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Central sidonicity for compact Lie groups


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0. Introduction.

Suppose $G$ is a compact group with dual object $\hat{G}$. It is well known that if $G$ is an abelian group, then every infinite subset of $\hat{G}$ contains an infinite Sidon set [8]. In contrast, there are non-abelian groups which admit no infinite central Sidon sets [11]. For central $p$-Sidon sets the situation is quite different; even in the non-abelian setting these are plentiful. Indeed, Dooley [3] showed that every compact, connected group admits an infinite central $p$-Sidon set for all $p > 1$, however he was unable to determine if every infinite set contains an infinite central $p$-Sidon subset.

The main result of our paper answers this question affirmatively. In fact, we prove formally more. We study certain weighted generalizations of Sidon sets, introduced in [5], called (central) $(a,p)$-Sidon sets, which arise by considering classical Sidonicity with the Fourier transform weighted by the $a$'th powers of the representation degree: (central) $(1,p)$-Sidon sets are (central) $p$-Sidon sets. We prove that every infinite subset of the dual of a compact, connected group contains an infinite subset which is central $(a,p)$-Sidon for all $p \geq 1$ and $a < 2p - 1$. Our method is essentially constructive: we show that certain "lacunary-like" sets have the desired property.

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When $G$ is a compact, simply-connected, semisimple Lie group of rank $\ell$, the dual object can be identified with the set of dominant weights and consequently with $(\mathbb{Z}^+)^\ell$. Our examples of central $(a, p)$-Sidon sets in the duals of these groups correspond to Sidon sets in $\mathbb{Z}^\ell$. A natural question to ask is if all Sidon sets in $(\mathbb{Z}^+)^\ell$ correspond to Sidon-type sets in $\hat{G}$. We show that such sets are always central $(0, 1)$-Sidon, but need not be central $(a, 1)$-Sidon for any $a > 0$, and that there are central $(a, 1)$-Sidon sets in $\hat{G}$ which do not correspond to Sidon sets in $(\mathbb{Z}^+)^\ell$.

1. Preliminaries.

If $G$ is a compact group, $\hat{G}$ will denote a maximal set of pairwise inequivalent, unitary, irreducible representations of $G$. The degree of $\sigma \in \hat{G}$ will be denoted by $d_\sigma$.

The following generalization of Sidonicity was introduced in [5].

**Definition.** Let $a \in \mathbb{R}$, $1 < p < \infty$. A subset $E$ of $\hat{G}$ is called a (central) $(a, p)$-Sidon set if there is a constant $\kappa(a, p)$ so that whenever $f = \sum_{\sigma \in E} d_\sigma \text{Tr} A_\sigma \sigma$ is a (central) trigonometric polynomial on $G$, then

$$\|\hat{f}\|_{(a, p)} = \left( \sum_{\sigma \in E} d_\sigma^a \text{Tr} |A_\sigma|^p \right)^{1/p} \leq \kappa(a, p) \|f\|_\infty.$$  

(Central) $(1, p)$-Sidon sets are usually called (central) $p$-Sidon and (central) $1$-Sidon sets are simply referred to as (central) Sidon sets.

Obviously if $E$ consists of representations of bounded degree there is no distinction between $(a, p)$-Sidonicity for different values of $a$; if $G$ is abelian then central $p$-Sidon and $p$-Sidon properties coincide; and (for all groups) it is easier to be (central) $(a, p)$-Sidon as $a$ decreases or $p$ increases. There are other relationships between $(a, p)$-Sidon sets. For this paper we only need note that since $\ell^q \subset \ell^p$ if $q < p$, then any central $(a, q)$-Sidon set is central $(b, p)$-Sidon provided $(b + 1)/p \leq (a + 1)/q$. In particular any set which is central $(a, 1)$-Sidon for all $a < 1$ is also central $(b, p)$-Sidon for all $p \geq 1$ and $b < 2p - 1$.

One reason for the interest in $(a, p)$-Sidon sets is the scarcity of (central) Sidon sets: a compact, connected group admits an infinite central Sidon set if and only if it is not a semisimple Lie group [11], [12].
It is seen in [5] that if \( G \) is an infinite compact, connected group then \( \hat{G} \) is never central \((0, 1)\)-Sidon, but there are examples where \( \hat{G} \) is \((-\varepsilon, 1)\)-Sidon for any given \( \varepsilon > 0 \). Also, every central \((1 + \varepsilon, 1)\)-Sidon set for \( \varepsilon > 0 \) is a set of representations of bounded degree; consequently our interest (when \( p = 1 \)) is in the range \( 0 \leq a \leq 1 \).

There are a number of equivalent characterizations of (central) \((a, p)\)-Sidonicity (see [5]). For example, analogous to [6], 37.2 we have

**Proposition 1.1.** — Let \( G \) be a compact group. A subset \( E \) of \( \hat{G} \) is central \((a, 1)\)-Sidon if and only if whenever \( \phi \in \ell^\infty(E) \) there is a central measure \( \mu \) on \( G \) with

\[
\hat{\mu}(\sigma) \equiv \int_G \frac{\Tr \sigma}{d_\sigma} d\mu = \frac{\phi(\sigma)}{d_\sigma^{1-a}} \quad \text{for all } \sigma \in E.
\]

Next we recall some notation and basic facts from Lie theory. The reader is referred to [7] or [14] for more details. Let \( G \) denote a compact, simply-connected, semisimple Lie group of rank \( \ell \), \( T^\ell \) a maximal torus for \( G \) and \( t \) its Lie algebra. Let \( \Phi \) denote the set of roots for \((G, T^\ell)\) and \( \Phi^+ \) the positive roots relative to a fixed base \( \Delta \). To each \( \lambda = (n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell \) we associate the weight \( \lambda = \sum_{j=1}^\ell n_j \lambda_j \) where \( \lambda_j \) are the fundamental dominant weights relative to \( \Delta \), and we denote by \( \Lambda^+ \) the set of all dominant weights i.e. the set of all \( \lambda \) with non-negative integer coefficients. We view \( \Phi \) as a subset of \( i t^* \). The lattice of weights \( \Lambda \) is isomorphic to \( \widehat{T^\ell} : \lambda = \sum_{j=1}^\ell n_j \lambda_j \) determines a character on \( T^\ell \) by the map: \( H \mapsto e^{\lambda(H)} = e^{\sum j \lambda_j(H)} \) for \( H \in t \). The set \( \hat{G} \) is in a \( 1 \)-\( 1 \) correspondence with \( \Lambda^+ \); \( \sigma_\lambda \in \hat{G} \) is indexed by its highest weight \( \lambda \in \Lambda^+ \). Thus if \( E \) is a subset of \((\mathbb{Z}^+)\^\ell \), then \( E \) indexes a subset of \( \hat{G} \) in a canonical way, and we refer to this subset of \( \hat{G} \) by \( E \) as well. It should be clear from the context which set is actually meant.

