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SPECTRAL ASYMPTOTICS FOR MANIFOLDS WITH CYLINDRICAL ENDS

by T. CHRISTIANSEN AND M. ZWORSKI

1. Introduction and statement of results.

In this note we obtain spectral asymptotics for manifolds with cylindrical ends. Since the Laplacian has continuous spectrum one expects Weyl asymptotics for the sum of a term measuring the behaviour of the continuous spectrum and the counting function for embedded eigenvalues. Following ideas originating in mathematical physics the first term is expressed using an appropriately defined scattering matrix for manifolds with cylindrical ends [2]. In fact that term is an exact analogue of the scattering phase: see [1], [9], [14] for a discussion of the Euclidean situation, [8] and [11] for finite volume hyperbolic surfaces and [12], [13] for infinite volume hyperbolic manifolds. The consequent bound on $N(\lambda)$, the number of embedded eigenvalues less than or equal to λ^2,

$$N(\lambda) = \mathcal{O}(\lambda^n)$$

is optimal as shown by the example in Section 3. Here n is the dimension of the manifold.

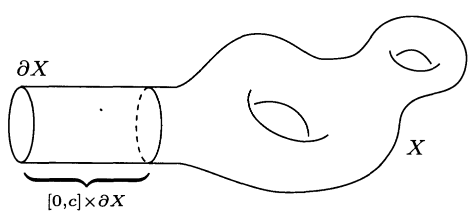

Figure 1. A manifold with a cylindrical end.

Manifolds with cylindrical ends (see Fig. 1) are a special case of b-manifolds [10] and since our results are expected to hold in that more

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general setting we recall the definition of the scattering matrix in that case. Thus let X be a compact manifold with boundary ∂X and let x be a defining function of ∂X, that is

$$x_{|X^{\circ}} > 0, \quad x_{|\partial X} = 0, \quad dx_{|\partial X} \neq 0.$$

Then a complete metric g on X is an exact b-metric if near the boundary it can be written as

$$g = \left(\frac{dx}{x}\right)^2 + h \tag{1}$$

where h is a semi-positive metric on X which restricts to a non-degenerate metric on ∂X.

If Δ_X is the Laplace operator for (X, g) then the spectrum decomposes as

$$\sigma(\Delta_X) = \sigma_{\text{ac}}(\Delta_X) + \sigma_{\text{pp}}(\Delta_X), \quad \sigma_{\text{ac}}(\Delta_X) = \bigsqcup_{\mu \in \sigma(\Delta_{\partial X})} [\mu, \infty),$$

where $\Delta_{\partial X}$ is the Laplacian of $(\partial X, h_{|\partial X})$. Hence the multiplicity of $\lambda^2 \in \sigma_{\text{ac}}(\Delta)$ is given by $N_{\partial X}(\lambda)$, the number of eigenvalues of $\Delta_{\partial X}$ less than or equal to λ^2 — see [10], Section 6.9.

If $\sigma_k^2 \in \sigma(\Delta_{\partial X})$ and $0 \leq \sigma_k < \lambda$, then there exists a generalized eigenfunction of the Laplacian, $\Phi^+_{x,k,\lambda}$, $(\Delta - \lambda^2)\Phi^+_{x,k,\lambda} = 0$, which has the following expansion at the boundary:

$$\begin{aligned} 2\Phi^+_{x,k,\lambda} &= x^{i\sqrt{\lambda^2 - \sigma_k^2}}\,\phi_k \\ &+ \sum_{0 \leq \sigma_m \leq \lambda} \left(\frac{\lambda^2 - \sigma_k^2}{\lambda^2 - \sigma_m^2}\right)^{1/4} \Psi_{x,mk}(\lambda) x^{-i\sqrt{\lambda^2 - \sigma_m^2}}\phi_m + \mathcal{O}(x^{\epsilon(\lambda)}), \end{aligned} \tag{2}$$

where $\epsilon(\lambda) > 0$. Here ϕ_m's are the orthonormal eigenfunctions on ∂X corresponding to σ_m's. The scattering matrix $\Psi_x(\lambda) = (\Psi_{x,mk}(\lambda))$ is an $N_{\partial X}(\lambda) \times N_{\partial X}(\lambda)$ unitary matrix whose entries are given above, for $\lambda > 0$. The relationship $\Psi_x(-\lambda) = \Psi^*_x(\lambda)$ gives the matrix for negative λ — see [2] for a detailed discussion. We recall here that the scattering matrix depends mildly on the choice of boundary-defining function x, as the notation indicates. This definition should be compared to that in the simpler finite volume situation (see [8] and [11]) where the scattering matrix is given by the coefficient in the expansion of the Eisenstein series: there, however, only $\sigma_0 = 0$ contributes.

