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ZHONGWEI SHEN

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L^p ESTIMATES FOR SCHRÖDINGER OPERATORS WITH CERTAIN POTENTIALS

by **Zhongwei SHEN** (*)

0. Introduction.

In this paper we consider the Schrödinger differential operator

$$P = -\Delta + V(x) \quad \text{on} \quad \mathbb{R}^n, n \geq 3 \tag{0.1}$$

where $V(x)$ is a nonnegative potential. We will assume that V belongs to the reverse Hölder class B_q for some $q \geq n/2$. We are interested in the L^p boundedness of the operators $(-\Delta+V)^{i\gamma}, \nabla(-\Delta+V)^{-1/2}, \nabla(-\Delta+V)^{-1}\nabla$ and $\nabla^2(-\Delta + V)^{-1}$ where $\gamma \in \mathbb{R}$.

Note that a nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to $B_q(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V dx\right) \tag{0.2}$$

holds for every ball B in $\mathbb{R}^n$ ([G], [M]).

One remarkable feature about the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (0.2), such that $V \in B_{q+\varepsilon}$ [G].

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We now state our main results in this paper.

THEOREM 0.3. — *Suppose $V \in B_q$ for some $q \geq n/2$. Then, for $1 < p \leq q$,*

$$\|\nabla^2(-\Delta + V)^{-1}f\|_p \leq C_p\|f\|_p$$

where C_p depends only on p, n and the constant in (0.2).

THEOREM 0.4. — *Suppose $V \in B_{n/2}$. Then, for $\gamma \in \mathbb{R}$, $(-\Delta + V)^{i\gamma}$ is a Calderón–Zygmund operator.*

It is well known that Calderón–Zygmund operators are bounded on L^p for $1 < p < \infty$.

THEOREM 0.5. — *Suppose $V \in B_q$ and $n/2 \leq q < n$. Then, for $1 < p \leq p_0$,*

$$\|\nabla(-\Delta + V)^{-1/2}f\|_p \leq C_p\|f\|_p$$

where $(1/p_0) = (1/q) - (1/n)$.

We remark that the ranges of p in Theorems 0.3 and 0.5 are optimal. This can be shown by considering the potential $V(x) = |x|^\alpha$ where $-2 < \alpha < 0$. See Section 7.

It follows easily from Theorem 0.5 that, if $V \in B_q$, $n/2 \leq q < n$,

$$\|(-\Delta + V)^{-1/2}\nabla f\|_p \leq C_p\|f\|_p \quad \text{for } p_0' \leq p < \infty \tag{0.6}$$

and

$$\|\nabla(-\Delta + V)^{-1}\nabla f\|_p \leq C\|f\|_p \quad \text{for } p_0' \leq p \leq p_0 \tag{0.7}$$

where $p_0' = p_0/(p_0 - 1)$.

THEOREM 0.8. — *Suppose $V \in B_n$. Then*

$$\nabla(-\Delta + V)^{-1/2}, (-\Delta + V)^{-1/2}\nabla \text{ and } \nabla(-\Delta + V)^{-1}\nabla$$

are Calderón–Zygmund operators.

We also obtain the L^p boundedness of the operators

$$V(-\Delta + V)^{-1}, \quad V^{1/2}(-\Delta + V)^{-1/2}, \quad V^{1/2}(-\Delta + V)^{-1}\nabla$$

and

$$V^{1/2}\nabla(-\Delta + V)^{-1}.$$

See Theorem 3.1, Theorem 5.10, Theorem 5.11 and Theorem 4.13 respectively. As a direct consequence of our L^p estimates, we have

COROLLARY 0.9. — *Suppose $V \in B_q$ for some $q \geq n/2$. Assume that $-\Delta u + Vu = f$ in $\mathbb{R}^n$. Then*

$$\|\nabla^2 u\|_p \leq C_p \|f\|_p \text{ for } 1 < p \leq q,$$
$$\|Vu\|_p \leq C\|f\|_p \quad \text{for } 1 \leq p \leq q,$$
$$\|V^{1/2}\nabla u\|_p \leq C\|f\|_p \quad \text{for } 1 \leq p \leq p_1$$

where $1/(p_1) = (3/2q) - (1/n)$ if $n/2 \leq q < n$; and $p_1 = 2q$ if $q \geq n$.

COROLLARY 0.10. — *Suppose $V \in B_q$ for some $q \geq n/2$. Assume that $-\Delta u + Vu = \operatorname{div} \vec{g}$ in $\mathbb{R}^n$. Then*

$$\|\nabla u\|_p \leq C_p \|\vec{g}\|_p \text{ for } p_0' \leq p \leq p_0 (1 < p < \infty \text{ if } q \geq n),$$
$$\|V^{1/2} u\|_p \leq C\|\vec{g}\|_p \quad \text{for } p_0' \leq p \leq 2q (1 \leq p \leq 2q \text{ if } q \geq n)$$

where $(1/p_0) = (1/q) - (1/n)$.

We now recall that an operator T taking $C_c^\infty(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is called a Calderón–Zygmund operator if

(a) T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$,

(b) there exists a kernel K such that for every $f \in L_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy \quad \text{a.e. on} \quad \{\operatorname{supp} f\}^c,$$

(c) the kernel K satisfies the Calderón–Zygmund estimates:

$$\text{(0.11)} \qquad \begin{cases} |K(x,y)| \leq \dfrac{C}{|x-y|^n}, \\ |K(x+h,y) - K(x,y)| \leq \dfrac{C|h|^\delta}{|x-y|^{n+\delta}}, \\ |K(x,y+h) - K(x,y)| \leq \dfrac{C|h|^\delta}{|x-y|^{n+\delta}} \end{cases}$$

for $x, y \in \mathbb{R}^n$, $|h| < |x-y|/2$ and for some $\delta > 0$. See [St2].