A partial order is defined on \( \Lambda \) by the positive roots: \( \mu < \sigma \) if and only if \( \sigma - \mu \) is a non-negative integral sum of positive roots. The Weyl group will be denoted by \( W \) and the weights of \( \sigma \in \Lambda^+ \) by

\[
\Pi(\sigma) \equiv \{ \mu \in \Lambda : w(\mu) < \sigma \quad \text{for all } w \in W \}.
\]

The set \( \Pi(\sigma) \) consists of all \( \mu \in \Lambda^+ \) with \( \mu < \sigma \), together with all their Weyl-conjugates. Lastly, we set \( \rho = \sum_{j=1}^\ell \lambda_j \); \( \rho \) is also half the sum of the positive roots.

One reason for the success in studying central \((a, p)\)-Sidon sets is that there are formulas for \( \Tr \sigma \) restricted to the torus. One of these is the Weyl character formula:
\[ \text{Tr} \sigma(x) = \sum_{w \in W} \frac{\det(w) e^{iw(\sigma+\rho)(x)}}{q(x)}, \quad x \in T^\ell \]

where

\[ q(x) = \sum_{w \in W} \det(w) e^{iw(\rho)(x)} = e^{-i\rho(x)} \prod_{\alpha \in \Phi^+} (e^{i\alpha(x)} - 1). \]

Related to this is the Weyl dimension formula which states:

\[ d_\sigma = \prod_{\alpha \in \Phi^+} \frac{\langle \sigma + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}. \]

A final fact which we will record here is that the weights in \( \Pi(\sigma) \) correspond to the irreducible subrepresentations of \( \sigma|_{T^\ell} \), and so we also have the formula

\[ \text{Tr} \sigma(x) = \sum_{\mu \in \Pi(\sigma)} m_\sigma(\mu) e^{i\mu(x)}, \quad x \in T^\ell \]

where \( m_\sigma(\mu) \) is the multiplicity of \( \mu \) in \( \sigma|_{T^\ell} \).

2. Main result.

In [3], Dooley constructs in the dual of any compact, connected, semisimple Lie group examples of infinite sets which are \( p \)-Sidon for all \( p > 1 \). By making the obvious modifications to his proof these examples can be seen to be central \( (a, p) \)-Sidon for all \( p \geq 1 \) and \( a < 2p-1 \). Consequently, every compact, connected group \( G \) admits infinite central \( (a, p) \)-Sidon sets for any \( a \) and \( p \) as above. The main objective of this section is to prove that these thin sets can be found in any infinite subset of \( \widehat{G} \).

We first construct examples in the case when \( G \) is semisimple.

**Theorem 2.1.** — Suppose \( G \) is a compact, simply-connected, semisimple Lie group of rank \( \ell \). Choose \( 0 < t < \ell/|\Phi^+| \) and \( 1 - t/2 \leq a < 1 \). There is a constant \( C = C(t, G) \) so that if \( \{\sigma_j\} \) is any set of representations in \( \widehat{G} \) whose degrees, \( d_{\sigma_j} \equiv d_j \), satisfy

1. \( d_j^{t/4} \geq 4^j \) for \( j \geq 1 \),
2. \( d_j^{(1-a)/2} \geq 4(C \log Cd_j)^\ell \) for \( j \geq 1 \), and
then \( \{\sigma_j\} \) is a central \((a,1)\)-Sidon set.

It is useful to prove two lemmas.

**Lemma 2.2.** — There is a constant \( C_1 = C_1(G) \) so that if \( \sigma \in \hat{G} \) and \( \mu \in \Pi(\sigma) \) with \( \mu = \sum_{i=1}^{\ell} i \mu_i \lambda_i \) and \( \sigma = \sum_{i=1}^{\ell} i \sigma_i \lambda_i \), then

\[
\max_i |\mu_i| \leq C_1 \max_i \sigma_i \leq C_1 d_\sigma.
\]

**Proof.** — The second inequality is immediate from the Weyl dimension formula so we only need prove the first.

Suppose \( \Delta = \{\alpha_j\}_{j=1}^{\ell} \). Then each \( \lambda_k = \sum_{j=1}^{\ell} a_{kj} \alpha_j \) for some \( a_{kj} = a_{kj}(G) \geq 0 \), and because \( \lambda_k \neq 0 \) there is an index \( j_k \) such that \( a_{kj_k} > 0 \). Since \( \mu < \sigma \) with respect to the partial order induced by the positive roots,

\[
\sum_{i=1}^{\ell} i \mu_i \alpha_{ij} \leq \sum_{i=1}^{\ell} i \sigma_i \alpha_{ij}
\]

for all \( j \).

If \( \mu \) is a dominant weight, taking \( j = j_k \) above we get

\[
0 \leq \mu_k \leq \frac{1}{a_{kj_k}} \sum_{i=1}^{\ell} i \sigma_i \alpha_{ij_k}
\]

\[
\leq C(G) \max_i \sigma_i.
\]

Otherwise \( \mu = w(v) \) for some dominant weight \( v \in \Pi(\sigma) \) and \( w \in W \). Since the Weyl action is linear,

\[
\max_i |\mu_i| \leq C'(G) \max_i \sigma_i
\]

for some constant \( C'(G) \). Now take \( C_1 = CC' \).

