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CENTRAL SIDONICITY FOR COMPACT LIE GROUPS

by Kathryn E. HARE (*)

0. Introduction.

Suppose G is a compact group with dual object $\widehat{G}$. It is well known that if G is an abelian group, then every infinite subset of $\widehat{G}$ contains an infinite Sidon set [8]. In contrast, there are non-abelian groups which admit no infinite central Sidon sets [11]. For central p-Sidon sets the situation is quite different; even in the non-abelian setting these are plentiful. Indeed, Dooley [3] showed that every compact, connected group admits an infinite central p-Sidon set for all $p > 1$, however he was unable to determine if every infinite set contains an infinite central p-Sidon subset.

The main result of our paper answers this question affirmatively. In fact, we prove formally more. We study certain weighted generalizations of Sidon sets, introduced in [5], called (central) (a, p)-Sidon sets, which arise by considering classical Sidonicity with the Fourier transform weighted by the a'th powers of the representation degree: (central) $(1, p)$-Sidon sets are (central) p-Sidon sets. We prove that every infinite subset of the dual of a compact, connected group contains an infinite subset which is central (a, p)-Sidon for all $p \geq 1$ and $a < 2p - 1$. Our method is essentially constructive: we show that certain "lacunary-like" sets have the desired property.

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When G is a compact, simply-connected, semisimple Lie group of rank ℓ, the dual object can be identified with the set of dominant weights and consequently with $(\mathbf{Z}^+)^\ell$. Our examples of central (a,p)-Sidon sets in the duals of these groups correspond to Sidon sets in $\mathbf{Z}^\ell$. A natural question to ask is if all Sidon sets in $(\mathbf{Z}^+)^\ell$ correspond to Sidon-type sets in $\widehat{G}$. We show that such sets are always central $(0,1)$-Sidon, but need not be central $(a,1)$-Sidon for any $a>0$, and that there are central $(a,1)$-Sidon sets in $\widehat{G}$ which do not correspond to Sidon sets in $(\mathbf{Z}^+)^\ell$.

1. Preliminaries.

If G is a compact group, $\widehat{G}$ will denote a maximal set of pairwise inequivalent, unitary, irreducible representations of G. The degree of $\sigma \in \widehat{G}$ will be denoted by d_σ.

The following generalization of Sidonicity was introduced in [5].

DEFINITION. — *Let $a \in \mathbf{R}$, $1 \le p < \infty$. A subset E of $\widehat{G}$ is called* a (central) (a,p)-Sidon set *if there is a constant $\kappa(a,p)$ so that whenever $f = \sum_{\sigma \in E} d_\sigma \operatorname{Tr} A_\sigma \sigma$ is a (central) trigonometric polynomial on G, then*

$$\|\hat{f}\|_{(a,p)} \equiv \left(\sum d_\sigma^a \operatorname{Tr} |A_\sigma|^p\right)^{1/p} \le \kappa(a,p)\|f\|_\infty .$$

(Central) $(1,p)$-Sidon sets are usually called (central) p-Sidon and (central) 1-Sidon sets are simply referred to as (central) Sidon sets.

Obviously if E consists of representations of bounded degree there is no distinction between (a,p)-Sidonicity for different values of a; if G is abelian then central p-Sidon and p-Sidon properties coincide; and (for all groups) it is easier to be (central) (a,p)-Sidon as a decreases or p increases. There are other relationships between (a,p)-Sidon sets. For this paper we only need note that since $\ell^q \subset \ell^p$ if $q<p$, then any central (a,q)-Sidon set is central (b,p)-Sidon provided $(b+1)/p \le (a+1)/q$. In particular any set which is central $(a,1)$-Sidon for all $a<1$ is also central (b,p)-Sidon for all $p \ge 1$ and $b < 2p-1$.

One reason for the interest in (a,p)-Sidon sets is the scarcity of (central) Sidon sets: a compact, connected group admits an infinite central Sidon set if and only if it is not a semisimple Lie group [11], [12].

It is seen in [5] that if G is an infinite compact, connected group then $\widehat{G}$ is never central $(0,1)$-Sidon, but there are examples where $\widehat{G}$ is $(-\varepsilon,1)$-Sidon for any given $\varepsilon > 0$. Also, every central $(1+\varepsilon,1)$-Sidon set for $\varepsilon > 0$ is a set of representations of bounded degree; consequently our interest (when $p=1$) is in the range $0 \le a \le 1$.