The *scattering phase* is defined as the winding number

$$\sigma_x(\lambda) = \frac{1}{2\pi i}\big(\log\det \Psi_x(\lambda) - \log\det \Psi_x(0)\big), \tag{3}$$

and we refer to Section 1.3 of [2] for its continuity properties.

A manifold is said to have *cylindrical ends* if it is a *b*-manifold and if for some $c > 0$ we have $h \equiv h_{|\partial X}$ for some identification $X_{|x<c} \simeq [0,c) \times \partial X$. By changing the defining function of the boundary we can change the value of c and below it will often be convenient to take $c > 1$. For manifolds with cylindrical ends we have:

THEOREM. — *For any boundary defining function x such that the metric on X is an exact product $(\mathrm{d}x/x)^2 + h_{|\partial X}$ for $x < \delta$, for some $\delta > 0$, we have*

$$N(\lambda) = \mathcal{O}(\lambda^n),$$

$$N(\lambda) + \sigma_x(\lambda) = c_n \lim_{\epsilon\downarrow 0}\Big[\int_{X_{|x>\epsilon}} 1 + \log\epsilon\,\mathrm{Vol}(\partial X)\Big]\lambda^n + \mathcal{O}(\lambda^{n-1})$$

as $\lambda \to \infty$, where n is the dimension of X, $c_n = \omega_n(2\pi)^{-n}$, and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Here, the leading term depends on the « *b*-volume »

$$\lim_{\epsilon\downarrow 0}\Big[\int_{X_{|x>\epsilon}} 1 + \log\epsilon\,\mathrm{Vol}(\partial X)\Big]$$

as one expects. The estimate improves the estimate of [4]:

$$N(\lambda) = \mathcal{O}(\lambda^{2n-1}).$$

It is natural to conjecture that the theorem holds for manifolds with exact *b*-metrics as well.

After this note was completed, we learned that L. Parnovski has independently obtained similar results.

We would like to thank Charlie Epstein, Richard Froese, Laurent Guillopé and Richard Melrose for their help in our work on this paper. The partial support of the second author by the National Science Foundation is also gratefully acknowledged.

2. Proof of the asymptotic formula.

The method of proof is inspired by the pseudo-Laplacian of Lax–Phillips [8] and Colin de Verdière [3]. If X is a manifold with cylindrical ends such that for some $\delta > 0$

$$\left(X_{|x<1+\delta}, g\right) = \left([0, 1+\delta) \times \partial X, \left(\frac{dx}{x}\right)^2 + h_{|\partial X}\right),$$

we decompose X into

$$X = X_0 \cup X_1, \quad X_0 = X_{|x>1}.$$

If u is an eigenfunction of the Laplacian, that is

$$u \in L^2_b(X), \quad \Delta_X u = \lambda^2 u$$

(here $L^2_b(X) = L^2(X, d\,\mathrm{vol}_g)$) then

$$(u_{|X_1})(x,y) = \sum_{j=0}^{\infty} u_j(x)\phi_j(y), \quad (x,y) \in [0,1] \times \partial X$$

and to have $u_{|X_1} \in L^2_b(X_1)$ the coefficients must satisfy

$$u_j(x) = \begin{cases} u_j(1)\, x^{\sqrt{\sigma_j{}^2 - \lambda^2}} & \text{if } \lambda^2 < \sigma_j^2, \\ 0 & \text{if } \lambda^2 \geq \sigma_j^2, \end{cases} \tag{4}$$

and that extends to $x < 1 + \delta$. This implies that

$$(\partial_\nu + (\Delta_{\partial X} - \lambda^2)_+^{1/2}) u_{|\partial X_0} = 0, \tag{5}$$

where ∂_ν is the outward unit normal derivative and where $(\Delta_{\partial X} - \lambda^2)_+^{1/2}$ is defined using the spectral decomposition as the positive square root of the positive part of $(\Delta_{\partial X} - \lambda^2)$.

