \( C \subseteq A \) in the decomposition.

(See [PS] and [KPS].)

**Remark.** — If \( S \) is moreover a structure on \((\mathbb{R}, +, \cdot)\), then the statement holds with \("C^N\) cells” and \("C^N\)” in place of “cells” and “continuous”, respectively, for every fixed positive integer \( N \); i.e., \( S \) admits \( C^N \) cell decomposition. It is an open question at present as to whether or not every \( o\)-minimal structure on \((\mathbb{R}, +, \cdot)\) admits \( C^\infty \) cell decomposition, or even analytic cell decomposition.

**Orders of growth of definable functions.**

A structure \( \mathcal{R} \) on \( \mathbb{R} \) is exponential if the exponential function \( e^x \) belongs to \( \mathcal{R} \); \( \mathcal{R} \) is polynomially bounded if for every function \( f : \mathbb{R} \to \mathbb{R} \) belonging to \( \mathcal{R} \), there exists some \( N \in \mathbb{N} \) such that ultimately \( |f(x)| \leq x^N \). (Ultimately abbreviates “for all sufficiently large positive arguments”.) If \( \mathcal{R} \) is generated by sets \( A_i \subseteq \mathbb{R}^{m(i)} \) (\( i \) in some index set \( I \)) and functions
Let $f_j : B_j \to \mathbb{R}$ with $B_j \subseteq \mathbb{R}^{n(j)}$ ($j$ in some index set $J$), then we also say that $(\mathbb{R}, (A_i)_{i \in I}, (f_j)_{j \in J})$ is exponential if $\mathcal{R}$ is exponential; similarly for polynomially bounded.

1.3. **Theorem (Growth Dichotomy).** — Let $\mathcal{R}$ be an o-minimal structure on $(\mathbb{R}, +, \cdot)$. Then either $\mathcal{R}$ is exponential, or $\mathcal{R}$ is polynomially bounded. If $\mathcal{R}$ is polynomially bounded, then for every $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{R}$, either $f$ is ultimately identically equal to 0, or there exist nonzero $c \in \mathbb{R}$ and a real power function $x^r$ belonging to $\mathcal{R}$ such that $f(x) = cx^r + o(x^r)$ as $x \to +\infty$.

(See [M1] for the proof.)

The first known example of an exponential o-minimal structure on $(\mathbb{R}, +, \cdot)$ is due to Wilkie [W], who established that the structure on $\mathbb{R}$ generated by addition, multiplication, all real constants, and exponentiation is o-minimal. The structure on $\mathbb{R}$ generated by addition, multiplication, exponentiation and all analytic functions $f : [-1, 1]^m \to \mathbb{R}$ for all $m \in \mathbb{N}$, is o-minimal and admits analytic cell decomposition; (see [DM] and [DMM]).

**Polynomially bounded o-minimal structures on $(\mathbb{R}, +, \cdot)$.**

We will be particularly concerned in this paper with the polynomially bounded case. For the remainder of this section, we assume that $\mathcal{S}$ is a polynomially bounded o-minimal structure on $(\mathbb{R}, +, \cdot)$.

The following variant of a result from [M2] is crucial to later developments:

1.4. **Theorem (Piecewise Uniform Asymptotics).** — Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^m$. Then there exist $r_1, \ldots, r_\ell \in \mathbb{R}$ such that for all $x \in A$, either $t \mapsto f(x, t) : \mathbb{R} \to \mathbb{R}$ vanishes identically for all sufficiently small (depending on $x$) positive $t$, or $f(x, t) = ct^{r_i} + o(t^{r_i})$ as $t \to 0^+$ for some $i \in \{1, \ldots, \ell\}$ and $c = c(x) \in \mathbb{R}$, $c \neq 0$.

**Remark.** — A "definable" version of the Lojasiewicz inequality follows from this fact; (see [M2]).

Let $U$ be an open subset of $\mathbb{R}^n$, $a \in U$, and let $f : U \to \mathbb{R}$ be given. If $f$ is $C^N$ at $a$ and all partial derivatives of $f$ of order less than or equal to $N$ vanish at $a$, then $f$ is said to be $N$-flat at $a$. If $f$ is $N$-flat at $a$ for all $N \in \mathbb{N}$ then $f$ is said to be flat at $a$. 
1.5. Theorem (Uniform Bounds on Orders of Vanishing). — Let $f : A \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^{m+n}$. Then there exists $N \in \mathbb{N}$ such that for all $(x,y) \in A$, if $y \in \text{int}(A_x)$ and $f(x,-)$ is $N$-flat at $y$, then $f(x,z) = 0$ for all $z \in A_x$ sufficiently close to $y$.

(See [M3] for the proof.)

In the special case that $m = 0$ and $A$ is open, we have that for all $y \in A$, if $f$ is flat at $y$, then $f$ vanishes identically in a neighborhood of $y$. It follows easily then that the set of all definable $C^\infty$ functions $f : U \to \mathbb{R}$, for a fixed connected definable open set $U \subseteq \mathbb{R}^n$, is an integral domain; we denote it by $C^\infty_{\text{df}}(U)$. Furthermore, $C^\infty_{\text{df}}(U)$ is a quasianalytic class; i.e., if $f \in C^\infty_{\text{df}}(U)$ and $f$ is flat at some $x_0 \in U$, then $f = 0$.

The descending chain condition on zero sets.

Given $f : A \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$, put $Z(f) := \{a \in A : f(a) = 0\}$. Note that if $f$ is definable, then so is $Z(f)$.

1.6. Proposition. — Assume that $S$ admits $C^\infty$ cell decomposition. Then given a family $(f_i : A \to \mathbb{R})_{i \in \mathbb{N}}$ of definable $C^\infty$ functions, $A \subseteq \mathbb{R}^n$, there exists $M \in \mathbb{N}$ such that

$$\bigcap_{i \in \mathbb{N}} Z(f_i) = \bigcap_{i \leq M} Z(f_i).$$

Proof. — To avoid trivialities, let us suppose that $\emptyset \neq Z(f_0) \neq A$. By taking a $C^\infty$ decomposition of $\mathbb{R}^n$ partitioning $A$, we may assume that $A$ is a $C^\infty$ cell; in particular, $A$ is connected. We proceed now by induction on $\dim(A)$ and $n$.

The result is trivial if $\dim(A) = 0$. So suppose that $\dim(A) = d > 0$, and that the result holds for all lower values of $d$ and $n$.

If $A$ is nonopen, then $A$ is $C^\infty$ diffeomorphic via some coordinate projection $\pi = \pi_i : A$ to an open cell $\pi(A) \subseteq \mathbb{R}^m$ with $m < n$; (see 1.2). By the inductive assumption, we have

$$\bigcap_{i \in \mathbb{N}} Z(f_i \circ \pi^{-1}) = \bigcap_{i \leq M} Z(f_i \circ \pi^{-1})$$

for some $M \in \mathbb{N}$; thus,

$$\bigcap_{i \in \mathbb{N}} Z(f_i) = \bigcap_{i \leq M} Z(f_i),$$
as desired.

Now suppose that $A$ is open. Take a partition $\mathcal{P}$ of $Z(f_0)$ into finitely many $C^\infty$ cells $B$; note that $\dim(B) < d$, since otherwise $f_0$ would vanish on a nonempty open subset of $A$, hence $f_0 = 0$ (by quasianalyticity). By the inductive assumption, for each $B \in \mathcal{P}$ there exists $M(B) \in \mathbb{N}$ such that

$$\bigcap_{i \in \mathbb{N}} Z(f_i \mid B) = \bigcap_{i \leq M(B)} Z(f_i \mid B).$$

Hence,

$$\bigcap_{i \in \mathbb{N}} Z(f_i) = \bigcap_{i \leq M} Z(f_i),$$

where $M := \max\{M(B) : B \in \mathcal{P}\}$.