We remark that in his thesis, which inspired the work in this paper, J. Zhong [Z] studied the Schrödinger operator $-\Delta + V(x)$, assuming that V is a nonnegative polynomial in $\mathbb{R}^n$. He showed that, for $\gamma \in \mathbb{R}$, $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-1/2}$ and $\nabla(-\Delta + V)^{-1}\nabla$ are Calderón–Zygmund operators with bounds depending only on the degree of V and the dimension n.

Related results can also be found in [HN] and [T]. In [HN], Helffer and Nourrigat considered the case of nonnegative polynomials. They proved the L^2 boundedness of $\nabla^2(-\Delta+V)^{-1}$ and $V^{1/2}\nabla(-\Delta+V)^{-1}$, based on a subelliptic estimate of Rothschild and Stein [RS]. In [T], the potential $V = |x|^2$ was considered in connection with the Hermite operator. Furthermore, it was pointed out by the referee that the L^p $(1 < p < \infty)$ boundedness of the operator $(-\Delta + V)^{i\gamma}$ in Theorem 0.4 follows from a general result of W. Hebisch [H].

Note that, if V is a nonnegative polynomial, then

$$\max_{x\in B} V(x) \le C\left(\frac{1}{|B|}\int_B V(x)dx\right) \tag{0.12}$$

for every ball B in $\mathbb{R}^n$, where C depends only on the degree of V and n. It follows that V satisfies (0.2), i.e., $V \in B_q$ for all q, $1 < q < \infty$, with the same constant as in (0.12). Hence, our Theorems 0.4 and 0.8 extend Zhong's results on the operator $(-\Delta+V)^{i\gamma}$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ to the general B_q class. Clearly, Theorem 0.3 implies that $\nabla^2(-\Delta+V)^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for all p, $1 < p < \infty$, if V satisfies (0.12). But it seems that extra conditions are needed to assure that the kernel function for the operator $\nabla^2(-\Delta + V)^{-1}$ satisfies the Calderón–Zygmund estimate (0.11).

It is interesting to notice that, if $V(x) = |x_n - \varphi(x')|^\alpha$, $\alpha > 0$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, then V satisfies (0.12) with a constant depending only on n, α and $\|\nabla\varphi\|_\infty$. Also, if $V(x) = |x|^\alpha$, $\alpha q > -n$, then $V \in B_q$. In particular, $V \in B_{n/2}$ if $\alpha > -2$; and $V \in B_n$ if $\alpha > -1$.

The Schrödinger operator $-\Delta + V$ with nonnegative potentials is useful in the study of certain subelliptic operators. Indeed, by taking the partial Fourier transform in the t variable, the operator $-\Delta_x - V(x)\partial_t^2$ is reduced to $-\Delta_x + V(x)\xi^2$. See [Sm].

We briefly sketch the argument we will use in this paper. First, we note that, in the case that $V = P(x)$ is a nonnegative polynomial, the weight function

$$M(x,V) = \sum_{|\alpha|\le k} |\partial_x^\alpha P(x)|^{1/(|\alpha|+2)} \tag{0.13}$$

plays a key role. In (0.13), k = degree of $P(x)$. See [Sm], [Z]. Here, for $V \in B_{n/2}$, we define the function $m(x, V)$ by

$$\frac{1}{m(x,V)} = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\right\}. \tag{0.14}$$

The function $m(x, V)$ was introduced in [Sh] for the potential V satisfying the condition (0.12) to study the Neumann problem for the operator

$-\Delta+V(x)$ in the region above a Lipschitz graph. Note that, if $r=\dfrac{1}{m(x,V)}$, then

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy=1.$$

In the case that $V=P(x)\geq 0$ is a polynomial, it can be shown that

$$m(x,V)\sim M(x,V).$$

Next, we use a lemma due to C. Fefferman and D.H. Phong [F] to show that, if $V\in B_{n/2}$,

$$|\Gamma(x,y)|\leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k}\cdot\frac{1}{|x-y|^{n-2}} \tag{0.15}$$

and, if $V\in B_n$,

$$|\nabla\Gamma(x,y)|\leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k}\cdot\frac{1}{|x-y|^{n-1}} \tag{0.16}$$

for any $k>0$, where $\Gamma(x,y)$ denotes the fundamental solution for the operator $-\Delta+V(x)$ in $\mathbb{R}^n$. We remark that similar approaches were used in [Sm] and [Z]. Also see [Sh].

Estimates (0.15) and (0.16) are essential to deal with the kernels of operators in the part where $|x-y|>\dfrac{1}{m(x,V)}$. For the part where $|x-y|\leq\dfrac{1}{m(x,V)}$, the key observation is that, if $V\in B_q$, $q>(n/2)$,

$$|\Gamma(x,y)-\Gamma_0(x,y)|\leq\frac{C\{|x-y|m(x,V)\}^{2-(n/q)}}{|x-y|^{n-2}} \tag{0.17}$$

where $\Gamma_0(x,y)$ is the fundamental solution for $-\Delta$ in $\mathbb{R}^n$.

Estimates like (0.15), (0.16) and (0.17) will enable us to control the operator $(-\Delta+V)^{i\gamma}$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ by the Hardy–Littlewood maximal function and the corresponding (maximal) singular integral operators associated with $-\Delta$.