Let T_λ be defined as the Laplacian Δ_{X_0} on X_0 with the boundary condition given by (5) and with the domain $C^\infty(X_0)$. The operator T_λ is symmetric and we define its *Friedrichs extension* using the form

$$Q_\lambda(u,v) = \int_{X_0} \nabla u \overline{\nabla v} + \int_{\partial X_0} (\Delta_{\partial X} - \lambda^2)_+^{1/4} u \, \overline{(\Delta_{\partial X} - \lambda^2)_+^{1/4} v} \tag{6}$$

with the form domain

$$\mathcal{D}_\lambda = \{u \in L^2(X_0) : Q_\lambda(u,u) < \infty\} = H^1(X_0).$$

We will denote the corresponding self-adjoint operator by T_λ as well. For Dirichlet and Neumann problems on X_0 the forms and form domains are given by

$$Q_D(u,v) = \int_{X_0} \nabla u \overline{\nabla v}, \quad \mathcal{D}_D = H_0^1(X_0);$$
$$Q_N(u,v) = \int_{X_0} \nabla u \overline{\nabla v}, \quad \mathcal{D}_N = H^1(X_0).$$

Hence $Q_\lambda(u,u) \geq Q_N(u,u)$ for $u \in H^1(X_0)$ and $Q_\lambda(u,v) = Q_D(u,v)$ for $u, v \in H_0^1(X_0) \subset H^1(X_0) = \mathcal{D}_\lambda$. Thus the spectrum of T_λ is discrete and if $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots \leq \mu_j(\lambda) \leq \cdots$ are the eigenvalues, then the max-min characterization

$$\mu_j(\lambda) = \sup_{\substack{M_{j-1} \\ \dim M_{j-1} < j}} \inf_{\substack{u \in H^1(X_0) \cap M_{j-1}^\perp \\ \|u\|=1}} Q_\lambda(u,u),$$

the fact that $Q_\lambda(u,u)$ is non-increasing in λ, and comparison with the Dirichlet and Neumann forms gives the following

LEMMA 1. — *The eigenvalues of T_λ depend continuously on λ and are non-increasing. If $N_\lambda(r), N_N(r), N_D(r)$ denote the number of eigenvalues less than or equal than r^2 for T_λ, the Neumann and the Dirichlet Laplacians on X_0 respectively, then*

$$N_D(r) \leq N_\lambda(r) \leq N_N(r). \tag{7}$$

By applying the standard Weyl law for the Dirichlet and Neumann problems, (7) immediately yields

$$N_\lambda(r) = c_n \mathrm{Vol}(X_0) r^n + \mathcal{O}(r^{n-1}). \tag{8}$$

We also note that the error estimate is independent of λ.

From the comments in the beginning of this section we see that if u is an L_b^2 eigenfunction of Δ_X, then $u_{|X_0}$ is in the domain of T_λ and $T_\lambda(u_{|X_0}) = \lambda^2(u_{|X_0})$, that is

$$\lambda^2 \in \sigma_{\mathrm{pp}}(\Delta_X) \implies \lambda^2 \in \sigma(T_\lambda),$$

with at least the same multiplicity. Other eigenvalues λ^2 of T_λ are characterized by

LEMMA 2. — *If λ^2 is in the spectrum of T_λ, but does not correspond to an eigenvalue of the Laplacian on X, then the scattering matrix $\Psi_x(\lambda)$ has 1 as an eigenvalue. Conversely, if 1 is an eigenvalue of $\Psi_x(\lambda)$, then λ^2 is in the spectrum of T_λ. The multiplicity of λ^2 as an eigenvalue of T_λ is equal to the sum of the multiplicity of λ^2 as an eigenvalue of Δ_X and the multiplicity of 1 as an eigenvalue of $\Psi_x(\lambda)$.*