Remark. — The assumption that $S$ is polynomially bounded and admits $C^\infty$ cell decomposition may be removed if one assumes that $A$ is a definable analytic submanifold of $\mathbb{R}^n$ and that each $f_i$ is analytic; (see Tougeron [To]).

2. Gateaux differentiability, analyticity and o-minimality.

In this section, we give a characterization of analyticity (at a point) for real functions that is a slight variant of a result of Bochnak and Siciak [BS].

First, we reformulate a result of Abhyankar and Moh on power series:

2.1. Proposition. — Let $F(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ and suppose that for all $x \in \mathbb{R}^n$ the series $F(x_1 T, \ldots, x_n T) \in \mathbb{R}[T]$ is convergent. Then $F(X_1, \ldots, X_n)$ is convergent.

Proof. — We proceed by induction on $n$; the case $n = 1$ is trivial. Assume the result for $n$. Let $F(X_1, \ldots, X_{n+1}) \in \mathbb{R}[X_1, \ldots, X_{n+1}]$, and suppose that for all $x_1, \ldots, x_{n+1} \in \mathbb{R}$, the series $F(x_1 T, \ldots, x_{n+1} T) \in \mathbb{R}[T]$ is convergent. Let $r \in \mathbb{R}$, and $x \in \mathbb{R}^n$. Then the series $F(x_1 T, \ldots, x_n T, r x_n T)$ is convergent. By the inductive assumption, the series $F(X_1, \ldots, X_n, r X_n)$ is convergent. It follows then from [AM] that $F(X_1, \ldots, X_n)$ is convergent.
Defining Gateaux differentiability for a function $f : U \to \mathbb{R}$ at a point $x$ in the open set $U$, we say that $f$ is Gateaux differentiable at $x$ if there exists a linear map $Df(x) : \mathbb{R}^n \to \mathbb{R}$ such that for all $y \in \mathbb{R}^n$, the limit
$$
\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}
$$
exists. If this limit exists for all $y$, we say that $f$ is Gateaux differentiable on $U$, denoted $f \in G^1(U)$. This is a generalization of the notion of directional derivative.

For $f$ and $x$ as in the preceding definition, if $f$ is $C^k$ at $x$, then $f$ is Gateaux differentiable at $x$, and indeed,
$$
D^k f(x)(y) = \frac{d^k}{dt^k} f(x + ty) \bigg|_{t=0}
$$
exists for all $y$. The converse fails; indeed, $f$ can be Gateaux differentiable at a point $x$, and yet not even be continuous at $x$. (For example, consider the characteristic function of $\{(x, x^2) : x > 0\}$, which is $G^\infty$ at $(0,0)$.)

Notation. — For $x \in \mathbb{R}^n$, $\|x\|$ denotes the usual euclidean norm of $x$.

2.2. Proposition. — Let $U \subseteq \mathbb{R}^n$ be open, let $x \in U$. Then $f : U \to \mathbb{R}$ is analytic at $x$ if and only if $f$ is Gateaux differentiable at $x$, and there exists $\varepsilon > 0$ such that for all $y \in \mathbb{R}^n$ with $\|y\| \leq 1$, the function $t \mapsto f(x + ty)$ is defined and analytic on $(-\varepsilon, \varepsilon)$.

Proof. — The forward implication is clear. For the other direction, it suffices to show the result for $U$ a neighborhood of 0, with $x = 0$ and $f(0) = 0$.

Since $f$ is Gateaux differentiable at 0, for all $k > 0$ the function $\delta_k : \mathbb{R}^n \to \mathbb{R}$ defined by
$$
\delta_k(y) := \frac{d^k}{dt^k} f(ty) \bigg|_{t=0}
$$
is given by a homogeneous real polynomial $\delta_k(Y_1, \ldots, Y_n)$ of degree $k$. Put
$$
F(Y_1, \ldots, Y_n) := \sum_{k=1}^{\infty} (1/k!) \delta_k(Y_1, \ldots, Y_n) \in \mathbb{R}[Y_1, \ldots, Y_n];
$$
(the “Taylor series” of $f$ at 0).

Let $y \in \mathbb{R}^n$, $\|y\| \leq 1$. Then, for the formal series $F$, we have
$$
F(y_1 T, \ldots, y_n T) = \sum_{k=1}^{\infty} (1/k!) \delta_k(y_1 T, \ldots, y_n T) = \sum_{k=1}^{\infty} (1/k!) \delta_k(y) T^k \in \mathbb{R}[T].
$$
Now there exists $\varepsilon > 0$ such that $f(ty)$ is defined and
$$
f(ty) = \sum_{k=1}^{\infty} (1/k!) \delta_k(y) t^k
$$
for all $|t| < \varepsilon$. Thus, $F(y_1T, \ldots, y_nT)$ is convergent. By the previous proposition, $F(Y_1, \ldots, Y_n)$ is convergent, say on some open neighborhood $V \subseteq (-\varepsilon, \varepsilon)^n$ of $0 \in \mathbb{R}^n$. Let $F$ also denote the analytic function on $V$ thus obtained. Then for every line $L \subseteq \mathbb{R}^n$ through the origin, we have $f \upharpoonright (V \cap L) = F \upharpoonright (V \cap L)$. Hence, $f \upharpoonright V = F \upharpoonright V$, and $f$ is analytic at $0$. \hfill \Box

We will need the following fact; (the proof is left to the reader).

2.3. Let $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$ there exist points $p(k,1), \ldots, p(k,\mu(k)) \in \mathbb{R}^n$ and linear functions $a_1, \ldots, a_{\mu(k)} : \mathbb{R}^{\mu(k)} \to \mathbb{R}$ such that for all $x \in \mathbb{R}^{\mu(k)}$,

$$P_k(x,Y) := \sum_{j=1}^{\mu(k)} a_j(x)M_j(Y) \in \mathbb{R}[Y]$$

is the unique homogeneous real polynomial $P(Y)$ of degree $k$ with $P(p(k,i)) = x_i$ for $i = 1, \ldots, \mu(k)$, where $\mu(k)$ is the dimension of the vector space of homogeneous polynomials in $Y := (Y_1, \ldots, Y_n)$ of degree $k$ over $\mathbb{R}$ and $M_1(Y), \ldots, M_{\mu(k)}(Y)$ are the monomials of degree $k$ in $Y$.

2.4. LEMMA. — Let $S$ be a structure on $(\mathbb{R}, +, \cdot)$, and let $f : A \to \mathbb{R}$ belong to $S$, $A \subseteq \mathbb{R}^{m+n}$, such that $A_x$ is open in $\mathbb{R}^n$ for all $x \in \mathbb{R}^m$. Then for all $k > 0$ there exists $w_k : A \times \mathbb{R}^n \to \mathbb{R}$ belonging to $S$ such that for all $(x, y) \in A$, $f(x, -)$ is $G^k$ at $y$ if and only if $w_k(x, y, z) = 0$ for all $z \in \mathbb{R}^n$.

Proof. — For positive integers $k$ define $\phi_k : A \times \mathbb{R}^n \to \mathbb{R}$ as follows: if $(x, y) \in A$ and $t \mapsto f(x, y + tz)$ is $k$-times differentiable at $0$ for all $z \in \mathbb{R}^n$, then put

$$\phi_k(x, y, z) := \frac{d^k f(x, y + tz)}{dt^k}(0);$$

otherwise, put $\phi_k(x, y, z) := 1$. Note that $\phi_k$ belongs to $S$.