Finally, we note that, to study $(-\Delta+V)^{i\gamma}$ and $\nabla(-\Delta+V)^{-1/2}$, by functional calculus, we actually need to deal with $\Gamma(x,y,\tau)$ and $\nabla\Gamma(x,y,\tau)$ where $\Gamma(x,y,\tau)$ is the fundamental solution for $-\Delta+V(x)+i\tau$, $\tau\in\mathbb{R}$. Moreover, if $V\in B_q$, $n/2\leq q<n$, we do not have pointwise estimates for $\nabla\Gamma(x,y,\tau)$. But, these difficulties are overcome by our basic idea that, the operators associated with $-\Delta+V$ can be viewed as a perturbation of the corresponding operators associated with $-\Delta$ in the scale less than $\{m(x,V)\}^{-1}$.

The paper is organized as follows. In Section 1 we introduce the auxiliary function $m(x, V)$ and study its properties. We also state and prove the Fefferman–Phong Lemma (Lemma 1.9) under the assumption $V \in B_{n/2}$. In Section 2 we establish the estimate (0.15) on the fundamental solutions (Theorem 2.7). Section 3 is devoted to the proof of Theorem 0.3. In Section 4 we give the proof of Theorem 0.4. Theorem 0.5 is proved in Section 5, while Theorem 0.8 is proved in Section 6. A counterexample is given in Section 7.

Throughout this paper, unless otherwise indicated, we will use C and c to denote constants, which are not necessarily the same at each occurrence, which depend at most on the constant in (0.2) and the dimension n. By $A \sim B$, we mean that there exist constants $C > 0$ and $c > 0$, such that $c \leq A/B \leq C$.

Finally, the author would like to thank the referee for pointing out the relevance of Hebish's work, and for the valuable comments concerning the limitations on p in Theorem 0.3 and 0.5, which lead to the counterexample in Section 7.

1. The auxiliary function $m(x, V)$.

Most of the results in this section were proved in [Sh] under the assumption (0.12). The extension to the case $V \in B_q$, $q > n/2$ is fairly straightforward. For the sake of completeness we provide the proofs.

Throughout the section we will assume that $V \in B_q$ for some $q > n/2$.

It is well known that $V \in B_q$, $q > 1$ implies that $V(x)dx$ is a doubling measure,

$$\int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy. \tag{1.1}$$

In fact, if $V \in B_q$, $q > 1$, then V is a Muckenhoupt A_∞ weight (e.g., see [St2]).

LEMMA 1.2. — *There exists $C > 0$ such that, for $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C \left(\frac{R}{r}\right)^{\frac{n}{q}-2} \cdot \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Proof. — By Hölder inequality,

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)} V dy &\le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q dy\right)^{1/q}\\ &\le \left(\frac{R}{r}\right)^{n/q}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} V^q dy\right)^{1/q}\\ &\le C\left(\frac{R}{r}\right)^{n/q}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} V dy\right)\end{aligned}$$

since $V \in B_q$.

The lemma then follows easily.

By Lemma 1.2 and the assumption $q > n/2$, we see that, for any $x \in \mathbb{R}^n$,

$$\lim_{r\to 0}\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = 0$$

and

$$\lim_{r\to\infty}\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = \infty.$$

DEFINITION 1.3. — *For $x \in \mathbb{R}^n$, the function $m(x,V)$ is defined by*

$$\frac{1}{m(x,V)} = \sup_{r>0}\left\{r\colon \frac{1}{r^{n-2}}\int_{B(x,r)} V dy \le 1\right\}.$$

Clearly, $0 < m(x,V) < \infty$ for every $x \in \mathbb{R}^n$ and if $r = \dfrac{1}{m(x,V)}$ then

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = 1.$$

Moreover, by Lemma 1.2, if

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V dy \sim 1, \quad \text{then } r \sim \frac{1}{m(x,V)}.$$

The following lemma is very useful to us.

LEMMA 1.4. — *There exist $C > 0$, $c > 0$ and $k_0 > 0$ such that, for x, y in $\mathbb{R}^n$,*

(a) $$m(x,V) \sim m(y,V) \ \text{ if } \ |x-y| \le \frac{C}{m(x,V)},$$

(b) $$m(y,V) \le C\{1+|x-y|m(x,V)\}^{k_0} m(x,V),$$

(c) $$m(y,V) \ge \frac{c\, m(x,V)}{\{1+|x-y|m(x,V)\}^{k_0/(k_0+1)}}.$$

Proof. — Let $r = \dfrac{1}{m(x,V)}$. Suppose $|x-y| \le Cr$. Since Vdx is a doubling measure,

$$\frac{1}{r^{n-2}} \int_{B(y,r)} V dz \sim \frac{1}{r^{n-2}} \int_{B(x,r)} V dz = 1.$$

It follows that

$$m(y,V) \sim \frac{1}{r} = m(x,V).$$

To see part (b), suppose $|y-x| \sim 2^j r, j \ge 1$. Let $0 < r_1 < r$ and $2^k r_1 \sim 2^j r$. Then, by Lemma 1.2,

$$\begin{aligned}
\int_{B(y,r_1)} V dz &\le C(2^k)^{(n/q)-n} \int_{B(y,2^k r_1)} V dz \\
&\le C(2^k)^{(n/q)-n} \int_{B(y,2^j r)} V dz \\
&\le C(2^k)^{(n/q)-n} \int_{B(x,2^j r)} V dz \\
&\le C(2^k)^{(n/q)-n} . C_0^j . \int_{B(x,r)} V dz \text{ (by the doubling condition (1.1))} \\
&= C(2^k)^{(n/q)-n} \cdot C_0^j \cdot r^{n-2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{r_1^{n-2}} \int_{B(y,r_1)} V dz &\le C(2^k)^{(n/q)-n} \cdot C_0^j \cdot \left(\frac{r}{r_1}\right)^{n-2} \\
&\le C(2^{(n/q)-n} C_0)^j \cdot \left(\frac{r}{r_1}\right)^{(n/q)-2} \\
&\le 1/2 \ \text{ if } \ r_1 \le C_1^{-j} r \ \text{ and } \ C_1 \ \text{ is large .}
\end{aligned}$$