Proof. — Suppose λ^2 is in the spectrum of T_λ, but does not correspond to an eigenvalue of the Laplacian on X. There is an eigenfunction u of T_λ, which, near the boundary of X_0, can be expanded in terms of the boundary eigenfunctions, with

$$\left(u_{|x<1+\delta}\right)(x,y) = \sum_{0\leq\sigma_j\leq\lambda}\left(c_{j,+}x^{i\sqrt{\lambda^2-\sigma_j^2}} + c_{j,-}x^{-i\sqrt{\lambda^2-\sigma_j^2}}\right)\phi_j + \sum_{\sigma_j>\lambda} \mathrm{d}_j x^{\sqrt{\sigma_j^2-\lambda^2}}\phi_j.$$

Clearly u can be smoothly extended to $\tilde{u}$ on X by allowing x to range down to 0 (as in the expansion above), and then $\Delta_X\tilde{u} = \lambda^2\tilde{u}$. The fact that u does not correspond to an L^2_b eigenfunction for the Laplacian on X means that not all the $c_{j,+}$, $c_{j,-}$ are 0, and the boundary condition implies that $c_{j,+} = c_{j,-}$. Since we can write $\tilde{u}$ as a linear combination of the generalized eigenfunctions $\Phi^+_{x,k,\lambda}$ (and possibly L^2_b eigenfunctions), this means that 1 is an eigenvalue of the matrix

$$S_x(\lambda) = \{S_{x,mk}(\lambda)\} = \left\{(\lambda^2-\sigma_k^2)^{1/4}(\lambda^2-\sigma_m^2)^{-1/4}\Psi_{x,mk}(\lambda)\right\},\quad \lambda\neq\sigma_j\,.$$

Then it follows that 1 is an eigenvalue of $\Psi_x(\lambda)$, if $\lambda \neq \sigma_j$, since $S_x(\lambda)$ and $\Psi_x(\lambda)$ are similar matrices. When $\lambda = \sigma_j$, using the fact that $\Psi_x(\lambda)$ and $S_x(\lambda)$ are continuous as $|\lambda| \downarrow |\sigma_j|$ (see [2], Section 1.3), we see the eigenvalues of $\Psi_x(\lambda)$ and $S_x(\lambda)$ are equal at these points as well.

Conversely, assume 1 is an eigenvalue of $\Psi_x(\lambda)$, and thus of $S_x(\lambda)$. That is, there is a v which is a linear combination of the generalized eigenfunctions $\Phi^+_{x,k,\lambda}$ of the Laplacian on X which is given, near the boundary, by

$$\sum_{\sigma_k^2\leq\lambda^2} d_k\left(x^{i\sqrt{\lambda^2-\sigma_k^2}} + x^{-i\sqrt{\lambda^2-\sigma_k^2}}\right)\phi_k + \sum_{\sigma_k^2>\lambda^2} c_k\, x^{\sqrt{\sigma_k^2-\lambda^2}}\phi_k$$

for some constants d_k and c_k, with not all $d_k = 0$. If we restrict v to X_0, we find $\Delta_{X_0}(v_{|X_0}) = \lambda^2 v_{|X_0}$, and $(\partial_\nu + (\Delta_{\partial X} - \lambda^2)^{1/2}_+)v_{|\partial X_0} = 0$.

This proves the lemma, except for the question of multiplicities. From the argument above it follows that to independent eigenfunctions of T_λ with eigenvalue λ^2 which do not correspond to eigenfunctions of Δ_X, we obtain the same number of eigenvectors of $\Psi_x(\lambda)$, corresponding to eigenvalue 1.

Thus it remains to show that the multiplicity of λ^2 as an eigenvalue of T_λ is at least as great as the multiplicity of 1 as an eigenvalue of $\Psi_x(\lambda)$. Suppose that 1 is an eigenvalue of $\Psi_x(\lambda)$ with multiplicity J. Then, since $\Psi_x(\lambda)$ is unitary, it has J linearly independent eigenvectors with eigenvalue 1. If $\lambda \neq \sigma_j$, any j, then we get J linearly independent eigenvectors of $S_x(\lambda)$ and thus, by the discussion above, J linearly independent eigenfunctions of T_λ.