**Lemma 2.3.** — There is a constant \( C_2 = C_2(t,G) \) so that if \( \sigma \in \hat{G} \) and \( \mu \in \Pi(\sigma) \), then \( m_\sigma(\mu) \leq C_2 d_\sigma^{\frac{1}{t}-1} \).

**Proof.** — This is a straight forward calculation. We begin with the fact that

\[
m_\sigma(\mu) = \int_{T^t} \text{Tr} \sigma(x) e^{-i\mu(x)} dx.
\]
By the Weyl character formula and standard inequalities
\[ m_\sigma(\mu) \leq \int_{T^\ell} \left| \sum_{w \in W} \frac{\det(w)e^{iw(\sigma+\rho)(x)}}{q(x)} \right| dx \]
\[ \leq \left\| \sum_{w \in W} \det(w)e^{iw(\sigma+\rho)(x)} \right\|_t q^{-t} \|q\|_{L^1(T^\ell)} \| \text{Tr} \sigma_{|T^\ell|} \|_{\infty}^{1-t} \]
\[ \leq |W|^t q^{-t} \|q\|_{L^1(T^\ell)} d_{\sigma}^{-1-t}. \]
Since \( q^{-t} \in L^1(T^\ell) \) for any \( t < \ell/|\Phi^+| \) ([13]) the proof is complete.

**Proof of Theorem 2.1.** Throughout the proof we will use the following notation: \( m_j(\mu) \equiv m_{\sigma_j}(\mu) \); \( \Pi_j \equiv \Pi(\sigma_j) \); and
\[ B_j \equiv \left\{ \sum_{i=1}^\ell m_i \lambda_i : |m_i| \leq C_1 d_j, \ m_i \in \mathbb{Z} \right\} \subseteq \Lambda. \]
For \( C_1 \) and \( C_2 \) as in the lemmas, put
\[ C = \max \left\{ \left( (2C_1 + 1)^\ell C_2 \right)^{2/t}, \sup_N \left( \frac{\left\| \sum_{n=-N}^{N} e^{inx} \right\|_{L^1(0,2\pi)}}{\log N} \right) \right\}. \]
Lemma 2.2 obviously implies \( \Pi_j \subseteq B_j \). The key idea of the proof (which we make precise below) is that “most” of \( \Pi_j \), counted by multiplicity, lies outside \( B_{j-1} \). This we are able to obtain from Lemma 2.3 and property (3). To be precise we have, if \( k > j \),
\[ \sum_{\mu \in \Pi_k \cap B_j} m_k(\mu) \leq |B_j| \max_{\mu \in \Pi_k} m_k(\mu) \]
\[ \leq (2C_1 d_j + 1)^\ell C_2 d_k^{1-t} \]
\[ \leq d_k^{1-t/2}, \]
and thus
\[ \sum_{\mu \in \Pi_k \setminus B_j} m_k(\mu) = \sum_{\mu \in \Pi_k} m_k(\mu) - \sum_{\mu \in \Pi_k \cap B_j} m_k(\mu) \]
\[ \geq d_k - d_k^{1-t/2} \]
\[ \geq \frac{1}{2} d_k. \]
Let \( D_n \) be the \( \ell \)-dimensional Dirichlet kernel supported by \( B_n \) (thinking now of \( B_n \) as a subset of \( \mathbb{Z}^\ell \) rather than of \( \Lambda \)),
\[ D_n(x_1, \ldots, x_\ell) = \prod_{k=1}^{\ell} \sum_{j=-C_1 d_n}^{C_1 d_n} e^{ij(x_k)}, \]
and let $H_n = D_n - D_{n-1}$. Then $\hat{H}_n = \chi_{B_n \setminus B_{n-1}}$ and $\|H_n\|_1 \leq 2(C \log C d_n)^t$.

Suppose $f = \sum_{j=1}^{N} d_j a_j \text{Tr} \sigma_j$ is a central trigonometric polynomial with $\|f\|_\infty \leq 1$. With our notation

$$f|_{T^t}(x) = \sum_{j=1}^{N} d_j a_j \sum_{\mu \in \Pi_j} m_j(\mu) e^{i \mu(x)}.$$ 

Notice that $H_n * \sum_{\mu \in \Pi_j} m_j(\mu) e^{i \mu(x)} = 0$ if $j < n$ (here the convolution is over $T^t$), and so if $n \leq N$,

$$1 \geq \frac{|f|_{T^t} * H_n|}{\|H_n\|_1} \geq \frac{|f|_{T^t} * H_n(0)|}{\|H_n\|_1} \geq \frac{|\sum_{j=n}^{N} d_j a_j \sum_{\mu \in \Pi_j} m_j(\mu) * H_n(0)|}{\|H_n\|_1} \geq \frac{2(C \log C d_n)^t}{2(C \log C d_n)^t}.$$

An application of the triangle inequality yields

$$d_n |a_n| \sum_{\mu \in \Pi_n \setminus B_{n-1}} m_n(\mu) \leq 2(C \log C d_n)^t + \sum_{j=n+1}^{N} d_j |a_j| \sum_{\mu \in \Pi_j \cap (B_n \setminus B_{n-1})} m_j(\mu).$$

Combined with our estimates (*) and (**), and property (2), this gives

$$d_n^2 |a_n| \leq d_n^{1-a}/2 + 2 \sum_{j=n+1}^{N} d_j^{2-t/2} |a_j|.$$ 

For $j = 1, 2, \ldots, N$ set $S_j = \sum_{k=0}^{j-1} d_{N-k}^{2-t/2} |a_{N-k}|$ and set $S_0 = 0$. This gives

$$d_n^2 |a_n| \leq d_n^{1-a}/2 + 2 S_{N-n},$$

and since (1) guarantees $d_{N-j}^{t/2} \geq 2$,

$$S_{j+1} \leq 2S_j + d_{N-j}^{\varepsilon-t/2}$$

where $\varepsilon = (1-a)/2$. By induction,

$$S_j \leq \sum_{i=1}^{j} 2^{i-1} d_{N-j+i}^{\varepsilon-t/2} \text{ for } j = 1, 2, \ldots, N.$$
Property (1) also ensures $S_j \leq 1$, thus
\[ d_2^2 |a_n| \leq d_1^{(1-a)/2} + 2. \]