For each $k > 0$, choose points $p(k,1), \ldots, p(k,\mu(k)) \in \mathbb{R}^n$ as in 2.3, and define $v_k : A \times \mathbb{R}^n \to \mathbb{R}$ by

$$v_k(x, y, z) := P_k(\phi_k(x, y, p(k,1)), \ldots, \phi_k(x, y, p(k,\mu(k))), z), \quad (P_k as in 2.3).$$

Define $w_k : A \times \mathbb{R}^n \to \mathbb{R}$ by $w_k := v_k - \phi_k$. Then $w_k$ belongs to $S$, and for all $(x, y) \in A$, $f(x, -)$ is $G^k$ at $y$ if and only if $w_k(x, y, z) = 0$ for all $z \in \mathbb{R}^n$. \hfill \Box

2.5. PROPOSITION. — Keep all assumptions and notation as in the preceding lemma and its proof. Assume in addition that
1. $S$ is o-minimal, polynomially bounded and admits $C^\infty$ cell decomposition;

2. $A \times \mathbb{R}^n$ is a union of sets $B_1, \ldots, B_\ell$, each belonging to $S$, such that $\phi_k \upharpoonright B_i$ is $C^\infty$ for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, \ell\}$.

Then there exists $N \in \mathbb{N}$ such that for all $(x, y) \in A$, if $f(x, -)$ is $G^N$ at $y$, then $f(x, -)$ is $G^\infty$ at $y$.

Proof. — Examining the proof above, we see that then $w_k \upharpoonright B_i$ is $C^\infty$ for all $k \geq 1$ and $i \in \{1, \ldots, \ell\}$. By 1.6, there exists $N \in \mathbb{N}$ such that

$$\bigcap_{k=1}^{\infty} Z(w_k) = \bigcap_{k=1}^{N} Z(w_k).$$

Hence, for all $(x, y) \in A$, $f(x, -)$ is $G^\infty$ at $y$ if and only if $w_i(x, y, z) = 0$ for all $i \leq N$ and $z \in \mathbb{R}^n$; i.e., if and only if $f(x, -)$ is $G^N$ at $y$. 

\[ \Box \]

3. Some results on $\mathbb{R}_n^K$.

Throughout the remainder of this paper, $K$ denotes some fixed subfield of $\mathbb{R}$; "definable" means "definable in $\mathbb{R}_n^K$" unless stated otherwise.

We will state here some facts (established in [M2]) about $\mathbb{R}_n^K$, and prove a lemma on definable functions that we will need in the next section.

**Definition.** — Let $G(X_1, \ldots, X_m)$ be a real power series converging on some open neighborhood $U$ of $[-1, 1]^m$ to an analytic function $g : U \to \mathbb{R}$. Then $\tilde{g} : \mathbb{R}^m \to \mathbb{R}$ given by

$$\tilde{g}(x) := \begin{cases} g(x), & \text{if } x \in [-1, 1]^m \\ 0, & \text{otherwise} \end{cases}$$

is a restricted analytic function. (For $m = 0$, $\tilde{g}$ is just the corresponding real constant.)

Note that $\tilde{g}$ is finitely subanalytic, hence definable.

**Definition.** — The $\mathbb{R}_n^K$-functions on $\mathbb{R}^n$ are defined inductively:

1. The projection functions $x \mapsto x_i : \mathbb{R}^n \to \mathbb{R}$ ($i = 1, \ldots, n$) are $\mathbb{R}_n^K$-functions on $\mathbb{R}^n$.

2. If $f : \mathbb{R}^n \to \mathbb{R}$ is an $\mathbb{R}_n^K$-function, then $-f$ is an $\mathbb{R}_n^K$-function on $\mathbb{R}^n$. 

(3) If \( f, g : \mathbb{R}^n \to \mathbb{R} \) are \( \mathbb{R}^K \)-functions, then both \( f + g \) and \( fg \) are \( \mathbb{R}^K \)-functions on \( \mathbb{R}^n \).

(4) If \( f : \mathbb{R}^n \to \mathbb{R} \) is an \( \mathbb{R}^K \)-function on \( \mathbb{R}^n \), then for each \( r \in K \), the function

\[
x \mapsto \begin{cases} f(x)^r, & \text{if } f(x) > 0 \\ 0, & \text{otherwise} \end{cases}
\]

is an \( \mathbb{R}^K \)-function on \( \mathbb{R}^n \).

(5) If \( f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R} \) are \( \mathbb{R}^K \)-functions on \( \mathbb{R}^n \) and \( \tilde{g} : \mathbb{R}^m \to \mathbb{R} \) is a restricted analytic function, then the composition \( \tilde{g}(f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R} \) is an \( \mathbb{R}^K \)-function on \( \mathbb{R}^n \).

Note that \( \mathbb{R}^K \)-functions are definable.

3.1. FACTS.

(1) \( \mathbb{R}^K \) is o-minimal, polynomially bounded, and admits analytic cell decomposition.

(2) Every definable set in \( \mathbb{R}^n \) is a finite union of (definable) sets of the form

\[
\{ x \in \mathbb{R}^n : f(x) = 0, g_1(x) < 0, \ldots, g_\ell(x) < 0 \},
\]

where \( f, g_1, \ldots, g_\ell \) are \( \mathbb{R}^K \)-functions on \( \mathbb{R}^n \).

(3) Given a definable function \( f : A \to \mathbb{R}, A \subseteq \mathbb{R}^n \), there are \( \mathbb{R}^K \)-functions \( f_1, \ldots, f_\ell \) on \( \mathbb{R}^n \) such that for all \( x \in \mathbb{R}^n \) there exists \( i \in \{1, \ldots, \ell\} \) with \( f(x) = f_i(x) \); (i.e., \( f \) is given piecewise by \( \mathbb{R}^K \)-functions).

(4) For every definable function \( f : (0, \varepsilon) \to \mathbb{R} \) with \( f(t) \neq 0 \) for all \( t \in (0, \varepsilon) \), there exist a convergent real power series \( F(Y_1, \ldots, Y_d) \) with \( F(0) \neq 0 \) and \( \tau_0, \tau_1, \ldots, \tau_d \in K \) with \( \tau_1, \ldots, \tau_d > 0 \) such that

\[
f(t) = t^{\tau_0} F(t^{\tau_1}, \ldots, t^{\tau_d})
\]

for all sufficiently small positive \( t \).

(These facts were previously known for the case \( K = \mathbb{Q} \); see [DD], [D2], [DM] and [DMM].)

Remark. — Item (2) above expresses a kind of Tarski-Seidenberg property for \( \mathbb{R}^K \), and presents definable sets in a form similar to semialgebraic sets.

3.2. LEMMA. — Let \( f : A \times (0,1) \to \mathbb{R} \) be definable, \( A \subseteq \mathbb{R}^m \) \( (m > 0) \). Then \( A \) is a disjoint union of definable sets \( A_1, \ldots, A_M \), and there exist definable analytic functions \( h_i : A_i \to (0,1) \), \( (i = 1, \ldots, M) \) and
(not necessarily distinct) $\mathbb{R}^k_{an}$-functions $f_1, \ldots, f_M : \mathbb{R}^{m+1} \to \mathbb{R}$ such that $f \upharpoonright (0, h_i)$ is analytic and $f \upharpoonright (0, h_i) = f_i \upharpoonright (0, h_i)$ for $i = 1, \ldots, M$.

(Here, $(0, h_i) := \{(x, t) : x \in A_i$ and $0 < t < h_i(x)\}$ for $i = 1, \ldots, M$.)