Hence, by Definition 1.3,

$$\frac{1}{m(y,V)} \ge C_1^{-j} r.$$

It follows that

$$m(y,V) \le C_1^j m(x,V) \le C\{1 + |x-y| m(x,V)\}^{k_0} m(x,V), k_0 = \log_2 C_1.$$

Finally, we need to show part (c). We may assume that

$$|x-y| \ge \frac{1}{m(y,V)},$$

for otherwise it follows easily from part (a).

By part (b),

$$\begin{aligned} m(x,V) &\le C\{1+|x-y|m(y,V)\}^{k_0} m(y,V) \\ &\le C|x-y|^{k_0} m(y,V)^{k_0+1}. \end{aligned}$$

Thus,

$$m(y,V) \ge \frac{c\,m(x,V)^{1/(k_0+1)}}{|x-y|^{k_0/(k_0+1)}} \ge \frac{c\,m(x,V)}{\{1+m(x,V)|x-y|\}^{k_0/(k_0+1)}}.$$

The proof is complete.

The following is an easy consequence of Lemma 1.4:

COROLLARY 1.5. — *There exist $C>0$, $c>0$ and $k_0>0$ such that, for any $x, y \in \mathbb{R}^n$,*

$$\begin{aligned} c\{1+|x-y|m(y,V)\}^{1/(k_0+1)} &\le 1+|x-y|m(x,V) \\ &\le C\{1+|x-y|m(y,V)\}^{k_0+1}. \end{aligned}$$

Using Hölder inequality and the B_q condition we see that

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2}}dy \le \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)dy. \tag{1.6}$$

Similarly, if $V \in B_n$, hence $V \in B_{n+\varepsilon}$,

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-1}}dy \le \frac{C}{R^{n-1}} \int_{B(x,R)} V(y)dy. \tag{1.7}$$

LEMMA 1.8. — *There exist $C>0$ and $k_0>0$ such that, if $Rm(x,V) \ge 1$, then*

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V dy \le C\{Rm(x,V)\}^{k_0}.$$

Proof. — Let $r = \dfrac{1}{m(x,V)}$. Suppose $2^j r \le R < 2^{j+1} r$, $j \ge 0$. Then, by the doubling condition (1.1),

$$\int_{B(x,R)} V dy \le C_0^{j+1} \int_{B(x,r)} V dy = C_0^{j+1} r^{n-2}.$$

It follows that

$$\begin{aligned} \frac{1}{R^{n-2}} \int_{B(x,R)} V dy &\le C_0^{j+1} \left(\frac{r}{R}\right)^{n-2} \le C_0 (C_0 2^{2-n})^j \\ &\le C(Rm(x,V))^{k_0}, k_0 = \log_2 C_0 + 2 - n. \end{aligned}$$

We end this section with a lemma due to C. Fefferman and D.H. Phong [F], which plays the crucial role in the estimates of the fundamental solution for the Schrödinger operator $-\Delta + V(x)$.

LEMMA 1.9 (C. Fefferman-D.H. Phong). — *Let $u \in C_c^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |u(x)|^2 m(x,V)^2 dx \le C \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^n} |u(x)|^2 V(x) dx \right\}.$$

Proof. — Fix $x_0 \in \mathbb{R}^n$, let $r_0 = \dfrac{1}{m(x_0,V)}$. Then

$$\int_B |\nabla u(x)|^2 dx \ge \frac{c}{r_0^{n+2}} \int_B \int_B |u(x) - u(y)|^2 dx dy$$

$$\int_B |u(x)|^2 V(x) dx \ge \frac{c}{r_0^n} \int_B \int_B |u(y)|^2 V(y) dx dy$$

where $B = B(x_0, r_0)$.

Adding two inequalities we obtain

$$\int_B |\nabla u|^2 dx + \int_B |u|^2 V dx \ge \frac{c}{r_0^n} \int_B \int_B \min_{y \in B} \left\{ \frac{c_0}{r_0^2}, V(y) \right\} |u(x)|^2 dx dy$$

where $c_0 > 0$ is a constant to be determined later.

Recall that V is an A_∞ weight. Hence, there exists $\varepsilon > 0$ such that, for every ball B in $\mathbb{R}^n$,

$$\left| \left\{ x \in B : V(x) \ge \frac{\varepsilon}{|B|} \int_B V dy \right\} \right| \ge \frac{1}{2} |B|.$$

Now, let $c_0 = \varepsilon |B(0,1)|^{-1}$, then

$$\int_B \min_{y \in B} \left\{ \frac{c_0}{r_0^2}, V(y) \right\} dy \ge c r_0^{n-2}.$$

It follows that

$$\int_B |u(x)|^2 m(x,V)^2 dx \le \frac{C}{r_0^2} \int_B |u|^2 dx \le C \left\{ \int_B |\nabla u|^2 dx + \int_B |u|^2 V dx \right\}$$

where we have used part (a) of Lemma 1.4. Moreover,

$$\int_{B(x_0,r_0)} |u(x)|^2 m(x,V)^{2+n} dx \le C \left\{ \int_{B(x_0,r_0)} |\nabla u(x)|^2 m(x,V)^n dx \right.$$

$$\left. + \int_{B(x_0,r_0)} |u(x)|^2 V(x) m(x,V)^n dx \right\}.$$

To finish the proof, we integrate both sides of the above inequality in x_0 over $\mathbb{R}^n$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x)|^2 m(x,V)^{2+n}\left(\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0\right) dx \\ \leq C\left\{\int_{\mathbb{R}^n} |\nabla u(x)|^2 m(x,V)^n \left(\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0\right) dx\right. \\ \left. + \int_{\mathbb{R}^n} |u(x)|^2 V(x) m(x,V)^n \left(\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0\right)\right\}.\end{aligned}$$

Finally, note that, by part (a) of Lemma 1.4,

$$\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \sim \left(\frac{1}{m(x,V)}\right)^n.$$

The lemma then follows easily.