It is now easy to see that \( \{\sigma_j\} \) is central \((a, 1)\)-Sidon:
\[
\|\hat{f}\|_{(a, 1)} = \sum_{n=1}^{N} d_1^{1+a} |a_n| \\
\leq \sum_{n=1}^{N} \frac{1}{d_1^{(1-a)/2} + \frac{2}{d_1^{1-a}}},
\]
and this sum is bounded over \( N \) since \( \{d_n\} \) is lacunary and \( a < 1 \). \( \square \)

Remark. — An application of the Weyl dimension formula shows that if \( \sigma_j = \sum_{i=1}^{\ell} \sigma_{ji} \lambda_i \), then \( \{(\sigma_{j1}, \ldots, \sigma_{j\ell})\}_j \) is the union of a finite set and a dissociate set in \( \mathbb{Z}^\ell \), and hence is a Sidon set in the dual of the torus.

**Corollary 2.4.** — If \( G \) is a compact, simply-connected, semisimple Lie group, then every infinite subset of \( \widehat{G} \) contains an infinite central \((a, p)\)-Sidon set for all \( p \geq 1 \) and \( a < 2p - 1 \).

**Proof.** — As remarked in the first section it suffices to prove this for \( p = 1 \) and all \( a < 1 \).

Let \( \ell = \text{rank } G \) and fix \( 0 < t < \ell/|\Phi^+| \). Set \( a_1 = 1 - \frac{t}{2} \) and choose an increasing sequence \( \{a_n\}_{n=1}^{\infty} \) with \( a_n < 1 \) and limit one. Let \( E \subseteq \widehat{G} \) be infinite. Since \( \widehat{G} \) contains only finitely many representations of any given degree we can choose an infinite subset \( \{\sigma_j\} \) of \( E \) satisfying (where \( C \) is as in the theorem):

1. \( \frac{1}{d_j^{j/4}} \geq 4^j \) for \( j \geq 1 \),
2. \( \frac{1}{d_j^{(1-a_j)/2}} \geq \frac{4(C \log C \max d_j)^\ell}{d_j^{1-a_j}} \) for \( j \geq 1 \), and
3. \( d_j \geq C(d_{j-1})^{2\ell/t} \) for \( j \geq 2 \).

Choose \( a < 1 \). Then \( a \leq a_j \) for all \( j \geq J \) and by the theorem \( \{\sigma_j\}_{j=1}^{\infty} \) is a central \((a, 1)\)-Sidon set. It is easy to see from Proposition 1.1 that the union of a finite set and a central \((a, 1)\)-Sidon set is again central \((a, 1)\)-Sidon, and therefore \( \{\sigma_j\}_{j=1}^{\infty} \) is central \((a, 1)\)-Sidon for any \( a < 1 \). \( \square \)

Remark. — As noted previously, these groups admit no infinite central Sidon sets. It is unknown if they admit infinite central \((2p-1, p)\)-Sidon sets for any \( p > 1 \).
The next step towards our main result is to consider the case when $G$ is an infinite product group.

**Theorem 2.5.** — Let $G = \prod_{j=1}^{\infty} G_j$ be a product of compact, simply-connected, semisimple Lie groups, and suppose $\sigma_j$ is a non-trivial representation of $G_j$. Then $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is a central $(a, 1)$-Sidon set for all $a < 1$.

**Proof.** — Suppose $f = \sum_{j=1}^{N} d_j a_j \text{Tr} \sigma_1 \times \cdots \times \sigma_j$ where $d_j = d_{\sigma_1} \cdots d_{\sigma_j}$. Without loss of generality assume $\|f\|_\infty \leq 1$. For the duration of this proof we will use the following notation: $m_j(\mu) \equiv m_j(\mu); m_j \equiv m_j(0); \Pi'_j \equiv \Pi(\sigma_j)\{0\}; T^{G_j} \equiv \text{torus of } G_j$; and $T_N = T^{G_1} \times \cdots \times T^{G_N}$. With this notation we have

$$f|_{T_N(x_1, \ldots, x_N)} = \sum_{j=1}^{N} d_j a_j \prod_{k=1}^{j} \left( m_k + \sum_{\mu \in \Pi'_k} m_k(\mu) e^{i\mu(x_k)} \right) \text{ for } x_k \in T^{G_k}.$$ 

Viewing $f$ as a function on $T_N$, we can read off the Fourier coefficients:

$$\hat{f}(0, \ldots, 0) = \sum_{j=1}^{N} d_j a_j m_1 \cdots m_j;$$

$$\hat{f}(\mu_1, \ldots, \mu_k, 0, \ldots, 0) = \left( d_k a_k + \sum_{j=k+1}^{N} d_j a_j m_{k+1} \cdots m_j \right) m_1(\mu_1) \cdots m_k(\mu_k)$$

if $\mu_i \in \Pi(\sigma_i)$ for $i = 1, \ldots, k - 1$, $\mu_k \in \Pi'_k$; and $\hat{f}(\mu_1, \ldots, \mu_N) = 0$ otherwise.