There are a number of equivalent characterizations of (central) (a,p)-Sidonicity (see [5]). For example, analogous to [6], 37.2 we have

PROPOSITION 1.1. — *Let G be a compact group. A subset E of $\widehat{G}$ is central $(a,1)$-Sidon if and only if whenever $\phi \in \ell^\infty(E)$ there is a central measure μ on G with*

$$\hat{\mu}(\sigma) \equiv \int_G \frac{\operatorname{Tr}\sigma}{d_\sigma} d\mu = \frac{\phi(\sigma)}{d_\sigma^{1-a}} \quad \text{for all } \sigma \in E.$$

Next we recall some notation and basic facts from Lie theory. The reader is referred to [7] or [14] for more details. Let G denote a compact, simply-connected, semisimple Lie group of rank ℓ, T^ℓ a maximal torus for G and t its Lie algebra. Let Φ denote the set of roots for (G,T^ℓ) and Φ^+ the positive roots relative to a fixed base Δ. To each $\lambda = (n_1,\dots,n_\ell) \in \mathbf{Z}^\ell$ we associate the weight $\lambda = \sum_{j=1}^{\ell} n_j\lambda_j$ where λ_j are the fundamental dominant weights relative to Δ, and we denote by Λ^+ the set of all dominant weights i.e. the set of all λ with non-negative integer coefficients. We view Φ as a subset of it^*. The lattice of weights Λ is isomorphic to $\widehat{T^\ell}$: $\lambda = \sum n_j\lambda_j$ determines a character on T^ℓ by the map: $\exp H \mapsto e^{\lambda(H)} = e^{\Sigma n_j\lambda_j(H)}$ for $H \in t$. The set $\widehat{G}$ is in a $1-1$ correspondence with Λ^+; $\sigma_\lambda \in \widehat{G}$ is indexed by its highest weight $\lambda \in \Lambda^+$. Thus if E is a subset of $(\mathbf{Z}^+)^\ell$, then E indexes a subset of $\widehat{G}$ in a canonical way, and we refer to this subset of $\widehat{G}$ by E as well. It should be clear from the context which set is actually meant. A partial order is defined on Λ by the positive roots: $\mu \prec \sigma$ if and only if $\sigma - \mu$ is a non-negative integral sum of positive roots. The Weyl group will be denoted by W and the weights of $\sigma \in \Lambda^+$ by

$$\Pi(\sigma) \equiv \{\mu \in \Lambda : w(\mu) \prec \sigma \ \text{ for all } w \in W\}.$$

The set $\Pi(\sigma)$ consists of all $\mu \in \Lambda^+$ with $\mu \prec \sigma$, together with all their Weyl-conjugates. Lastly, we set $\rho = \sum_{j=1}^{\ell} \lambda_j$; ρ is also half the sum of the positive roots.

One reason for the success in studying central (a,p)-Sidon sets is that there are formulas for $\operatorname{Tr}\sigma$ restricted to the torus. One of these is the Weyl character formula:

$$\operatorname{Tr}\sigma(x) = \frac{\sum_{w\in W} \det(w) e^{iw(\sigma+\rho)(x)}}{q(x)}, \ x \in T^{\ell}$$

where

$$\begin{aligned} q(x) &= \sum_{w\in W} \det(w) e^{iw(\rho)(x)} \\ &= e^{-i\rho(x)} \prod_{\alpha\in\Phi^+} (e^{i\alpha(x)} - 1). \end{aligned}$$

Related to this is the Weyl dimension formula which states:

$$d_\sigma = \prod_{\alpha\in\Phi^+} \frac{\langle \sigma+\rho, \alpha\rangle}{\langle \rho, \alpha\rangle}.$$

A final fact which we will record here is that the weights in $\Pi(\sigma)$ correspond to the irreducible subrepresentations of $\sigma|_{T^\ell}$, and so we also have the formula

$$\operatorname{Tr}\sigma(x) = \sum_{\mu\in\Pi(\sigma)} m_\sigma(\mu) e^{i\mu(x)}, \ x \in T^{\ell}$$

where $m_\sigma(\mu)$ is the multiplicity of μ in $\sigma|_{T^\ell}$.

2. Main result.

In [3], Dooley constructs in the dual of any compact, connected, semisimple Lie group examples of infinite sets which are p-Sidon for all $p > 1$. By making the obvious modifications to his proof these examples can be seen to be central (a,p)-Sidon for all $p \geq 1$ and $a < 2p-1$. Consequently, every compact, connected group G admits infinite central (a,p)-Sidon sets for any a and p as above. The main objective of this section is to prove that these thin sets can be found in any infinite subset of $\widehat{G}$.