2. Estimates of fundamental solutions.

This section is devoted to the estimates of fundamental solutions for the operator $-\Delta + (V + i\tau)$ on $\mathbb{R}^n$ where $\tau \in \mathbb{R}$.

We will assume that $V \in B_q$ for some $q > n/2$ throughout this section.

LEMMA 2.1. — *Suppose* $-\Delta u + (V + i\tau)u = 0$ *in* $B(x_0, 2R)$ *for some* $x_0 \in \mathbb{R}^n, R > 0$ *and* $\tau \in \mathbb{R}$. *Then*

$$\begin{aligned}\sup\{|u(x)| : x \in B(x_0,R)\} \\ \leq \frac{C_k}{\{1 + R|\tau|^{1/2}\}^k \{1 + Rm(x_0,V)\}^k} \left\{\frac{1}{R^n}\int_{B(x_0,2R)} |u|^2 dy\right\}^{1/2}\end{aligned}$$

for any integer $k > 0$, *where* C_k *depends only on* k, n *and the constant in the reverse Hölder inequality (0.2).*

Proof. — Since

$$\Delta(|u|^2) = 2Re\Delta u \cdot \overline{u} + 2|\nabla u|^2 = 2V|u|^2 + 2|\nabla u|^2 \geq 0,$$

$|u|^2$ is subharmonic. It follows that

$$\sup\{|u(x)| : x \in B(x_0,R)\} \leq C\left(\frac{1}{R^n}\int_{B(x_0,3R/2)} |u|^2 dy\right)^{1/2}. \tag{2.2}$$

By Caccioppoli's inequality,

$$\int_{B(x_0,3R/2)} |\nabla u|^2 dx + \int_{B(x_0,3R/2)} |u|^2 V dx + |\tau| \int_{B(x_0,3R/2)} |u|^2 dx \leq \frac{C}{R^2} \int_{B(x_0,2R)} |u|^2 dx. \tag{2.3}$$

Now, let $\eta \in C_0^\infty(B(x_0,3R/2))$ such that $\eta \equiv 1$ on $B(x_0,R)$ and $|\nabla \eta| \leq C/R$. Applying Lemma 1.9 to the function $u\eta$, we obtain

$$\begin{aligned}\int_{B(x_0,R)} m(x,V)^2 |u|^2 dx &\leq C \left\{ \int_{B(x_0,3R/2)} |\nabla u|^2 dx + \int_{B(x_0,3R/2)} |u|^2 V dx + \frac{1}{R^2} \int_{B(x_0,2R)} |u|^2 dx \right\} \\ &\leq \frac{C}{R^2} \int_{B(x_0,2R)} |u|^2 dx\end{aligned}$$

where we used (2.3) in the last inequality.

Note that, by part (c) of Lemma 1.4, for $x \in B(x_0,R)$,

$$m(x,V) \geq \frac{c\, m(x_0,V)}{\{1+Rm(x_0,V)\}^{k_0/(k_0+1)}}.$$

It follows that

$$\int_{B(x_0,R)} |u|^2 dx \leq C \cdot \frac{\{1+Rm(x_0,V)\}^{2k_0/(k_0+1)}}{\{m(x_0,V)R\}^2} \int_{B(x_0,2R)} |u|^2 dx.$$

Hence,

$$\int_{B(x_0,R)} |u|^2 dx \leq \frac{C}{\{1+Rm(x_0,V)\}^{2/(k_0+1)}} \int_{B(x_0,2R)} |u|^2 dx.$$

Clearly, by repeating above argument, we have

$$\int_{B(x_0,3R/2)} |u|^2 dx \leq \frac{C_k}{\{1+Rm(x_0,V)\}^k} \int_{B(x_0,2R)} |u|^2 dx \text{ for any } k>0. \tag{2.4}$$

Similarly, by Caccioppoli's inequality (2.3),

$$\int_{B(x_0,3R/2)} |u|^2 dx \leq \frac{C_k}{(1+|\tau|^{1/2}R)^k} \int_{B(x_0,2R)} |u|^2 dx \quad \text{for any } k>0. \tag{2.5}$$

The lemma then follows from (2.2), (2.4) and (2.5).

Let $\Gamma(x,y,\tau)$ denote the fundamental solution for the Schrödinger operator $-\Delta + (V(x)+i\tau), \tau \in \mathbb{R}$. Clearly,

$$\Gamma(x,y,\tau) = \Gamma(y,x,-\tau).$$

Since $V \geq 0$ and $V \in L^{n/2}_{\mathrm{loc}}$, it is well known that

$$|\Gamma(x,y,\tau)| \leq \frac{C}{|x-y|^{n-2}} \quad \text{for } x,y \in \mathbb{R}^n \tag{2.6}$$

where C depends only on n.

Since $V \in B_{n/2}$ implies that $V \in B_q$ for some $q > n/2$, the following theorem follows easily from Lemma 2.1 and (2.6).