For $x_j = (x_{j1}, \ldots, x_{j\ell(j)}) \in T^{G_j}$ (here $\ell(j) = \text{rank } G_j$), and $M$ very large, let

$$H_j(x_j) = \prod_{k=1}^{\ell(j)} \sum_{n=-M}^{M} \left( 1 - \frac{|n|}{M+1} \right) e^{in(x_{jk})},$$

and for $n \leq N$ let

$$K_n(x_1, \ldots, x_N) = \prod_{j=1}^{n-1} H_j(x_j)(H_n(x_n) - 1).$$

Observe that if $\mu_j \in \widehat{T^{G_j}}$ for $j = 1, \ldots, N$, and $\widehat{K_n}(\mu_1, \ldots, \mu_N) \neq 0$, then $\mu_j = 0$ for $j > n$ and $\mu_n \neq 0$. Consider the convolution of $K_n$ and $f|_{T_N}$. If $M$ is chosen sufficiently large then

$$2 \geq |f * K_n(0)| \geq \frac{1}{2} \left| d_n a_n + \sum_{j=n+1}^{N} d_j a_j m_{n+1} \cdots m_j \right| \sum_{\mu_n} m_1(\mu_1) \cdots m_n(\mu_n)$$
where $\sum'$ denotes the sum over all $\mu_j \in \Pi(\sigma_j)$ for $j = 1, \ldots, n - 1$, and $\mu_n \in \Pi'_n$. Clearly
\[
\sum'_n m_1(\mu_1) \cdots m_n(\mu_n) = d_{\sigma_1} \cdots d_{\sigma_{n-1}}(d_{\sigma_n} - m_n) = d_{n-1}(d_{\sigma_n} - m_n).
\]
Thus
\[
|d_n a_n| \leq \frac{4}{d_{n-1}(d_{\sigma_n} - m_n)} + \left| \sum_{j=n+1}^N d_j a_j m_{n+1} \cdots m_j \right|.
\]
Furthermre,
\[
\left| \sum_{j=n+1}^N d_j a_j m_{n+1} \cdots m_j \right| = \left| d_{n+1} a_{n+1} + \sum_{j=n+2}^N d_j a_j m_{n+2} \cdots m_j \right| m_{n+1} \leq \frac{4m_{n+1}}{d_n(d_{\sigma_{n+1}} - m_{n+1})}
\]
(where the empty sum and $m_{N+1}$ equal 0). Thus
\[
|d_n a_n| \leq \frac{4}{d_{n-1}(d_{\sigma_n} - m_n)} + \frac{4m_{n+1}}{d_n(d_{\sigma_{n+1}} - m_{n+1})}.
\]
In [4] Gallagher proves that if $\sigma$ is any non-trivial representation of a compact, simply-connected, semisimple Lie group then $\text{Tr}\sigma$ has a root, say $x$, in the maximal torus. Evaluating $\text{Tr}\sigma$ at $x$ we derive the formula
\[
m_{\sigma}(0) = -\sum_{\mu \in \Pi(\sigma) \setminus \{0\}} m_{\sigma}(\mu)e^{i\mu(x)},
\]
from which one readily sees that $m_{\sigma}(0) \leq d_{\sigma}/2$. Hence $|d_n a_n| \leq 12/d_n$, and so
\[
||\hat{f}||_{(a,1)} = \sum_{j=1}^N a_n^{1+a}|a_n| \leq \sum_{j=1}^N \frac{12}{d_n^{1-a}}.
\]
Since $d_n \geq 2^n$, this sum converges provided $a < 1$, and thus $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^\infty$ is a central $(a,1)$-Sidon set for all $a < 1$. $\square$

This set of representations is independent in the sense that if
\[
\int_G \prod_{j=1}^N (\text{Tr}\sigma_1 \times \cdots \times \sigma_j)^{\epsilon_j} \neq 0
\]
for some $N \in \mathbb{N}$ and $\epsilon_j = 0, \pm 1$ for $j = 1, \ldots, N$, then necessarily all $\epsilon_j = 0$. This independence condition is not sufficient to be Sidon [1]. It is not sufficient for central-Sidon either as the next example demonstrates.
Example 2.6. — Suppose \( G_j = SU(2) \), \( G = \prod_{j=1}^{\infty} G_j \) and \( \sigma_j = 2\lambda_1 \).
The set \( \{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty} \) is not central \((2p-1,p)\)-Sidon for any \( p \geq 1 \).

Proof. — It is well known that the torus of \( SU(2) \) is the circle group \( T \) and that \( \text{Tr} 2\lambda_1|_T = 1 + e^{ix} + e^{-ix} \) ([6], 29.25). Therefore \( \text{Tr} 2\lambda_1|_T \) takes on precisely the values in \([-1,3]\). Let
\[
 f_N = \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p}3^j} \text{Tr} \sigma_1 \times \cdots \times \sigma_j;
\]
\[
 \|f_N\|_{(2p-1,p)} = \sum_{j=1}^{N} \frac{1}{j} \text{ which diverges as } N \to \infty.
\]
Being a central function, \( \|f_N\|_{\infty} = \|f_N|_T\|_{\infty} \), and from the remark above the latter equals
\[
 \sup_{w_i \in [-\frac{1}{3},1]} \left| \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p}3^j} w_1 \cdots w_j \right|.
\]
We will now prove that this supremum is bounded over \( N \) which certainly suffices to prove \( \{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty} \) is not central \((2p-1,p)\)-Sidon.

Set \( j_1 = 1 \) and inductively define \( j_k \) to be the least integer greater than \( j_{k-1} \) with
\[
 (-1)^{j_k}w_1 \cdots w_{j_k} (-1)^{j_{k-1}}w_1 \cdots w_{j_{k-1}} \leq 0.
\]
Consider first the alternating sum
\[
 \sum_{j} \frac{(-1)^j}{j^{1/p}3^j} w_1 \cdots w_j.
\]
Since \( \left| w_1 \cdots w_{j_k} \right| \) decreases to zero, this sum is bounded in absolute value by \( \frac{1}{j_{k}^{1/p}} = 1 \).

If \( j \notin \{j_i\} \) then \((-1)^jw_1 \cdots w_j\) and \((-1)^{j-1}w_1 \cdots w_{j-1}\) have the same sign. This can occur only if \( w_j < 0 \), but then \( |w_1 \cdots w_j| \leq \frac{1}{3}|w_1 \cdots w_{j-1}|. \)
As \( |w_i| \leq 1 \) for all \( i \), it follows that
\[
 \left| \sum_{j \notin j_i} (-1)^j \frac{w_1 \cdots w_j}{j^{1/p}} \right| \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2}.
\]
These estimates clearly combine to give
\[
 \sup_{w_i \in [-\frac{1}{3},1]} \left| \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p}3^j} w_1 \cdots w_j \right| \leq \frac{3}{2}.
\]
THEOREM 2.7. — Let $G = \prod G_\alpha$ be a product (possibly finite) of compact, simply-connected, simple Lie groups. Then any infinite subset of $G$ contains an infinite central $(a, p)$-Sidon set for all $p \geq 1$ and $a < 2p - 1$.