We first construct examples in the case when G is semisimple.

THEOREM 2.1. — *Suppose G is a compact, simply-connected, semisimple Lie group of rank ℓ. Choose $0 < t < \ell/|\Phi^+|$ and $1-t/2 \leq a < 1$. There is a constant $C = C(t,G)$ so that if $\{\sigma_j\}$ is any set of representations in $\widehat{G}$ whose degrees, $d_{\sigma_j} \equiv d_j$, satisfy*

(1) $d_j^{t/4} \geq 4^j$ *for* $j \geq 1$,

(2) $d_j^{(1-a)/2} \geq 4(C \log C d_j)^{\ell}$ *for* $j \geq 1$, *and*

(3) $d_j \geq C(d_{j-1})^{2\ell/t}$ *for* $j \geq 2$,

then $\{\sigma_j\}$ *is a central* $(a,1)$*-Sidon set.*

It is useful to prove two lemmas.

LEMMA 2.2. — *There is a constant* $C_1 = C_1(G)$ *so that if* $\sigma \in \widehat{G}$ *and* $\mu \in \Pi(\sigma)$ *with* $\mu = \sum_{i=1}^{\ell} \mu_i \lambda_i$ *and* $\sigma = \sum_{i=1}^{\ell} \sigma_i \lambda_i$, *then*

$$\max_i |\mu_i| \leq C_1 \max_i \sigma_i \leq C_1 d_\sigma.$$

Proof. — The second inequality is immediate from the Weyl dimension formula so we only need prove the first.

Suppose $\Delta = \{\alpha_j\}_{j=1}^{\ell}$. Then each $\lambda_k = \sum_{j=1}^{\ell} a_{kj}\alpha_j$ for some $a_{kj} = a_{kj}(G) \geq 0$, and because $\lambda_k \neq 0$ there is an index j_k such that $a_{kj_k} > 0$. Since $\mu \prec \sigma$ with respect to the partial order induced by the positive roots,

$$\sum_{i=1}^{\ell} \mu_i a_{ij} \leq \sum_{i=1}^{\ell} \sigma_i a_{ij} \text{ for all } j.$$

If μ is a dominant weight, taking $j = j_k$ above we get

$$\begin{aligned} 0 \leq \mu_k &\leq \frac{1}{a_{kj_k}} \sum_{i=1}^{\ell} \sigma_i a_{ij_k} \\ &\leq C(G) \max_i \sigma_i. \end{aligned}$$

Otherwise $\mu = w(v)$ for some dominant weight $v \in \Pi(\sigma)$ and $w \in W$. Since the Weyl action is linear,

$$\max_i |\mu_i| \leq C'(G) \max_i v_i$$

for some constant $C'(G)$. Now take $C_1 = CC'$. □

LEMMA 2.3. — *There is a constant* $C_2 = C_2(t, G)$ *so that if* $\sigma \in \widehat{G}$ *and* $\mu \in \Pi(\sigma)$, *then* $m_\sigma(\mu) \leq C_2 d_\sigma^{1-t}$.

Proof. — This is a straight forward calculation. We begin with the fact that

$$m_\sigma(\mu) = \int_{T^\ell} \operatorname{Tr} \sigma(x) e^{-i\mu(x)} dx.$$

By the Weyl character formula and standard inequalities

$$\begin{aligned} m_\sigma(\mu) &\leq \int_{T^\ell} \left| \frac{\sum_{w\in W} \det(w) e^{iw(\sigma+\rho)(x)}}{q(x)} \right| dx \\ &\leq \left\| \sum_{w\in W} \det(w) e^{iw(\sigma+\rho)(x)} \right\|_\infty^t \|q^{-t}\|_{L^1(T^\ell)} \| \operatorname{Tr} \sigma|_{T^\ell} \|_\infty^{1-t} \\ &\leq |W|^t \|q^{-t}\|_{L^1(T^\ell)} d_\sigma^{1-t}. \end{aligned}$$

Since $q^{-t} \in L^1(T^\ell)$ for any $t < \ell/|\Phi^+|$ ([13]) the proof is complete. □