THEOREM 2.7. — *Suppose $V \in B_{n/2}$. Then, for any $x, y \in \mathbb{R}^n$,*

$$|\Gamma(x,y,\tau)| \leq \frac{C_k}{\{1+|\tau|^{1/2}|x-y|\}^k\{1+m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2}}$$

where C_k is a constant depending only on n, k and the constant in (0.2).

COROLLARY 2.8. — *Suppose $V \in B_{n/2}$. Then there exists $C > 0$ depending on n and the constant in (0.2) such that, for every $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$,*

$$\|m(\cdot,V)^2(-\Delta+V)^{-1}f\|_p \leq C\|f\|_p.$$

Proof. — It follows from Theorem 2.7 that

$$\begin{aligned}\int_{\mathbb{R}^n} |\Gamma(x,y)|dy &\leq C\int_{\mathbb{R}^n} \frac{dy}{\{1+|x-y|m(x,V)\}^3|x-y|^{n-2}} \\ &\leq \frac{C}{m(x,V)^2}\end{aligned}$$

where $\Gamma(x,y) = \Gamma(x,y,0)$ is the fundamental solution for $-\Delta+V(x)$. Thus, if $f \in L^p(\mathbb{R}^n)$,

$$u(x) = (-\Delta+V)^{-1}f(x) = \int_{\mathbb{R}^n} \Gamma(x,y)f(y)dy,$$

we have

$$\begin{aligned}|u(x)| &\leq \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)|dy\right\}^{1/p'} \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)||f(y)|^p dy\right\}^{1/p} \\ &\leq \frac{C}{m(x,V)^{2/p'}} \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)||f(y)|^p dy\right\}^{1/p}, \quad p' = \frac{p}{p-1}.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^n} |m(x,V)^2u(x)|^p dx \leq C\int_{\mathbb{R}^n} |f(y)|^p \left\{\int_{\mathbb{R}^n} m(x,V)^2|\Gamma(x,y)|dx\right\} dy.$$

Finally, note that, by part (b) of Lemma 1.4 and Theorem 2.7,

$$\int_{\mathbb{R}^n} m(x,V)^2|\Gamma(x,y)|dx \leq C_k \int_{\mathbb{R}^n} \frac{(m(y,V))^2}{\{1+|x-y|m(y,V)\}^{k-2k_0}|x-y|^{n-2}}dx$$
$$\leq C \quad \text{if we choose} \quad k = 2k_0 + 3.$$

Corollary 2.8 then follows.

Remark 2.9. — If V satisfies (0.12), then $V(x) \leq Cm(x,V)^2$ a.e. on $\mathbb{R}^n$. It follows from Corollary 2.8 that, for $1 \leq p \leq \infty$,

$$\|V(-\Delta+V)^{-1}f\|_p \leq C\|f\|_p.$$

Also, for $1 < p < \infty$, by the L^p boundedness of the Riesz transforms,

$$\begin{aligned}\|\nabla^2(-\Delta+V)^{-1}f\|_p &\leq C\|\Delta(-\Delta+V)^{-1}f\|_p \\ &\leq C\|V(-\Delta+V)^{-1}f\|_p + C\|f\|_p \\ &\leq C\|f\|_p.\end{aligned}$$

3. The proof of Theorem 0.3.

In this section we give the proof of Theorem 0.3.

THEOREM 3.1. — *Suppose $V \in B_q$ for some $q \geq n/2$. Then, for $1 \leq p \leq q$,*

$$\|V(-\Delta+V)^{-1}f\|_p \leq C\|f\|_p$$

where C depends only on n and the constant in (0.2).

Proof. — Let $f \in L^p(\mathbb{R}^n), 1 \leq p \leq q$ and

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x,y)f(y)dy.$$

We need to show that

$$\|Vu\|_p \leq C\|f\|_p.$$

Write

$$u(x) = \int_{|y-x|<r} \Gamma(x,y)f(y)dy + \int_{|y-x|\geq r} \Gamma(x,y)f(y)dy = u_1(x) + u_2(x)$$

where $r = \dfrac{1}{m(x,V)}$.

By the properties of B_q class, $V \in B_{q_0}$ for some $q_0 > q$. We have

$$|u_1(x)| \le \int_{|y-x|<r} \frac{|f(y)|}{|x-y|^{n-2}} dy$$

$$\le Cr^{2-\frac{n}{q_0}} \left(\int_{|y-x|<r} |f(y)|^{q_0} dy \right)^{1/q_0}$$

where we have used Hölder inequality and the fact $q_0 > n/2$. Thus,

$$\int_{\mathbb{R}^n} |V(x)u_1(x)|^{q_0} dx \le C \int_{\mathbb{R}^n} \left\{ \int_{|y-x|<\frac{1}{m(x,V)}} |f(y)|^{q_0} dy \right\} V(x)^{q_0} m(x,V)^{n-2q_0} dx$$

$$= C \int_{\mathbb{R}^n} |f(y)|^{q_0} \left\{ \int_{|x-y|<\frac{1}{m(x,V)}} V(x)^{q_0}\, m(x,V)^{n-2q_0} dx \right\} dy.$$

Now, let $R = \dfrac{1}{m(y,V)}$. Then

$$\int_{|x-y|<\frac{1}{m(x,V)}} V(x)^{q_0} m(x,V)^{n-2q_0} dx \le CR^{2q_0-n} \int_{|x-y|\le CR} V(x)^{q_0} dx$$

$$\le CR^{2q_0} \left\{ \frac{1}{R^n} \int_{|x-y|\le CR} V(x) dx \right\}^{q_0}$$

$$\le C$$

where we used the part (a) of Lemma 1.4, (0.2) and Lemma 1.8.