First we introduce some notation and prove a lemma.

**Notation.** — Let $\sigma_j \in \hat{G}$. Then $\sigma_j = \times \sigma_{j\alpha}$ where $\sigma_{j\alpha} \in \hat{G}_\alpha$ and only finitely many $\sigma_{j\alpha}$ are non-trivial. Denote by supp $\sigma_j$ the set $\{\alpha : \sigma_{j\alpha} \neq 1\}$. We will say $\sigma_j$ is orthogonal to $\sigma_k$, and write $\sigma_j \perp \sigma_k$, if supp $\sigma_j \cap$ supp $\sigma_k$ is empty. Recall that Parker [9] has shown that if $\{\sigma_j\}$ consists of mutually orthogonal, non-trivial representations then $\{\sigma_j\}$ is a central Sidon set.

**Lemma 2.8.** — Let $a < 1$ and suppose $\{\sigma_j\}$ is a central $(a, 1)$-Sidon set in $\hat{G}$. Suppose $\{\tau_j\} \subseteq \hat{G}$ and $\tau_j \perp \sigma_k$ for all $j, k$. Then $\{\tau_j \times \sigma_j\}$ is another central $(a, 1)$-Sidon set.

**Proof.** — This is an easy consequence of the fact that $\|f\|_\infty \geq \sup \{ |f(x)| : x = (x_\alpha) \text{ and } x_\alpha = 1 \text{ if } \alpha \in \bigcup_j \text{supp } \tau_j \}$.

**Proof of Theorem 2.7.** — It suffices to show that any countably infinite set, $E = \{\sigma_j\}_{j=1}^\infty$, contains an infinite subset which is central $(a, 1)$-Sidon for all $a < 1$.

Suppose first $\{\sigma_{j\alpha} : \sigma_j \in E\}$ is infinite for some $\alpha$. By Corollary 2.4 we can find an infinite subset of $\{\sigma_{j\alpha}\}_{j=1}^\infty$ which is a central $(a, 1)$-Sidon subset of $\hat{G}_\alpha$, for all $a < 1$. The corresponding subset of $E$ has the same property.

So we may assume $\{\sigma_{j\alpha} : \sigma_j \in E\}$ is finite for each $\alpha$.

**Case 1.** — For each index $\alpha$, $\{\sigma_j : \sigma_{j\alpha} \neq 1\}$ is finite.

Set $j_1 = 1$ and inductively assume mutually orthogonal representations $\sigma_{j_1}, \ldots, \sigma_{j_n} \subseteq E$ have been picked. Since there are only finitely many representations $\sigma_j$ with $\sigma_{j\alpha} \neq 1$ for $\alpha \in \bigcup_{k=1}^n \text{supp } \sigma_k$, we can choose $\sigma_{j_{n+1}}$ orthogonal to each of $\sigma_{j_1}, \ldots, \sigma_{j_n}$. By Parker [9] $\{\sigma_{j_k}\}_k$ is central Sidon.

**Case 2.** — $\{\sigma_j : \sigma_{j\alpha} \neq 1\}$ is infinite for some $\alpha$, say $\alpha = \alpha_1$.

Since $\{\sigma_{j\alpha_1} : \sigma_j \in E\}$ is finite there must be a non-trivial representation $\phi_1$ of $\hat{G}_{\alpha_1}$, with $\phi_1 = \sigma_{j_\alpha_1}$ for all $\sigma_j \in F_1$, an infinite subset of $E$. Select $\sigma_{j_1} \in F_1$. 
If \( \{\sigma_j \in F_1 : \sigma_{j\alpha} \neq 1\} \) is finite for all \( \alpha \not\in \text{supp} \sigma_{j1} \), then by arguments similar to case 1 we can obtain an infinite subset of \( F_1 \) of the form \( \{\tau_k \times w_k\}_{k=2}^{\infty} \) where \( \text{supp} \tau_k \subseteq \text{supp} \sigma_{j1} \) and the representations \( w_k \) are non-trivial, mutually orthogonal, and all orthogonal to \( \sigma_{j1} \). By [9] and the lemma this set is central Sidon.

Otherwise we repeat the argument to produce infinite sets \( F_n \subseteq F_{n-1} \) \( (F_0 = E) \), representations \( \sigma_{jn} \in F_n \) and \( \phi_n \) orthogonal to \( \sigma_{jk} \) for \( k \leq n-1 \), and an index \( \alpha_n \) with the property that \( \sigma_{j\alpha_n} = \phi_n \) for all \( \sigma_j \in F_n \). If \( \{\sigma_j \in F_n : \sigma_{j\alpha} \neq 1\} \) is finite for all \( i \not\in \bigcup_{k=1}^{n} \text{supp} \sigma_{jk} \) we quit this process and produce an infinite central Sidon set in \( F_n \) by standard arguments. Otherwise, as in the first step of case 2, we choose \( F_{n+1}, \alpha_{n+1}, \phi_{n+1} \) and \( \sigma_{jn+1} \) with the properties above.

If this process never stops we produce an infinite set \( \{\sigma_{jn}\} \subseteq E \). By construction \( \sigma_{jn} = \phi_1 \times \cdots \times \phi_n \times \tau_n \) where \( \phi_n \perp \phi_1 \times \cdots \times \phi_j \times \tau_j \) for all \( n > j \). From Theorem 2.5 \( \{\phi_1 \times \cdots \times \phi_n\}_{n=1}^{\infty} \) is central \((a,1)\)-Sidon for all \( a < 1 \) and hence so is \( \{\sigma_{jn}\} \).