Proof of Theorem 2.1. — Throughout the proof we will use the following notation: $m_j(\mu) \equiv m_{\sigma_j}(\mu)$; $\Pi_j \equiv \Pi(\sigma_j)$; and

$$B_j \equiv \left\{ \sum_{i=1}^{\ell} m_i \lambda_i : |m_i| \leq C_1 d_j,\ m_i \in \mathbf{Z} \right\} \subseteq \Lambda.$$

For C_1 and C_2 as in the lemmas, put

$$C = \max \left\{ \left((2C_1+1)^\ell C_2 \right)^{2/t},\ \sup_N \left(\frac{\left\| \sum_{n=-N}^{N} e^{inx} \right\|_{L^1[0,2\pi)}}{\log N} \right) \right\}.$$

Lemma 2.2 obviously implies $\Pi_j \subseteq B_j$. The key idea of the proof (which we make precise below) is that "most" of Π_j, counted by multiplicity, lies outside B_{j-1}. This we are able to obtain from Lemma 2.3 and property (3). To be precise we have, if $k > j$,

$$(*) \qquad \begin{aligned} \sum_{\mu \in \Pi_k \cap B_j} m_k(\mu) &\leq |B_j| \max_{\mu \in \Pi_k} m_k(\mu) \\ &\leq (2C_1 d_j + 1)^\ell C_2 d_k^{1-t} \\ &\leq d_k^{1-t/2}, \end{aligned}$$

and thus

$$(**) \qquad \begin{aligned} \sum_{\mu \in \Pi_k \setminus B_j} m_k(\mu) &= \sum_{\mu \in \Pi_k} m_k(\mu) - \sum_{\mu \in \Pi_k \cap B_j} m_k(\mu) \\ &\geq d_k - d_k^{1-t/2} \\ &\geq \frac{1}{2} d_k. \end{aligned}$$

Let D_n be the ℓ-dimensional Dirichlet kernel supported by B_n (thinking now of B_n as a subset of $\mathbf{Z}^\ell$ rather than of Λ),

$$D_n(x_1, \ldots, x_\ell) = \prod_{k=1}^{\ell} \sum_{j=-C_1 d_n}^{C_1 d_n} e^{ij(x_k)},$$

and let $H_n = D_n - D_{n-1}$. Then $\widehat{H}_n = \chi_{B_n \setminus B_{n-1}}$ and $\|H_n\|_1 \leq 2(C \log C d_n)^\ell$.

Suppose $f = \sum_{j=1}^{N} d_j a_j \operatorname{Tr} \sigma_j$ is a central trigonometric polynomial with $\|f\|_\infty \leq 1$. With our notation

$$f|_{T^\ell}(x) = \sum_{j=1}^{N} d_j a_j \sum_{\mu \in \Pi_j} m_j(\mu) e^{i\mu(x)}.$$

Notice that $H_n * \sum_{\mu \in \Pi_j} m_j(\mu) e^{i\mu(x)} = 0$ if $j < n$ (here the convolution is over T^ℓ), and so if $n \leq N$,

$$\begin{aligned} 1 \geq \frac{\|f|_{T^\ell} * H_n\|_\infty}{\|H_n\|_1} &\geq \frac{|f|_{T^\ell} * H_n(0)|}{\|H_n\|_1} \\ &= \frac{\left| \sum_{j=n}^{N} d_j a_j \sum_{\mu \in \Pi_j} m_j(\mu) e^{i\mu(x)} * H_n(0) \right|}{\|H_n\|_1} \\ &\geq \frac{\left| \sum_{j=n}^{N} d_j a_j \sum_{\mu \in \Pi_j \cap (B_n \setminus B_{n-1})} m_j(\mu) \right|}{2(C \log C d_n)^\ell}. \end{aligned}$$

An application of the triangle inequality yields

$$d_n |a_n| \sum_{\mu \in \Pi_n \setminus B_{n-1}} m_n(\mu) \leq 2(C \log C d_n)^\ell + \sum_{j=n+1}^{N} d_j |a_j| \sum_{\mu \in \Pi_j \cap (B_n \setminus B_{n-1})} m_j(\mu).$$