Proof. — By 3.1(3), there exist $\mathbb{R}^k_{an}$-functions $f_1, \ldots, f_\ell : \mathbb{R}^{m+1} \to \mathbb{R}$ such that for all $(x, t) \in A \times (0,1)$, there is an $i \in \{1, \ldots, \ell\}$ with $f(x, t) = f_i(x, t)$. Then for $i = 1, \ldots, \ell$ the sets

$$B_i := \{(x, t) \in A \times (0, 1) : f(x, t) = f_i(x, t)\}$$

are definable, and $A \times (0, 1) = B_1 \cup \ldots \cup B_\ell$. By 3.1(1), there exists a decomposition $C$ of $\mathbb{R}^{m+1}$ into (definable) analytic cells partitioning $B_1, \ldots, B_\ell$ such that $f \upharpoonright C$ is analytic and is the restriction to $C$ of an $\mathbb{R}^k_{an}$-function, for each cell $C \in C$ with $C \subseteq A \times (0,1)$. Let $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ be the projection onto the first $m$ coordinates. Then $\pi C$ is a decomposition of $\mathbb{R}^m$ partitioning $A$, say that $A$ is the disjoint union of analytic cells $A_1, \ldots, A_M \in \pi C$. It suffices to consider the case that $M = 1$. By 3.1(1), we have

$$A \times (0, 1) = \bigcup_{i=1}^{k-1} \Gamma(g_i) \cup \bigcup_{i=1}^{k} (g_{i-1}, g_i)$$

where $g_0 < \cdots < g_k : A \to \mathbb{R}$ are analytic and definable, with $g_0 = 0$ and $g_k = 1$. Now put $h := g_1$. \(\square\)

4. The Expansion Theorem.

The goal of this section is to prove a “parametric” version of 3.1(4).

It will be convenient to introduce some working definitions and notation.

Given tuples of distinct variables $X := (X_1, \ldots, X_m)$ and $Y := (Y_1, \ldots, Y_d)$, we let $M(X; Y)$ denote the ring of all power series $F \in \mathbb{R}[X, Y]$ that converge on an open neighborhood of $[-1, 1]^m \times [-\varepsilon, \varepsilon]^d$, for some $\varepsilon > 0$ that depends on $F$. For $d = 0$, we just write $M(X)$. From now on, we assume that we have chosen such an $\varepsilon$ for each $F \in M(X_1, \ldots, X_m; Y_1, \ldots, Y_d)$. Given $(x, y) \in \mathbb{R}^{m+d}$, we let $F(x, y)$ be the value given by the power series if $(x, y) \in [-1, 1]^m \times [-\varepsilon, \varepsilon]^d$, and put $F(x, y) := 0$ otherwise. The resulting function $F$ is finitely subanalytic, hence definable.
Notation. — Given a property $P(t)$ of positive real numbers $t$, we say that $P(t)$ holds at $0^+$ if there exists $\varepsilon > 0$ such that $P(t)$ holds for all $t \in (0, \varepsilon)$. When the property $P(x, t)$ also depends on a parameter $x$ ranging over a set $A \subseteq \mathbb{R}^p$, then we allow $\varepsilon = \varepsilon(x) > 0$ also to depend on this parameter.

Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^p$. We wish to expand $f(x, t)$ at $0^+$ in a power series in $t$ with exponents from $K$, uniformly in the parameter $x \in A$.

**Definition.** — The function $f$ has a uniform expansion on $A$ if there exist

1. $F \in M(X_1, \ldots, X_m; Y_1, \ldots, Y_d)$ for some $m, d \in \mathbb{N}$,
2. $r_0, r_1, \ldots, r_d \in K$ with $r_1, \ldots, r_d > 0$,
3. definable analytic maps $a : A \to (0, \infty)$, $b = (b_1, \ldots, b_m) : A \to [-1, 1]^m$ and $c : A \to [1, \infty)$, such that for each $x \in A$, $F(b(x), 0) \neq 0$ and $f(x, t) = a(x)t^{r_0}F(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d})$ at $0^+$. (In particular, $f(x, t) \neq 0$ at $0^+$.)

**Remarks.**

1. Suppose that $f$ has a uniform expansion on $A$. Then there is a sequence $(f_n : A \to \mathbb{R})_{n \geq 0}$ of definable analytic functions and an unbounded strictly increasing sequence $(\alpha_n)_{n \geq 0}$ of real numbers such that for all $x \in A$, $f_0(x) \neq 0$ and there exists $\varepsilon(x) > 0$ such that $f(x, t) = \sum_{n \geq 0} f_n(x)t^{\alpha_n}$ for $t \in (0, \varepsilon(x))$, where the convergence is absolute and uniform on each subinterval $(0, \delta) \subseteq (0, \varepsilon(x))$. Consequently, $f(x, -)$ is analytic on $(0, \varepsilon(x))$; (see §4 of [M2]).

2. For the case $K = \mathbb{Q}$, if $f$ has a uniform expansion on $A$, then there exist a rational number $q$, a positive integer $k$, a power series $F \in M(X; Y_1)$, and finitely subanalytic, analytic maps $a : A \to (0, \infty)$, $c : A \to [1, \infty)$ and $b = (b_1, \ldots, b_m) : A \to [-1, 1]^m$ such that for each $x \in A$ we have $F(b(x), 0) \neq 0$ and $f(x, t) = a(x)t^qF(b(x), (c(x)t)^{1/k})$ at $0^+$. Thus, there exists a sequence $(f_n : A \to \mathbb{R})_{n \geq 0}$ of finitely subanalytic, analytic functions such that for all $x \in A$, $f_0(x) \neq 0$ and there exists $\varepsilon(x) > 0$ such that $f(x, t) = t^q \sum_{n \geq 0} f_n(x)t^{n/k}$ for $t \in (0, \varepsilon(x))$.

**Definition.** — Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^p$. For definable $B \subseteq A$, $f$ has a uniform expansion on $B$ if $f \upharpoonright B \times \mathbb{R}$ has a
uniform expansion on $B$; $f$ has a piecewise uniform expansion on $A$ if $A$ is a union of definable sets $A_1, \ldots, A_\ell$ such that for $i = 1, \ldots, \ell$, either $f$ has a uniform expansion on $A_i$, or $f(x, t)$ vanishes identically at $0^+$ for all $x \in A_i$.

(Note that we can take $A_1, \ldots, A_\ell$ to be disjoint.)

We can now state the main result of this section:

**Expansion Theorem.** Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^p$. Then $f$ has a piecewise uniform expansion on $A$.

We have some work to do before we begin the proof.

Given a tuple of variables $X := (X_1, \ldots, X_m)$ and $\nu \in \mathbb{N}^m$, we let $X^\nu$ denote the monomial $X_1^{\nu_1} \cdots X_m^{\nu_m}$. We note here a fact from analysis that we will need:

(*) Let $F \in M(X)$, $X := (X_1, \ldots, X_m)$, and let $Y := (Y_1, \ldots, Y_m)$ be a tuple of new variables. Then there exists $\varepsilon > 0$ such that the power series

$$G(X, Y) := \sum_{\nu} \frac{\partial^{\left|\nu\right|}}{\partial X^\nu} Y^\nu \quad (\nu \in \mathbb{N}^m)$$

converges on a neighborhood of $[-1,1]^m \times [-\varepsilon, \varepsilon]^m$ (i.e., $G \in M(X; Y)$), and such that for all $(u, v) \in [-1,1]^m \times [-\varepsilon, \varepsilon]^m$, with $|u + v| \leq 1$, we have $F(u + v) = G(u, v)$.

We are thus justified in denoting the power series $G$ by $F(X + Y)$.

**N.B.** The following reductions will be used throughout this section, often without mention.

1. Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable with $A \subseteq \mathbb{R}^p$. Then the set

$$\{x \in A : f(x, t) = 0 \text{ at } 0^+\}$$

is definable. Thus, in order to show that $f$ has a piecewise uniform expansion on $A$, we may remove this set from $A$ and assume (by 1.1) that $f(x, t) \neq 0$ at $0^+$; i.e., for all $x \in A$ there exists $\varepsilon(x) > 0$ such that $f(x, t) \neq 0$ for all $t \in (0, \varepsilon(x))$.