Hence, we have proved that

$$\int_{\mathbb{R}^n} |V(x)u_1(x)|^{q_0} dx \le C \int_{\mathbb{R}^n} |f(x)|^{q_0} dx \quad \text{for some } \ q_0 > q \ge n/2.$$

Next, note that, by (1.6),

$$\int_{\mathbb{R}^n} |V(x)u_1(x)| dx \le C \int_{\mathbb{R}^n} |f(y)| \left\{ \int_{|x-y|<\frac{1}{m(x,V)}} \frac{V(x)}{|x-y|^{n-2}} dx \right\} dy$$

$$\le C \int_{\mathbb{R}^n} |f(y)| dy.$$

Therefore, by interpolation,

$$\|Vu_1\|_p \le C\|f\|_p \ \text{ for } \ 1 \le p \le q_0.$$

To finish the proof, we note that, by Theorem 2.7 and the Hölder inequality,

$$|u_2(x)| \le C \int_{|y-x|\ge r} \frac{|f(y)| dy}{\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}}$$

$$\le Cr^{2/p'} \left\{ \int_{|y-x|\ge r} \frac{|f(y)|^p dy}{\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}} \right\}^{1/p}$$

where $1 \leq p \leq q_0$ and $r = \dfrac{1}{m(x,V)}$.

Thus,

$$\int_{\mathbb{R}^n} |V(x)u_2(x)|^p dx \leq C \int_{\mathbb{R}^n} |f(y)|^p$$

$$\left\{ \int_{|y-x| \geq \frac{1}{m(x,V)}} \frac{|V(x)|^p dx}{m(x,V)^{2(p-1)}\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}} \right\} dy.$$

Now, fix $y \in \mathbb{R}^n$, let $R = \dfrac{1}{m(y,V)}$. By Lemma 1.4,

$$\int_{|y-x| \geq \frac{1}{m(x,V)}} \frac{|V(x)|^p dx}{m(x,V)^{2(p-1)}\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}}$$

$$\leq C \int_{|y-x| \geq CR} \frac{|V(x)|^p dx}{R^{2(1-p)}(1+R^{-1}|x-y|)^{k_1}|x-y|^{n-2}},$$

$$\left(\text{where } k_1 = \frac{k-2(q-1)k_0}{k_0+1}\right)$$

$$\leq C \sum_{j=1}^{\infty} \frac{1}{(2^j R)^n} \int_{|x-y| \leq 2^j R} V^p dx \cdot (2^j)^{-k_1+2} \cdot R^{2p}$$

$$\leq C \sum_{j=1}^{\infty} \left(\frac{1}{(2^j R)^n} \int_{|x-y| \leq 2^j R} V(x) dx \right)^p \cdot (2^j)^{-k_1+2} \cdot R^{2p}$$

$$\leq C \sum_{j=1}^{\infty} \left(\frac{1}{R^{n-2}} \int_{|x-y| \leq R} V(x) dx \right)^p \cdot C^j \cdot (2^j)^{-k_1+2}$$

$$\leq C \quad \text{if we choose } k \text{ sufficiently large.}$$

From this we have

$$\int_{\mathbb{R}^n} |V(x)u_2(x)|^p dx \leq C \int_{\mathbb{R}^n} |f(x)|^p dx \quad \text{for} \quad 1 \leq p \leq q_0.$$

The theorem is then proved.

We are now in a position to give

The proof of Theorem 0.3. — Suppose $V \in B_q$ for some $q \geq n/2$. By Theorem 3.1

$$\|V(-\Delta+V)^{-1}f\|_p \leq C\|f\|_p \quad \text{for} \quad 1 \leq p \leq q.$$

It follows that

$$\|\Delta(-\Delta+V)^{-1}f\|_p \leq C\|f\|_p \quad \text{for} \quad 1 \leq p \leq q.$$

Then, by the L^p boundedness of Riesz transforms, for $1 < p \le q$,

$$\|\nabla^2(-\Delta + V)^{-1} f\|_p \le C_p \|\Delta(-\Delta + V)^{-1} f\|_p \le C_p \|f\|_p.$$

4. The proof of Theorem 0.4.

In this section we give the proof of Theorem 0.4 stated in the Introduction. We also prove an L^p estimate for the operator $V^{1/2}\nabla(-\Delta+V)^{-1}$ in the section (Theorem 4.13).

By the functional calculus, we may write, for $\gamma \in \mathbb{R}$,

$$(-\Delta + V)^{i\gamma} = -\frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{i\gamma}(-\Delta + V + i\tau)^{-1} d\tau. \tag{4.1}$$

Thus,

$$\begin{aligned}(-\Delta + V)^{i\gamma} f(x) &= -\frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{i\gamma}(-\Delta + V + i\tau)^{-1} f(x) d\tau \\ &= \int_{\mathbb{R}^n} K(x,y) f(y) dy\end{aligned}$$

where

$$K(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{i\gamma}\Gamma(x,y,\tau) d\tau. \tag{4.2}$$

Note that, by Theorem 2.7,

$$|K(x,y)| \le \frac{C_k e^{\pi|\gamma|/2}}{\{1 + |x-y| m(x,V)\}^k} \cdot \frac{1}{|x-y|^n}, \quad \text{for any } k > 0. \tag{4.3}$$

Let $\Gamma_0(x,y,\tau)$ denote the fundamental solution for the operator $-\Delta + i\tau$ in $\mathbb{R}^n$. It is well known that

$$\begin{cases} |\Gamma_0(x,y,\tau)| \le \dfrac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \dfrac{1}{|x-y|^{n-2}} \\ |\nabla_x \Gamma_0(x,y,\tau)| \le \dfrac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \dfrac{1}{|x-y|^{n-1}} \\ |\nabla_x^2 \Gamma_0(x,y,\tau)| \le \dfrac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \dfrac{1}{|x-y|^{n}} \end{cases} \tag{4.4}$$

where C_k is independent of $\tau \in \mathbb{R}$.