In either case we can find an infinite central \((a,1)\)-Sidon subset of \( E \) and thus the proof of the theorem is complete. \( \square \)

The main result will now be seen to follow from the structure theorem ([10], 6.5.6): If \( G \) is a compact, connected group then there is a continuous epimorphism \( \phi : T \times \prod G_{\alpha} \to G \) where \( T \) is a compact abelian group and each \( G_{\alpha} \) is a compact, simply-connected, simple Lie group.

**Theorem 2.9.** — If \( G \) is a compact, connected group then any infinite subset of \( \hat{G} \) contains an infinite central \((a,p)\)-Sidon set for all \( p \geq 1 \) and \( a < 2p - 1 \).

We need only one additional lemma whose proof is obvious.

**Lemma 2.10.** — If \( \phi : H \to G \) is a continuous epimorphism of compact groups then \( E \subseteq \hat{G} \) is a (central) \((a,p)\)-Sidon set if and only if the same is true for \( E \circ \phi = \{\sigma \circ \phi : \sigma \in E\} \subseteq \hat{H} \).

**Proof of Theorem 2.9.** — Let \( E \subseteq \hat{G} \) be an infinite set and let \( \phi : T \times \prod G_{\alpha} \to G \) be the structure theorem epimorphism. Since \( \phi \) is onto \( E \circ \phi \) is also infinite. For \( \sigma \circ \phi \in E \circ \phi \), write \( \sigma \circ \phi = \tau_{\sigma} \times \psi_{\sigma} \) where \( \tau_{\sigma} \in \hat{T} \) and \( \psi_{\sigma} \in \prod \hat{G}_{\alpha} \). If \( \{\tau_{\sigma} : \sigma \in E\} \) is infinite, then since \( T \) is an
abelian group there is an infinite Sidon subset of \( \{ \tau_\sigma \} \), and by Lemma 2.8 the corresponding subset of \( E \circ \phi \) is central Sidon. If \( \{ \psi_\alpha \} \) is infinite we appeal to Theorem 2.7 and Lemma 2.8 to obtain an infinite central \((a, 1)\)-Sidon set for all \( a < 1 \). In either case the corresponding infinite subset of \( E \) has the required property.

\[ \square \]

**Corollary 2.11.** — Suppose \( G \) is a compact, connected group. Any infinite subset of \( \hat{G} \) contains an infinite set which is central \( p \)-Sidon for all \( p > 1 \).

**Remark.** — This answers the open problem left in [3].

### 3. Central \((0, 1)\)-Sidon sets.

In this section we investigate the relationship between weighted central Sidonicity for a Lie group \( G \) and Sidonicity for its abelian torus. This investigation is motivated in part by the fact that both Dooley's examples [3] of central \( p \)-Sidon sets and our examples from Theorem 2.1 correspond to Sidon sets in \( \mathbb{Z}^{\text{rank} G} \).

**Theorem 3.1.** — Let \( G \) be a compact, simply-connected, semisimple Lie group of rank \( \ell \), with torus \( T^\ell \). If \( E \subseteq (\mathbb{Z}^+)^\ell \) is a Sidon set for \( T^\ell \), then \( E \) viewed as a subset of \( \hat{G} \) is central \((0, 1)\)-Sidon.

**Proof.** — Let \( f = \sum_{\sigma \in E} d_\sigma a_\sigma \text{Tr} \sigma \) be a central trigonometric polynomial on \( G \). Since \( |q(x)| \leq |W| \), the Weyl character formula implies

\[
\|f\|_\infty \geq \frac{1}{|W|} \sup_{x \in T^\ell} \left| \sum_{\sigma \in E} d_\sigma a_\sigma \sum_{w \in W} \det(w)e^{iw(\sigma + \rho)(x)} \right|.
\]

Because the representations \( \sigma + \rho, \sigma \in \hat{G} \), belong to the fundamental Weyl chamber, the weights \( w(\sigma + \rho) \) are distinct as \( w \) varies over \( W \) and \( \sigma \) over \( E \) ([7], ch. 10). Furthermore, the family of Sidon sets in an abelian group is closed under linear transformations and finite unions ([8], p. 44) so this set of distinct elements, \( \bigcup_{w \in W} \{w(\sigma + \rho) : \sigma \in E\} \), forms a Sidon set in \( \mathbb{Z}^\ell \) (with the natural identification). With these observations it is straightforward to check that \( E \) is central \((0, 1)\)-Sidon.

\[ \square \]
Our next result shows that Theorem 3.1 cannot be improved. Recall that $SU(2)$ has one fundamental weight so its dual can be identified with $\mathbb{Z}^+$. The degree of the representation indexed by $n$ is $n+1$.

**Proposition 3.2.** — There is a Sidon set in $\mathbb{Z}$, which is contained in $\mathbb{Z}^+$, and is not a central $(a, 1)$-Sidon set in $SU(2)$ for any $a > 0$.

**Proof.** — Let $E$ be any infinite Sidon subset of $\mathbb{Z}$ contained in $\{2, 3, 4, \ldots\}$ and disjoint from $E-2$. Certainly $E \cup E-2$ is a Sidon set in $\mathbb{Z}$. If it was a central $(a, 1)$-Sidon set in $SU(2)$ for some $a > 0$, by Proposition 1.1 there would be a measure $\mu$ on $SU(2)$ satisfying

$$\hat{\mu}(n) = \begin{cases} \frac{1}{(n+1)^{1-a}} & \text{for } n \in E \\ 0 & \text{for } n \in E-2. \end{cases}$$

Coifman and Weiss [2] have shown that $\mu$ is a measure on $SU(2)$ if and only if

$$\sum_{n \geq 2} ((n+1)\hat{\mu}(n) - (n-1)\hat{\mu}(n-2)) \cos n\theta$$

represents a measure $\nu$ on $T$. But for $n \in E$, $\hat{\nu}(n) = (n+1)^a$ which tends to infinity, so this is an impossibility. \(\square\)

It is natural to ask if the converse to Theorem 3.1 is true. It is not.

**Theorem 3.3.** — There are subsets of $\mathbb{Z}^+$ containing arbitrarily long arithmetic progressions which are central $(a, 1)$-Sidon sets in $SU(2)$, for all $a < 1$; consequently a central $(a, 1)$-Sidon set need not be a Sidon set in $\mathbb{Z}$.