2. Suppose $r_0, r_1, \ldots, r_d$, $F$ and $a, b, c$ are as in the definition of “uniform expansion on $A$”, except that instead of requiring $a, b$ and $c$ to be definable and analytic, we only assume that they are definable. It follows then from 3.1(1) that $f$ has a piecewise uniform expansion on $A$. (We do not actually need this observation, but it will relieve us in the coming pages
of doing the easy, but quite frequently occurring, verifications that certain
definable functions are analytic.)

**Lemma.** Let \( r_1, \ldots, r_d \in K \cap (0, \infty), d \geq 1 \). Let \( 0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots \) be the elements of the monoid \( r_1 \mathbb{N} + \cdots + r_d \mathbb{N} \) in increasing order. Then given \( N \in \mathbb{N} \), there exist \( s_1, \ldots, s_c \in K \cap (0, \infty) \) such that \( \{\alpha_n - \alpha_N : n \geq N\} \subseteq s_1 \mathbb{N} + \cdots + s_c \mathbb{N} \).

(See [M2], e.g., for a proof.)

**Main Lemma.** Let \( r_1, \ldots, r_d \in K \cap (0, \infty), F \in M(X_1, \ldots, X_m; Y_1, \ldots, Y_d) \), and let \( b = (b_1, \ldots, b_m) : A \to [-1,1]^m \) and \( c_1, \ldots, c_d : A \to [1,\infty) \) be definable maps, \( A \subseteq \mathbb{R}^p \). Then the definable function \( f : A \times \mathbb{R} \to \mathbb{R} \) given by

\[
    f(x,t) := F(b(x), (c_1(x)t)^{r_1}, \ldots, (c_d(x)t)^{r_d})
\]

has a piecewise uniform expansion on \( A \).

**Proof.** First, we do the case that \( c_1 = \ldots = c_d \).

Note that \( f \) has a piecewise uniform expansion on the definable set

\[
    \{x \in A : F(b(x),0) \neq 0\},
\]

so we may reduce to the case that \( F(b(x),0) = 0 \) and \( f(x,t) \neq 0 \) at \( 0^+ \) for each \( x \in A \). Write

\[
    F(X,Y) = \sum_{\nu} F_\nu(X)Y^\nu, \quad F_\nu(X) \in M(X),
\]

where the sum is taken over \( \nu \in \mathbb{N}^d \). Let \( 0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots \) be the elements of the monoid \( r_1 \mathbb{N} + \cdots + r_d \mathbb{N} \) in increasing order. For all \( n \in \mathbb{N} \) put

\[
    G_n(X) := \sum_{\nu} F_\nu(X) \in M(X),
\]

where the (finite) sum is taken over all \( \nu \in \mathbb{N}^d \) such that \( r_1 \nu_1 + \cdots + r_d \nu_d = \alpha_n \). Then for each \( x \in A \), we have

\[
    F(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}) = \sum_{n \geq 0} G_n(b(x))(c(x)t)^{\alpha_n} \text{ at } 0^+.
\]

Note that \( G_0(b(x)) = F(b(x),0) = 0 \) for all \( x \in A \), and that \( x \mapsto G_n(b(x)) : A \to \mathbb{R} \) is definable for all \( n \in \mathbb{N} \). Since \( f \) is definable, and we assume that \( f(x,t) \neq 0 \) at \( 0^+ \), there exists by 1.4 some \( N \in \mathbb{N} \) such that for all \( x \in A \), there is an \( n \leq N \) with \( G_n(b(x)) \neq 0 \). Now for each \( N > 0 \), the set

\[
    \{x \in A : G_0(b(x)) = \ldots = G_{N-1}(b(x)) = 0, G_N(b(x)) \neq 0\}
\]
is definable. Partitioning $A$ suitably, we may thus reduce to the case that there exists $N > 0$ such that for all $x \in A$, we have $G_0(b(x)) = \cdots = G_{N-1}(b(x)) = 0$ and $G_N(b(x)) \neq 0$. Then for each $x \in A$ we have

$$f(x, t) = (c(x)t)^{\alpha N} \left( G_N(b(x)) + \sum_{n > N} G_n(b(x))(c(x)t)^{\alpha n - \alpha N} \right)$$

at $0^+$. Let $s = (s_1, \ldots, s_e) \in \mathbb{N}^e$ be as in the previous lemma. For each $n > N$, choose $\mu(n) \in \mathbb{N}^e$ such that $s_1 \mu(n)_1 + \cdots + s_e \mu(n)_e = \alpha_n - \alpha_N$. Put

$$H(X, Z) := G_N(X) + \sum_{n > N} G_n(X)Z^{\mu(n)} \in M(X; Z),$$

where $Z := (Z_1, \ldots, Z_e)$ is a tuple of new variables. Put $a(x) := (c(x))^{\alpha N}$ for all $x \in A$. Note that $H(b(x), 0) = G_N(b(x)) \neq 0$ and $a(x) > 0$ for all $x \in A$. Then for each $x \in A$ we have

$$f(x, t) = a(x)^{\alpha N} H(b(x), (c(x)t)^{s_1}, \ldots, (c(x)t)^{s_e})$$

at $0^+$, as desired.

Next, let $c_1, \ldots, c_d : A \to [1, \infty)$ be definable functions. For each $i = 1, \ldots, d$, the set

$$A_i := \{x \in A : c_i(x) \geq c_j(x) \text{ for } j = 1, \ldots, d\}$$

is definable; thus, we may reduce to the case that, say, $A = A_1$. Define $b_{m+j} : A \to [-1, 1]$ for $j = 1, \ldots, d$ by $b_{m+j}(x) := (c_j(x)/c_1(x))^{r_j}$. Put

$$G(X, X_{m+1}, \ldots, X_{m+d}, Y) := F(X, X_{m+1}Y_1, \ldots, X_{m+d}Y_d).$$

Then $G \in M(X, X_{m+1}, \ldots, X_{m+d}; Y)$ and

$$F(b(x), (c_1(x)t)^{r_1}, \ldots, (c_d(x)t)^{r_d}) = G(b'(x), (c_1(x)t)^{r_1}, \ldots, (c_1(x)t)^{r_d})$$

at $0^+$ for each $x \in A$, where $b' := (b_1, \ldots, b_{m+d})$. By the previous case, we are done.

Proof of the Expansion Theorem.

To show that a definable function $f : A \times \mathbb{R} \to \mathbb{R}$ with $A \subseteq \mathbb{R}^p$ has a piecewise uniform expansion on $A$, we may (by 3.2) reduce to the case that $f$ is the restriction to $A \times \mathbb{R}$ of an $\mathbb{R}_a^K$-function on $\mathbb{R}^{p+1}$. We now proceed by "induction on complexity of $\mathbb{R}_a^K$-functions" to show that each $\mathbb{R}_a^K$-function $f$ on $\mathbb{R}^{p+1}$ has a piecewise uniform expansion on the definable set $A \subseteq \mathbb{R}^p$. As usual, we will assume that $f(x, t) \neq 0$ at $0^+$ for each $x \in A$. Throughout the proof, the parameter $x$ will range over $A$.\[\square\]
Case. \( f \) is a projection function \( \mathbb{R}^{p+1} \rightarrow \mathbb{R} \).
(Trivial.)

Case. \( f = -g \), where \( g \) is an \( \mathbb{R}^K_{\text{an}} \)-function having a piecewise uniform expansion on \( A \).
(Easy; details omitted.)

Case. \( f = g + h \), where \( g \) and \( h \) are \( \mathbb{R}^K_{\text{an}} \)-functions having piecewise uniform expansions on \( A \).