If $\lambda^2 = \sigma_M^2$, we proceed as follows: if $c = (c_1, \ldots c_m)$ is an eigenvector of $\Psi_x(\lambda)$, let

$$d_i = \begin{cases} (\lambda^2 - \sigma_i^2)^{-1/4} c_i & \text{if } i \neq M, \\ c_i & \text{if } i = M. \end{cases}$$

Then

$$\Big(2 \sum d_k \Phi^+_{x,k,\lambda}\Big)_{|x<1+\delta} = \sum_{\sigma_k^2 < \sigma_M^2} d_k \big(x^{i\sqrt{\lambda^2 - \sigma_k^2}} + x^{-i\sqrt{\lambda^2-\sigma_k^2}}\,\big)\phi_k + \sum_{\sigma_\ell^2 = \sigma_M^2} \tilde{c}_\ell \phi_\ell + \sum_{\sigma_\ell^2 > \sigma_M^2} \tilde{c}_\ell x^{\sqrt{\sigma_\ell^2 - \lambda^2}} \phi_\ell .$$

Restricting $\sum d_k \Phi^+_{x,k,\lambda}$ to X_0, we get a function that satisfies the boundary conditions for T_λ, and thus is an eigenfunction for T_λ. Additionally, if we start with J linearly independent eigenvectors of $\Psi_x(\lambda)$ with eigenvalue 1, we get J linearly independent eigenfunctions of T_λ. □

This motivates the following definition:

$$P_x(\lambda) = \sum_{0 \le \xi \le \lambda} \dim\big\{v \in \mathbb{C}^{N_{\partial X}(\xi)} \colon\ \Psi_x(\xi) v = v\big\}, \tag{9}$$

that is, $P_x(\lambda)$ is the number of times 1 is an eigenvalue of $\Psi_x(\xi)$ for $0 \le \xi \le \lambda$ (counted with multiplicities). Lemma 2 shows that

$$N(r) + P_x(r) = \sharp\big\{\lambda^2 \colon \lambda^2 \in \sigma(T_\lambda),\ 0 \le \lambda \le r\big\}, \tag{10}$$

where the eigenvalues are counted with their multiplicities.

For the right hand side we get:

LEMMA 3. — *If N_r is defined as in Lemma 1 then*

$$\sharp\big\{\lambda^2 : \lambda^2 \in \sigma(T_\lambda),\ 0 \le \lambda \le r\big\} = N_r(r).$$

Proof. — By Lemma 1, $\mu_j(\lambda) \in \sigma(T_\lambda)$ are continuous and non-increasing in λ. Hence if $\mu_j(\lambda) = \lambda^2 \leq r^2$ for some j then $\mu_j(r) \leq \mu_j(\lambda) \leq r^2$. Conversely, if $\mu_j(r) \leq r^2$, then $\mu_j(\lambda) = \lambda^2$ for exactly one $0 \leq \lambda \leq r$, since $\mu_j(\lambda) - \lambda^2$ is a strictly decreasing function, and $\mu_j(0) \geq 0$. □

The next lemma gives the desired upper bound for the sum of the counting function $P_x(\lambda)$ and the number of embedded eigenvalues:

LEMMA 4. — *If the metric on X is of exact product type for $x < 1+\delta$, some $\delta > 0$, then*

$$N(\lambda) + P_x(\lambda) = c_n\Big(\int_{X,x>1} 1\Big)\lambda^n + \mathcal{O}(\lambda^{n-1}).$$

Proof. — This follows from (8), (10) and Lemma (3). □

The following lemma bounds the variation of σ_x.

LEMMA 5. — *For any boundary-defining function x such that the metric has the form* (1) *near the boundary,*

$$\big|\sigma_x(\lambda+1) - \sigma_x(\lambda)\big| \leq P_x(\lambda+1) - P_x(\lambda) + \mathcal{O}(\lambda^{n-1}).$$