**Proof.** — The second statement follows from the first since sets containing arbitrarily long arithmetic progressions are never Sidon sets in $\mathbb{Z}$ ([8], p. 77). We follow the strategy of [3] to produce examples of central $(a, 1)$-Sidon sets with this property.

Let $\{n_j\}_{j=1}^\infty$ be a sequence of positive integers, $\sigma_j$ the representation of $SU(2)$ indexed by $2n_j$, and let $f = \sum_{j=1}^N (2n_j + 1)a_j \text{Tr} \sigma_j$ be a central trigonometric polynomial on $SU(2)$. It is well known ([6], 29.25) that for $t_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \in T$,

$$\text{Tr} \sigma_j(t_\theta) = \sin(n_j + \frac{1}{2})\theta / \sin \frac{\theta}{2} = D_{n_j}(\theta),$$
where $D_n$ is the $n$'th Dirichlet kernel. Thus

$$\|f\|_\infty = \|f|_T\|_\infty = \sup_{\theta \in [0,2\pi]} \left| \sum_{j=1}^{N} (2n_j + 1)a_jD_{n_j}(\theta) \right|.$$  

For even integers $a < b$, let $F_{ab}$ denote the translated Fejér kernel with transform supported on $(a, b)$. One easily sees that

(i) if $k < a$ then $F_{ab} \ast D_k = 0$; while

(ii) if $b \leq k$ then $F_{ab} \ast D_k(0) = F_{ab}(0) = \frac{b-a}{2}$.

To simplify notation we write $F'_j = F_{n_j-1,n_j}$ (taking $n_0 = 0$), $B_j = (n_{N-j} - n_{N-j-1})/2$ and $X_j = (2n_{N-j} + 1)|a_{N-j}|$. With this notation

$$(*) \quad \|\hat{f}\|_{(a,1)} = \sum_{j=0}^{N-1} (2n_{N-j} + 1)^a X_{N-j}.$$  

Without loss of generality we may assume $\|f\|_\infty = 1$, so, for $m = N - k$,

$$1 \geq |f \ast F'_m(0)| = \left| \sum_{j=1}^{N} (2n_j + 1)a_jF'_m \ast D_{n_j}(0) \right|$$

$$\geq (2n_m + 1)|a_m| \left( \frac{n_m - n_{m-1}}{2} \right) - \sum_{j>m} (2n_j + 1)|a_j| \left( \frac{n_m - n_{m-1}}{2} \right)$$

$$\geq X_k B_k - \sum_{j=0}^{k-1} X_j B_k.$$  

Thus

$$X_k \leq \frac{1}{B_k} + \sum_{j=0}^{k-1} X_j,$$

and simplifying this yields the estimate

$$X_k \leq \frac{1}{B_k} + \sum_{j=0}^{k-1} \frac{2^j}{B_{k-1-j}}.$$  

Obviously there are many ways to choose a sequence $\{n_j\}$ containing arbitrarily long arithmetic progressions, and yet have $X_k$ sufficiently small so that (*) bounded over all $N$ and all $a < 1$. One choice, whose verification is routine, and is left for the reader, is to set $n_{2^k + i} = A^k (1 + i)$ for $i = 0, 1, \ldots, 2^k - 1$, where $A$ is sufficiently large.

There is however a partial converse to Theorem 3.1. We state it for $SU(2)$, the context in which we will apply it to show the failure of the
union property, but similar results hold more generally for all compact, simply-connected, semisimple Lie groups.

**Proposition 3.4.** — Suppose $E$ and $E - 2$ are disjoint subsets of $\mathbb{Z}^+$ and that $E \cup E - 2$ is a central $(0, 1)$-Sidon set in $SU(2)$. Then $E$ is a Sidon set in $\mathbb{Z}$.

**Proof.** — Let $\phi \in \ell^\infty(E)$. Since $E \cup E - 2$ is central $(0, 1)$-Sidon, there exists a central measure $\mu$ on $SU(2)$ with $\hat{\mu}(n) = \phi(n)/(n + 1)$ for $n \in E$ and $\hat{\mu} = 0$ on $E - 2$. As in Proposition 3.2, [2] implies that there is a measure $\nu$ on $T$ with $\hat{\nu}(\pm n) = (n + 1)\hat{\mu}(n) - (n - 1)\hat{\mu}(n - 2)$ if $n \in \mathbb{Z}^+$. For $n \in E$, $\hat{\nu}(n) = \hat{\phi}(n)$, and consequently $E$ is a Sidon set in $\mathbb{Z}$.

In contrast to the situation for abelian groups it is known that the union of two central Sidon sets need not be central Sidon [12]. This extends to central $(a, 1)$-Sidon sets.

**Proposition 3.5.** — The union of two sets which are central $(a, 1)$-Sidon for all $a < 1$, need not be a central $(0, 1)$-Sidon set.

**Proof.** — Consider the example $E = \{n_i\}$, where $n_{2^k+i} = A^{a^k}(1+i)$ for $i = 0, 1, \cdots, 2^k - 1$ and $A$ sufficiently large. This example is seen in Theorem 3.3 to be a non-Sidon set in $\mathbb{Z}^+$ which is a central $(a, 1)$-Sidon set for all $a < 1$. The set $E - 2$ clearly has the same properties and is disjoint from $E$. By the previous proposition their union is not central $(0, 1)$-Sidon.

**Remark.** — Our understanding of weighted Sidon sets is much less satisfactory in the non-central case. It is known that any set of representations whose degrees tend to infinity sufficiently fast is $(-\varepsilon, 1)$-Sidon for any given $\varepsilon > 0$, and that a compact Lie group admits no $(\varepsilon, 1)$-Sidon set for $\varepsilon > 0$ [5], but we do not know if any of our examples of central $(a, 1)$-Sidon sets, or any other infinite sets in the dual of a compact, simple-connected semisimple Lie group, are $(0, 1)$-Sidon.
BIBLIOGRAPHY


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