Now \( f \) has a piecewise uniform expansion on the set
\[
\{ x \in A : g(x,t) = 0 \text{ at } 0^+ \} \cup \{ x \in A : h(x,t) = 0 \text{ at } 0^+ \},
\]
so we may reduce to the case that \( g \) and \( h \) have uniform expansions on \( A \); say that
\[
g(x,t) = a(x) t^{r_0} G(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}) \text{ at } 0^+
\]
and
\[
h(x,t) = a'(x) t^{s_0} H(b'(x), (c'(x)t)^{s_1}, \ldots, (c'(x)t)^{s_e}) \text{ at } 0^+
\]
of the required form. We may assume that \( a = a' \). (To see this, note that the sets \( A_1 := \{ x \in A : a(x) \leq a'(x) \} \) and \( A_2 := \{ x \in A : a(x) > a'(x) \} \) are definable; so we may assume that either \( A = A_1 \) or \( A = A_2 \). If \( A = A_1 \), then
\[
g(x,t) = a'(x) t^{r_0} G'(b(x), (a(x)/a'(x)), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}) \text{ at } 0^+,
\]
where \( G'(X, X_{m+1}, Y) := X_{m+1} G(X, Y) \). We use the same trick in case \( A = A_2 \).) Put \( b'' := (b, b') : A \rightarrow [-1,1]^{m+n} \), for appropriate \( m \) and \( n \). Suppose without loss of generality that \( s_0 \leq r_0 \).

Subcase. \( s_0 = r_0 \).

Introducing new variables as needed, put
\[
F(X, Y) := G(X_1, \ldots, X_m, Y_1, \ldots, Y_d)
+H(X_{m+1}, \ldots, X_{m+n}, Y_{d+1}, \ldots, Y_{d+e}).
\]
Then at \( 0^+ \) we have
\[
f(x,t) = a(x) t^{s_0} F(b''(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}, (c'(x)t)^{s_1}, \ldots, (c'(x)t)^{s_e}).
\]
Apply the Main Lemma.
Subcase. $s_0 < r_0$.

Put $c''(x) := 1$ for $x \in A$, and put $F(X,Y)$ equal to

$$Y_{d+e+1}G(X_1, \ldots, X_m, Y_1, \ldots, Y_d) + H(X_{m+1}, \ldots, X_{m+n}, Y_{d+1}, \ldots, Y_{d+e}).$$

Then at $0^+$, $f(x,t)$ is equal to

$$a(x)t^{s_0}F(b''(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}, (c'(x)t)^{s_1}, \ldots, (c'(x)t)^{s_e},$$

$$(c''(x)t)^{r_0-s_0}) \cdot$$

Apply the Main Lemma.

Case. $f = gh$, where $g$ and $h$ are $R_{an}^K$-functions having piecewise uniform expansions on $A$.

This case is similar to, but easier than, the previous case, and we omit the details.

Case. $f = h^s$, where $s \in K$, and $h$ is an $R_{an}^K$-function having a piecewise uniform expansion on $A$. (Recall that we put $h(x,t)^s := 0$ for $h(x,t) \leq 0$; see §3.)

For $s = 0$, the result is trivial, so suppose that $s \neq 0$. Since $f(x,t) \neq 0$ at $0^+$, we have $h(x,t) > 0$ at $0^+$. We may assume that $h$ has a uniform expansion on $A$; say that

$$h(x,t) = a(x)t^{r_0}H(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d})$$

of the required form. Then

$$f(x,t) = a(x)^st^{s_0}H(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d})^s$$

Thus, it suffices to show that the definable function $u : A \times \mathbb{R} \to \mathbb{R}$ given by

$$u(x,t) := (H(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}))^s$$

has a piecewise uniform expansion on $A$.

Write

$$H(X,Y) = \sum H_\nu(X)Y^\nu, H_\nu(X) \in M(X),$$

where the sum is taken over all $\nu \in \mathbb{N}^d$. Note that $H_0(b(x)) = H(b(x), 0) > 0$, and that the sets $A_1 := \{x \in A : H_0(b(x)) \geq 1\}$ and $A_2 := \{x \in A : H_0(b(x)) < 1\}$ are definable. So we may assume that either $A = A_1$ or $A = A_2$. 

Subcase. $H_0(b(x)) \geq 1$ for all $x$.

At $0^+$ we have
\[ H(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}) = H_0(b(x))(1 + H'(b'(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d})), \]
where $b' := (b_1, \ldots, b_m, 1/(H_0(b)))$ and
\[ H'(X, X_{m+1}, Y) := X_{m+1} \sum_{\nu \neq 0} H_\nu(X)Y^\nu. \]

Then
\[ u(x, t) = H_0(b(x))^s F(b'(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}) \text{ at } 0^+, \]
where
\[ F(X, X_{m+1}, Y) := \sum_{k \geq 0} \binom{s}{k} (H'(X, X_{m+1}, Y))^k \in M(X, X_{m+1}; Y). \]

Subcase. $0 < H_0(b(x)) < 1$ for all $x$.

For $i = 1, \ldots, d$, put $c_i(x) := c(x)(H_0(b(x)))^{-1/r_i}$. Note that $c_i(x) > 1$ and $(c_i(x)t)^{r_i} = (c(x)t)^{r_i}(1/H_0(b(x)))$. Then at $0^+$ we have
\[ u(x, t) = H_0(b(x))^s F(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}, (c_1(x)t)^{r_1}, \ldots, (c_d(x)t)^{r_d}), \]
where
\[ F(X, Y_1, \ldots, Y_{2d}) := \sum_{k \geq 0} \binom{s}{k} (H^*(X, Y_1, \ldots, Y_{2d}))^k, \]
and
\[ H^*(X, Y_1, \ldots, Y_{2d}) := \sum_{\nu_1 \neq 0} H_\nu(X)Y_{d+1}^{\nu_1-1}Y_2^{\nu_2} \cdots Y_d^{\nu_d} \]
\[ + \sum_{\nu_1 = 0} \sum_{\nu_2 \neq 0} H_\nu(X)Y_{d+2}^{\nu_1}Y_2^{\nu_2-1}Y_3^{\nu_3} \cdots Y_d^{\nu_d} + \]
\[ \cdots + \sum_{\nu_1 = \ldots = \nu_{d-1} = 0} \sum_{\nu_d \neq 0} H_\nu(X)Y_2^{\nu_d-1}. \]

(Note that $F, H^* \in M(X; Y_1, \ldots, Y_{2d})$ and $H^*(X, 0) = 0$.)

Apply the Main Lemma.

(We alert the reader here that we will use again in the next case the construction of the series $H^*$.)
Case. \( f = \tilde{g}(h_1, \ldots, h_\ell), \) where \( \tilde{g} \) is a restricted analytic function and \( h_1, \ldots, h_\ell \) are \( \mathbb{R}^K_{an} \)-functions each having piecewise uniform expansions on \( A \).

For each \( i = 1, \ldots, \ell \), the set \( A_i := \{ x \in A : h_i(x, t) = 0 \text{ at } 0^+ \} \) is definable, so we may assume by the monotonicity theorem that \( h_i(x, t) \neq 0 \) at \( 0^+ \) for \( i = 1, \ldots, \ell \), and that each \( h_i \) has a uniform expansion on \( A \). Since \( f(x, t) \neq 0 \) at \( 0^+ \), by the definition of "restricted analytic function" we must have \( |h_i(x, t)| \leq 1 \) at \( 0^+ \) for \( i = 1, \ldots, \ell \).