Proof. — We note that the unitarity of $\Psi_x(\lambda)$ implies that

$$\sigma_x(\lambda) = \frac{1}{2\pi}\sum_{0\leq\sigma_j\leq\lambda}\theta_j(\lambda), \tag{11}$$

where the $\theta_j(\lambda)$ are the arguments of the eigenvalues of $\Psi_x(\lambda)$. We recall (see [2], Section 1.3) that $\Psi_x(\lambda)$ is continuous, except where λ^2 crosses a point in the spectrum of the boundary Laplacian $\Delta_{\partial X}$. The magnitude of the jump in $\sigma_x(\lambda)$ at $\lambda = \sigma_j$ is no worse than $\frac{1}{2}$ times the multiplicity of σ_j^2 as an eigenvalue of $\Delta_{\partial X}$, a contribution which is $\mathcal{O}(\lambda^{n-1})$. Since there are $\mathcal{O}(\lambda^{n-1})$ θ_j's in (11), and if any is to change by more than 2π it must cross an integral multiple of 2π (and hence a point where 1 is an eigenvalue of the scattering matrix), we get the lemma. □

For completeness, we include the following lemma which describes the variation of the scattering phase under a change of a boundary defining function.

LEMMA 6. — *If $a > 0$ is a constant and if the metric on X is of the form* (1) *for $x < \epsilon$, some $\epsilon > 0$, then*

$$\sigma_{ax}(\lambda) - \sigma_x(\lambda) = \frac{1}{\pi} \log a \sum_{0 \leq \sigma_k \leq \lambda} \sqrt{\lambda^2 - \sigma_k^2}.$$

Proof. — Recall that the entries in the scattering matrix corresponding to the boundary defining function x are determined by the leading terms of a generalized eigenfunction of the Laplacian at the boundary, as in (2):

$$\begin{aligned}
&x^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k + \sum_{0\leq\sigma_m\leq\lambda} \Big(\frac{\lambda^2 - \sigma_k^2}{\lambda^2 - \sigma_m^2}\Big)^{1/4} \Psi_{x,mk}(\lambda) x^{-i\sqrt{\lambda^2-\sigma_m^2}}\phi_m + \mathcal{O}(x^{\epsilon(\lambda)}) \\
&= \Big[(ax)^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k \\
&\quad + \sum_{0\leq\sigma_m\leq\lambda} \Big(\frac{\lambda^2 - \sigma_k^2}{\lambda^2 - \sigma_m^2}\Big)^{1/4} \Psi_{x,mk}(\lambda) a^{i\sqrt{\lambda^2-\sigma_k^2}+i\sqrt{\lambda^2-\sigma_m^2}} (ax)^{-i\sqrt{\lambda^2-\sigma_m^2}}\phi_m \\
&\qquad\qquad + \mathcal{O}\Big((ax)^{\epsilon(\lambda)}\Big)\Big] a^{-i\sqrt{\lambda^2-\sigma_k^2}}.
\end{aligned}$$

The sum in square brackets on the third line is the expansion, at the boundary, of the generalized eigenfunction which determines the entries in the scattering matrix for the boundary defining function ax. Therefore,

$$\Psi_{ax,mk}(\lambda) = \Psi_{x,mk}(\lambda)\, a^{i\sqrt{\lambda^2-\sigma_k^2}+i\sqrt{\lambda^2-\sigma_m^2}},$$

and a straight-forward computation completes the proof of the lemma. □

Proof of Theorem. — We recall the trace formula of [2]:

$$\begin{aligned}
\mathcal{F}\Big[b\text{-}\mathrm{Tr}_x\big(\cos(t\sqrt{\Delta})\big)\Big](\lambda) &= \pi \frac{\mathrm{d}}{\mathrm{d}\lambda}\sigma_x(\lambda) \;+\; \tfrac{1}{4}\pi \sum_{\substack{\sigma_k^2 \in \mathrm{spec}\,\Delta_{\partial X} \\ \sigma_k \neq 0}} \delta(\lambda - \sigma_k) \\
&\quad + \tfrac{1}{2}\pi \,\mathrm{Tr}\,\Psi_x(0)\delta(\lambda) \;+\; \pi \sum_{\lambda_j^2 \in \mathrm{ppSpec}\,\Delta} \delta(\lambda - \lambda_j)
\end{aligned} \tag{12}$$

and the behaviour of $b\text{-}\mathrm{Tr}_x \cos(t\sqrt{\Delta}\,)$ near $t = 0$ (see [2], Lemma 1.1)