LEMMA 4.5. — *Suppose $V \in B_{n/2}$. Then there exists $C_k > 0$ such that*

$$|\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)| \le \frac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \frac{\{|x-y| m(x,V)\}^{2-(n/q_0)}}{|x-y|^{n-2}}$$

for $x, y \in \mathbb{R}^n$, $|x-y| \leq \dfrac{1}{m(x,V)}$ *and for some* $q_0 > n/2$.

Proof. — Note that

$$-\Delta_x\{\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)\}+i\tau\{\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)\} = -V(x)\Gamma(x,y,\tau).$$

We have

$$\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau) = -\int_{\mathbb{R}^n}\Gamma_0(x,z,\tau)V(z)\Gamma(z,y,\tau)dz.$$

Now, let $R=|x-y|$ and suppose $R \leq \dfrac{1}{m(x,V)}$. By Theorem 2.7,

$$\begin{aligned}
&|\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)| \\
&\leq C_k \int_{\mathbb{R}^n} \frac{(1+|\tau|^{1/2}|x-z|)^{-k}(1+|\tau|^{1/2}|z-y|)^{-k}V(z)\,dz}{|x-z|^{n-2}\{1+m(y,V)|z-y|\}^k|z-y|^{n-2}} \\
&= C_k\int_{|z-x|<R/2} + C_k\int_{|z-y|<R/2} + C_k\int_{\substack{|z-x|\geq R/2\\ |z-y|\geq R/2}} \\
&= I_1+I_2+I_3.
\end{aligned}$$

Since $V \in B_{n/2}$, $V \in B_{q_0}$ for some $q_0 > n/2$. We obtain

$$\begin{aligned}
I_1 &\leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\cdot\int_{B(x,R/2)}\frac{V(z)dz}{|z-x|^{n-2}} \\
&\leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\cdot\frac{1}{R^{n-2}}\int_{B(x,R/2)}V(z)dz \\
&\leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{\{Rm(x,V)\}^{2-(n/q_0)}}{R^{n-2}}
\end{aligned}$$

where we have used (1.6) in the second inequality and Lemma 1.2 in the third.

Similarly,

$$I_2 \leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{\{Rm(x,V)\}^{2-(n/q_0)}}{R^{n-2}}.$$

Finally, we need to estimate I_3.

It is easy to see that

$$\begin{aligned}
I_3 &\leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\int_{|z-y|\geq R/2}\frac{V(z)dz}{|z-y|^{2n-4}\{1+m(y,V)|z-y|\}^k} \\
&\leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\left\{\int_{r>|z-y|\geq\frac{R}{2}}\frac{V(z)dz}{|z-y|^{2n-4}}+r^k\int_{|z-y|\geq r}\frac{V(z)dz}{|z-y|^{2n-4+k}}\right\}
\end{aligned}$$

where $r = \dfrac{1}{m(y,V)} \sim \dfrac{1}{m(x,V)}$.

Using Hölder inequality and the B_{q_0} condition (0.2), we obtain

$$\begin{aligned}
\int_{r>|z-y|\geq R/2} \frac{V(z)dz}{|z-y|^{2n-4}}
&\leq C\left\{\int_{B(y,r)} V(z)^{q_0}dz\right\}^{1/q_0}\left\{\int_{R/2}^{r} t^{(n-1)-2(n-2)q_0'}dt\right\}^{1/q_0'}\\
&\leq C\left(\frac{R}{r}\right)^{2-(n/q_0)}\cdot\frac{1}{R^{n-2}}\cdot\left\{\int_{1/2}^{r/R} t^{(n-1)-2(n-2)q_0'}dt\right\}^{1/q_0'}\\
&\leq \frac{C\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}
\end{aligned}$$

where we have assumed, without loss of generality, that $(1/q_0) > (4-n)/n$ for $n \geq 3$, so $n-2(n-2)q_0' < 0$.

Also, using the doubling condition and taking k sufficiently large, we see that

$$\begin{aligned}
r^k\int_{|z-y|\geq r}\frac{V(z)dz}{|z-y|^{2n-4+k}} &\leq Cr^k\sum_{j=1}^{\infty}\frac{1}{(2^jr)^{2n-4+k}}\int_{|z-y|\leq 2^jr}V(z)dz\\
&\leq Cr^{4-2n}\sum_{j=1}^{\infty}\frac{1}{(2^j)^{2n-4+k}}\cdot C_0^j\int_{|z-y|\leq r}V(z)dz\\
&\leq \frac{C}{r^{n-2}}\\
&\leq \frac{C\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}.
\end{aligned}$$

Thus,

$$I_3 \leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}.$$

The proof is then finished since $m(y,V) \sim m(x,V)$ when $|x-y| \leq \dfrac{1}{m(x,V)}$.

We need another lemma before we carry out the proof of Theorem 0.4.

LEMMA 4.6. — *Suppose $V \in B_{q_0}$, $(n/2) < q_0 < n$. Assume that $-\Delta u + (V+i\tau)u = 0$ in $B(x_0,2R)$. Then*

$$\left(\int_{B(x_0,R)}|\nabla u|^t dx\right)^{1/t} \leq CR^{(n/q_0)-2}\{1+Rm(x_0,V)\}^{k_0}\sup_{B(x_0,2R)}|u|$$