For simplicity, we do the case \( \ell = 1 \); the case \( \ell > 1 \) is similar, but notationally cumbersome. Put \( h := h_1 \). Then

\[
h(x, t) = a(x)t^{s_0}H(b(x), (c(x)t)^{s_1}, \ldots, (c(x)t)^{s_\ell}) \quad \text{at } 0^+,
\]

of the required form. Note that we must have \( r_0 \geq 0 \), since \( |h(x, t)| \leq 1 \) at \( 0^+ \). The sets \( A_1 := \{ x \in A : a(x) \leq 1 \} \) and \( A_2 := \{ x \in A : a(x) > 1 \} \) are definable, so we may as well assume that either \( A = A_1 \) or \( A = A_2 \). We must also consider separately the cases \( r_0 = 0 \) and \( r_0 > 0 \). Thus, there are four subcases to treat. We will show that in each subcase, \( h \) can be represented at \( 0^+ \) in the form

\[
h(x, t) = B'(x) + H'(B(x), (C_1(x)t)^{s_1}, \ldots, (C_\ell(x)t)^{s_\ell}),
\]

where the maps \( B : A \to [-1,1]^n \) for some \( n \in \mathbb{N} \), \( B' : A \to [-1,1] \), and \( C_1, \ldots, C_\ell : A \to [1,\infty) \) are definable, \( s_1, \ldots, s_\ell \in K \cap (0,\infty) \), and \( H'(X,Y) \in M(X;Y) \) for \( X = (X_1, \ldots, X_n) \) and \( Y = (Y_1, \ldots, Y_\ell) \), with \( H'(X,0) = 0 \). It follows then from Fact (*) that

\[
f(x, t) = F(B(x), B'(x), (C_1(x)t)^{s_1}, \ldots, (C_\ell(x)t)^{s_\ell}) \quad \text{at } 0^+,
\]

where \( G \) is the Taylor series at \( 0 \) of \( \tilde{g} \) and

\[
F := G(H'(X, Y) + X_{n+1}) \in M(X, X_{n+1}; Y).
\]

The Main Lemma then applies, finishing each subcase, and thus finishing the proof as well.

Subcase. \( r_0 > 0 \) and \( a(x) \leq 1 \) for all \( x \).

Put \( B := (b, a) : A \to [-1,1]^{m+1}, \ c' := 1 : A \to \mathbb{R}, \) and \( B' := 0 : A \to \mathbb{R}. \) Then

\[
h(x, t) = B'(x) + H'(B(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_\ell}, (c'(x)t)^{s_0}) \quad \text{at } 0^+,
\]

where

\[
H'(X, X_{m+1}, Y, Y_{d+1}) := X_{m+1}Y_{d+1}H(X, Y) \in M(X, X_{m+1}; Y, Y_{d+1}).
\]
Subcase. $r_0 > 0$ and $a(x) > 1$ for all $x$.

Put $B' := 0 : A \to \mathbb{R}$ and $c'(x) := a(x)^{1/r_0}$; note that $c'(x) > 1$ and $(c'(x)t)^{r_0} = a(x)t^{r_0}$. Then

$$h(x, t) = B'(x) + H'(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}, (c'(x)t)^{r_0}) \text{ at } 0^+,$$

where

$$H'(X, Y, Y_{d+1}) := Y_{d+1}H(X, Y) \in M(X; Y, Y_{d+1}).$$

Subcase. $r_0 = 0$ and $a(x) > 1$ for all $x$.

Write

$$H(X, Y) = \sum H_{\nu}(X)Y^\nu, H_{\nu}(X) \in M(X),$$

where the sum is taken over $\nu \in \mathbb{N}^d$. Note that we must have $|a(x)H_0(b(x))| \leq 1$ for $x \in A$. Put $B'(x) := a(x)H_0(b(x))$ and put $c_i(x) := a(x)^{1/r_i}c(x)$ for $i = 1, \ldots, d$; then $c_i(x) > 1$ and $(c_i(x)t)^{r_i} = a(x)(c(x)t)^{r_i}$. Let $H' := H^*$, where $H^*$ is constructed from $H$ as in the previous case. Hence, at $0^+$ we have

$$h(x, t) = B'(x) + H'(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}, (c_1(x)t)^{r_1}, \ldots, (c_d(x)t)^{r_d}).$$

Subcase. $r_0 = 0$ and $a(x) \leq 1$ for all $x$.

With $H_0$ as in the previous subcase, let $B'$ also be as in the previous subcase and put

$$H' := X_{m+1}(H(X, Y) - H_0(X))$$

and

$$B := (b, a) : A \to [-1, 1]^{m+1}.$$

\[\square\]

5. Proof of the Main Theorem.

Let $f : A \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^{m+n}$. Replacing $A$ by

$$\{(x, y) \in A : y \in \text{int}(A_x)\},$$

and $f$ by its restriction to this definable set, we may assume that $A_x \subseteq \mathbb{R}^n$ is open for all $x \in \mathbb{R}^m$. We must show that there exists $N > 0$ such that for all $(x, y) \in A$, if $f(x, -)$ is $C^N$ at $y$, then $f(x, -)$ is analytic at $y$. 

Consider the definable function $F : A \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by

$$F(x, y, z, t) := \begin{cases} f(x, y + tz), & \text{if } y + tz \in A_x \\ 0, & \text{otherwise.} \end{cases}$$

For notational convenience, let the variable $v$ range over $\mathbb{R}^{m+n+n}$.

**Claim.** There exists $N > 0$ such that for all $v \in A \times \mathbb{R}^n$, if $F(v, -)$ is $C^N$ at 0, then $F(v, -)$ is analytic at 0.

**Proof of Claim.** First, suppose that $D \subseteq A \times \mathbb{R}^n$ is definable, and that $F$ has a uniform expansion on $D$. Arguing as in the proof of the Main Lemma, we write

$$F(v, t) = a(v)t^{r_0} \sum_{n \geq 0} G_n(b(v))(c(v)t)^{\alpha_n} \text{ at } 0^+,$$

with $v$ ranging over $D$. Note that if there exist $v \in D$ and a positive integer $p > r_0$ such that $F(v, -)$ is $C^p$ at 0, then $r_0 \in \mathbb{N}$, (since $a(v)F(b(v), 0) \neq 0$). So we may as well assume that $r_0 \in \mathbb{N}$. Put

$$G(X, Y)_1 := \sum G_n(X)Y_1^{\alpha_n} \in M(X; Y_1),$$

where the sum is taken over all $n \in \mathbb{N}$ with $\alpha_n \in \mathbb{N}$. Then the function $g : D \times \mathbb{R} \to \mathbb{R}$ given by $g(v, t) := a(v)t^{r_0}G(b(v), c(v)t)$ is definable. Furthermore, there exists an $\varepsilon > 0$ (depending only on $F$) such that $g$ is analytic on the set

$$\{(v, t) \in D \times \mathbb{R} : |t| < \varepsilon/c(v)\};$$

in particular, $g(v, -)$ is analytic at 0 for all $v \in D$. Note also that for each $k \in \mathbb{N}$, the function

$$v \mapsto \frac{d^k g(v, t)}{dt^k}(0) : D \to \mathbb{R}$$

is analytic. Put $h := F \upharpoonright (D \times \mathbb{R}) - g$. Then $h$ is definable, and we have

$$h(v, t) = \sum a(v)G_n(b(v))(c(v)t)^{\alpha_n}t^{\alpha_n + r_0} \text{ at } 0^+,$$

where the sum is taken over all $n \in \mathbb{N}$ with $\alpha_n \notin \mathbb{N}$. Thus,

$$h(v, t) = \sum_{k=0}^{\infty} h_k(v)t^{\beta_k} \text{ at } 0^+,$$

where $(\beta_k)$ is a strictly increasing sequence of nonintegral real numbers (since $r_0 \in \mathbb{N}$) and each $h_k : D \to \mathbb{R}$ is definable. By 1.5, there is an $N \in \mathbb{N}$ such that for all $v \in D$, if $h(v, t) = O(t^N)$ as $t \to 0^+$, then $h(v, t) = 0$ at $0^+$. Thus, there exists $N > 0$ such that for all $v \in D$, if $F(v, -)$ is $C^N$ at 0, then
\[ F(v, \cdot) = g(v, \cdot) \text{ on some interval (depending on } v \text{) about } 0; \text{ thus, } F(v, \cdot) \text{ is analytic at } 0. \] (Note: For each } k \in \mathbb{N} \text{ the function}

\[ v \mapsto \frac{d^k F(v, t)}{dt^k}(0) : B \to \mathbb{R} \]

is definable and analytic, where \( B := \{v \in D : F(v, \cdot) \text{ is } C^N \text{ at } 0\}\).