$$\begin{aligned}
&\frac{1}{\pi}\mathcal{F}\Big[\widehat{\rho}(t)\, b\text{-}\mathrm{Tr}_x \cos\big(t\sqrt{\Delta}\,\big)\Big](\lambda) \\
&\qquad = nc_n \lim_{\epsilon\downarrow 0} \Big[\int_{X|_{x>\epsilon}} 1 + \log\epsilon\,\mathrm{Vol}(\partial X)\Big]\lambda^{n-1} + \mathcal{O}(\lambda^{n-2}),
\end{aligned} \tag{13}$$

where $\widehat{\rho} \in C_c^\infty(\mathbb{R})$, $\widehat{\rho}(0) = 1$, and $\widehat{\rho}(t)$ has sufficiently small support. We will also require that $\rho > 0$. Let

$$e_x(\lambda) = \sigma_x(\lambda) + N(\lambda). \tag{14}$$

Then for $\lambda \geq 0$, the formula (12) shows that

$$\rho * \mathrm{d}\, \mathrm{e_x}(\lambda) = \frac{1}{\pi} \mathcal{F}\big(\widehat{\rho}(\mathrm{t})\, \mathrm{b\text{-}Tr_x} \cos(\mathrm{t}\sqrt{\Delta}\,)\big)(\lambda) + \mathcal{O}(\lambda^{\mathrm{n}-2}), \tag{15}$$

where we used the standard estimate $N_{\partial X}(\lambda) = \mathcal{O}(\lambda^{n-1})$ to estimate the contribution of the second term.

We will now apply Hörmander's Tauberian argument [7] to obtain asymptotics for $e_x(\lambda)$ in the special case where x is chosen so that the metric is a product for $x < 1 + \delta$, some $\delta > 0$. We do not, however, have the positivity of $d\, e_x(\lambda)$, and to circumvent this problem we use the asymptotics of Lemma 4 and the estimate of Lemma 5. In fact, they yield immediately

$$e_x(\lambda + 1) - e_x(\lambda) = \mathcal{O}(\lambda^{n-1}). \tag{16}$$

Integrating (15) from 0 to λ gives the theorem in this special case, since (16) shows that

$$\left| \int \rho(\lambda - \mu) e_x(\mu)\, \mathrm{d}\mu - e_x(\lambda) \right| = \mathcal{O}(\lambda^{n-1}). \tag{17}$$

The proof for the more general choice of x satisfying the conditions of the theorem follows from the result in the special case above, along with (13), (15), Lemma 6, and Hörmander's argument. □

3. An example.

This section describes a class of examples of compact manifolds with boundary and exact *b*-metrics which have an infinite number of eigenvalues. In fact, the manifolds are n-dimensional, and $N(\lambda)$ grows like λ^n. This shows that the bound on the growth of the number of eigenvalues proved in the previous section is sharp. These examples were motivated by the example of [5].

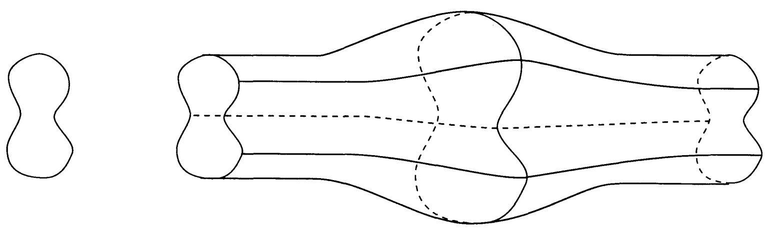
It will be helpful to consider manifolds with two infinite cylindrical ends instead of two boundary components. Let (Y, h) be a smooth, compact, $(n-1)$ dimensional Riemannian manifold without boundary, and let

$$(X, g) = \big(\mathbb{R} \times Y,\ \mathrm{d}t^2 + f(t)^{4/(n-1)} h\big),$$

where $f \in C^\infty(\mathbb{R})$ — see Figure 2. We choose $f(t) \equiv 1$ outside a compact set; then a change of variables makes X a compact manifold with two boundary components and an exact b-metric. The Laplacian on X is

$$\Delta = f(t)^{-1}\Big(D_t^2 + \frac{f''(t)}{f(t)} + \frac{1}{f(t)^{4/(n-1)}}\Delta_Y\Big)f(t), \tag{18}$$

where Δ_Y is the Laplacian on Y.



(Y, h) $\qquad$ $(X, g) \simeq \big(\mathrm{R} \times Y,\ \mathrm{d}t^2 + f(t)^{4/(n-1)}h\big)$

Figure 2. A manifold with many embedded eigenvalues: $f(t)$ large on a compact set.

If $\{\sigma_j^2\}$ are the eigenvalues of the Laplacian on Y, listed with multiplicity, and $\{\phi_j\}$ are a set of corresponding orthonormal eigenfunctions, then we may expand (18) in terms of the eigenfunctions on Y. If $\chi(t, y) \in C_c^\infty(X)$, then

$$\begin{aligned}\Delta\chi(t,y) &= \frac{1}{f(t)}\sum_j\Big(D_t^2 + \frac{f''(t)}{f(t)} + \frac{1}{f(t)^{4/(n-1)}}\sigma_j^2\Big) \\ &\qquad\qquad \times f(t)\,\phi_j(y)\int_Y \overline{\phi_j(y')}\chi(t,y') \\ &= \frac{1}{f(t)}\sum_j (D_t^2 + V_j + \sigma_j^2) f(t)\phi_j(y)\int_Y \overline{\phi_j(y')}\,\chi(t,y')\end{aligned} \tag{19}$$

where

$$V_j = \frac{f''(t)}{f(t)} + \big(f(t)^{-4/(n-1)} - 1\big)\sigma_j^2.$$

If we choose $f(t) \geq 1$, and $f(t) > 1$ on a compact set, then V_j has compact support, and $V_j \ll 0$ on a fixed compact set for j sufficiently large. Using (19), we see that λ_k^2 is an eigenvalue of the Laplacian on X if and only if $(\lambda_k^2 - \sigma_j^2)$ is an eigenvalue of $D_t^2 + V_j$ for some j.

To find the asymptotic behaviour of $N(\lambda)$ for X, we need to know the asymptotic behaviour both of the σ_j^2 and of the eigenvalues of $D_t^2 + V_j$, as j goes to infinity. The standard Weyl law for ∂X gives that

$$\sum_{\sigma_j \leq \lambda} \sigma_j = \frac{n-1}{n} c_{n-1} \operatorname{Vol}(Y)\, \lambda^n + \mathcal{O}(\lambda^{n-1}), \tag{20}$$

where $c_m = \omega_m (2\pi)^{-m}$ and ω_m is the volume of the unit ball in $\mathbb{R}^m$.

To study the behaviour of the one dimensional Schrödinger operator we take the semi-classical point of view with $h = \sigma_j^{-1}$ — see [6]. As $j \to \infty$, the operator $D_t^2 + V_j$ has

$$\frac{\sigma_j}{\pi} \Big(\int \sqrt{1 - f^{-4/(n-1)}(t)}\, \mathrm{d}t + o(1) \Big) \tag{21}$$

eigenvalues, and they lie in the interval $[-\sigma_j^2 a + c, 0]$, where

$$a = \max\big(1 - f^{-4/(n-1)}(t)\big).$$

Finally, using (20) and (21), we can bound $N(\lambda)$ from above and below as $\lambda \to \infty$. Summing over σ_j^2 such that $\sigma_j^2 \leq \lambda^2$, we get, as $\lambda \to \infty$:

$$\frac{N(\lambda)}{\lambda^n} \geq \frac{(n-1)c_{n-1}}{n\pi} \operatorname{Vol}(Y) \int \sqrt{1 - f^{-4/(n-1)}(t)}\, \mathrm{d}t + o(1).$$

This shows that the order of growth of embedded eigenvalues given by the theorem in Section 1 is indeed optimal. Similarly, summing over those σ_j^2 such that $\sigma_j^2 \leq \lambda^2/(1-a)$, we can bound $N(\lambda)$, as $\lambda \to \infty$, by:

$$\frac{N(\lambda)}{\lambda^n} \leq \frac{(n-1)c_{n-1}}{n\pi} \operatorname{Vol}(Y)(1-a)^{-n/2} \int \sqrt{1 - f^{-4/(n-1)}(t)}\, \mathrm{d}t + o(1).$$

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