Next, suppose that \( D \subseteq A \times \mathbb{R}^n \) is definable and that \( F(v, t) = 0 \) at \( 0^+ \) for each \( v \in D \). Note that for any positive integer \( p \) and \( v \in D \), if \( F(v, \cdot) \) is \( C^p \) at \( 0 \), then \( F(v, \cdot) \) is \( p \)-flat at \( 0 \). By 1.5, there exists \( N > 0 \) such that for all \( v \in D \), if \( F(v, \cdot) \) is \( C^N \) at \( 0 \), then \( F(v, \cdot) \) vanishes identically on some interval about \( 0 \), hence is analytic at \( 0 \). (Note: For each } k \in \mathbb{N} \text{ the function}

\[ v \mapsto \frac{d^k F(v, t)}{dt^k}(0) : B \to \mathbb{R} \]

is identically zero, where \( B := \{v \in D : F(v, \cdot) \text{ is } C^N \text{ at } 0\}\).

The claim now follows easily from the Expansion Theorem applied to \( F \).

**Proof of Main Theorem from Claim.** — Let \( N > 0 \) be as in the claim. Put

\[ A' := \{(x, y) \in A : F(x, y, z, \cdot) \text{ is } C^N \text{ at } 0 \in \mathbb{R} \text{ for all } z \in \mathbb{R}^n \}. \]

Note that if \( (x, y) \in A - A' \), then \( f(x, \cdot) \) is not \( C^N \) at \( y \), hence not analytic at \( y \). So, we may replace \( A \) by its definable subset \( A' \) and assume that for all \( v \in A \times \mathbb{R}^n \), the function \( F(v, \cdot) \) is \( C^N \) at \( 0 \in \mathbb{R} \), and hence analytic at \( 0 \) by the claim. The arguments in the proof of the claim then establish that there exist definable sets \( B_1, \ldots, B_\ell \) with \( A \times \mathbb{R}^n = B_1 \cup \cdots \cup B_\ell \) such that

\[ v \mapsto \frac{d^k F(v, t)}{dt^k}(0) : B_i \to \mathbb{R} \]

is analytic for all \( k \in \mathbb{N} \) and \( i \in \{1, \ldots, \ell\} \). Increasing (if necessary) \( N \) as in the claim and applying 2.5 and 3.1(1), we may assert that for all \( (x, y) \in A \), if \( f(x, \cdot) \) is \( C^N \) at \( y \), then \( f(x, \cdot) \) is \( G^\infty \) at \( y \).

Now let \( (x, y) \in A \) and suppose that \( f(x, \cdot) \) is \( C^N \) on some open euclidean ball \( U \) centered at \( y \) of radius \( \varepsilon > 0 \). Then \( f(x, \cdot) \) is \( G^\infty \) at \( y \). Furthermore, \( F(x, y', z, \cdot) \) is \( C^N \), and thus analytic, at \( t = 0 \) for all \( (y', z) \in U \times \mathbb{R}^n \). Thus, \( t \mapsto f(x, y + tz) \) is defined and analytic on \( (-\varepsilon, \varepsilon) \) for all \( z \in \mathbb{R}^n \) with \( ||z|| \leq 1 \). Hence, by 2.2, \( f(x, \cdot) \) is analytic at \( y \). \( \square \)

**Note.** — The result clearly holds for definable maps \( F : A \to \mathbb{R}^p \), \( A \subseteq \mathbb{R}^{m+n} \).
COROLLARY. — With assumptions as in the Main Theorem, the set
\{(x, y) \in A : f(x, -) \text{ is analytic at } y\}
is definable.

Proof. — The set \{(x, y) \in A : f(x, -) \text{ is } C^M \text{ at } y\} is definable for any fixed positive integer \(M\).

DEFINITION. — Let \(X \subseteq \mathbb{R}^P\). Then \(x \in X\) is a smooth point of \(X\) of dimension \(k\) if \(X \cap U\) is an analytic submanifold of \(\mathbb{R}^P\) of dimension \(k\) for some open neighborhood \(U\) of \(x\). The singular set of \(X\), denoted \(\text{Sing}(X)\), is the complement in \(X\) of the smooth points of highest dimension.

COROLLARY. — Let \(X \subseteq \mathbb{R}^P\) be definable. Then for each \(k \in \mathbb{N}\), the set of smooth points of \(X\) of dimension \(k\) is definable; in particular, \(\text{Sing}(X)\) is a closed definable subset of \(X\).

Proof. — Let \(k \in \{0, \ldots, p\}\). Given \(\varepsilon > 0\) and \(x \in \mathbb{R}^P\), put
\[B(x, \varepsilon) := \{y \in \mathbb{R}^P : |x - y| < \varepsilon\}.
Note that \(x \in X\) is a smooth point of dimension \(k\) if and only if for some \(\varepsilon > 0\) and some \(i = (i_1, \ldots, i_k)\) with \(1 \leq i_1 < \cdots < i_k \leq p\), the coordinate projection \(\pi_i\) (as in 1.2) maps \(X \cap B(x, \varepsilon)\) bijectively onto an open subset \(C\) of \(\mathbb{R}^k\), and the inverse of \(\pi_i | A \cap B(x, \varepsilon)\) (as a map \(C \to \mathbb{R}^P\)) is analytic at \(\pi_i(x)\). Now use the previous corollary.

COROLLARY. — Let \(A \subseteq \mathbb{R}^{m+n}\) be definable. Then \\{(x, y) \in \mathbb{R}^{m+n} : y \in \text{Sing}(A_x)\}\ is definable.

The proof is similar to that of the preceding corollary.

Note. — Suppose that \(S\) is a structure on \((\mathbb{R}, +, \cdot)\) such that all sets in \(S\) are definable in \(\mathbb{R}^K\). Then the above corollaries (suitably rephrased) hold with the notion of definable in \(\mathbb{R}^K\) replaced by the notion of belonging to \(S\).

In closing, we point out that the results of this section never hold in o-minimal structures \(S\) on \((\mathbb{R}, +, \cdot)\) which are not polynomially bounded. By 1.3, the exponential function belongs to every such \(S\), and thus
\[t \mapsto \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}\]
belongs to $S$, which is $C^\infty$ on $\mathbb{R}$ but not analytic at $t = 0$. Also, the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x, y) := \begin{cases} \frac{|y|^{1/x} \cdot \exp (-1/(x^2 + y^2))}{0}, & \text{if } x > 0 \text{ and } y \neq 0 \\ \text{otherwise} & \end{cases}$$

belongs to $S$. Note that $F(x, -)$ is $C^\infty$ at $y = 0$ iff $x \in (-\infty, 0] \cup \{1/(2n) : n \geq 1\}$, which has infinitely many connected components; also, $F$ is $C^n$ at $(0, 0)$ for every $n > 0$, but not $C^\infty$ at $(0, 0)$.

BIBLIOGRAPHY


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