Combined with our estimates (*) and (**), and property (2), this gives

$$d_n^2 |a_n| \leq d_n^{(1-a)/2} + 2 \sum_{j=n+1}^{N} d_j^{2-t/2} |a_j|.$$

For $j = 1, 2, \ldots, N$ set $S_j = \sum_{k=0}^{j-1} d_{N-k}^{2-t/2} |a_{N-k}|$ and set $S_0 = 0$. This gives

$$d_n^2 |a_n| \leq d_n^{(1-a)/2} + 2 S_{N-n},$$

and since (1) guarantees $d_{N-j}^{t/2} \geq 2$,

$$S_{j+1} \leq 2 S_j + d_{N-j}^{\varepsilon - t/2}$$

where $\varepsilon = (1-a)/2$. By induction,

$$S_j \leq \sum_{i=1}^{j} 2^{i-1} d_{N-j+i}^{\varepsilon - t/2} \text{ for } j = 1, 2, \ldots, N.$$

Property (1) also ensures $S_j \leq 1$, thus
$$d_n^2|a_n| \leq d_n^{(1-a)/2} + 2.$$

It is now easy to see that $\{\sigma_j\}$ is central $(a,1)$-Sidon:
$$\begin{aligned}\|\hat{f}\|_{(a,1)} &= \sum_{n=1}^{N} d_n^{1+a}|a_n| \\ &\leq \sum_{n=1}^{N} \frac{1}{d_n^{(1-a)/2}} + \frac{2}{d_n^{1-a}},\end{aligned}$$
and this sum is bounded over N since $\{d_n\}$ is lacunary and $a < 1$. □

Remark. — An application of the Weyl dimension formula shows that if $\sigma_j = \sum_{i=1}^{\ell} \sigma_{ji}\lambda_i$, then $\{(\sigma_{j1}, \ldots, \sigma_{j\ell})\}_j$ is the union of a finite set and a dissociate set in $\mathbf{Z}^\ell$, and hence is a Sidon set in the dual of the torus.

COROLLARY 2.4. — *If G is a compact, simply-connected, semisimple Lie group, then every infinite subset of $\widehat{G}$ contains an infinite central (a,p)-Sidon set for all $p \geq 1$ and $a < 2p-1$.*

Proof. — As remarked in the first section it suffices to prove this for $p = 1$ and all $a < 1$.

Let ℓ=rank G and fix $0 < t < \ell/|\Phi^+|$. Set $a_1 = 1 - \dfrac{t}{2}$ and choose an increasing sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n < 1$ and limit one. Let $E \subseteq \widehat{G}$ be infinite. Since $\widehat{G}$ contains only finitely many representations of any given degree we can choose an infinite subset $\{\sigma_j\}$ of E satisfying (where C is as in the theorem):

(1) $d_j^{t/4} \geq 4^j$ for $j \geq 1$,

(2) $d_j^{(1-a_j)/2} \geq 4(C \log C d_j)^\ell$ for $j \geq 1$, and

(3) $d_j \geq C(d_{j-1})^{2\ell/t}$ for $j \geq 2$.

Choose $a < 1$. Then $a \leq a_j$ for all $j \geq J$ and by the theorem $\{\sigma_j\}_{j=J}^{\infty}$ is a central $(a,1)$-Sidon set. It is easy to see from Proposition 1.1 that the union of a finite set and a central $(a,1)$-Sidon set is again central $(a,1)$-Sidon, and therefore $\{\sigma_j\}_{j=1}^{\infty}$ is central $(a,1)$-Sidon for any $a < 1$. □

Remark. — As noted previously, these groups admit no infinite central Sidon sets. It is unknown if they admit infinite central $(2p-1,p)$-Sidon sets for any $p > 1$.

The next step towards our main result is to consider the case when G is an infinite product group.

THEOREM 2.5. — *Let $G = \prod_{j=1}^{\infty} G_j$ be a product of compact, simply-connected, semisimple Lie groups, and suppose σ_j is a non-trivial representation of G_j. Then $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is a central $(a,1)$- Sidon set for all $a < 1$.*

Proof. — Suppose $f = \sum_{j=1}^{N} d_j a_j \operatorname{Tr} \sigma_1 \times \cdots \times \sigma_j$ where $d_j = d_{\sigma_1} \cdots d_{\sigma_j}$. Without loss of generality assume $\|f\|_\infty \leq 1$. For the duration of this proof we will use the following notation: $m_j(\mu) \equiv m_{\sigma_j}(\mu)$; $m_j \equiv m_j(0)$; $\Pi_j' \equiv \Pi(\sigma_j)\backslash\{0\}$; $T^{G_j} \equiv$ torus of G_j; and $T_N = T^{G_1} \times \cdots \times T^{G_N}$. With this notation we have

$$f|_{T_N}(x_1, \ldots, x_N) = \sum_{j=1}^{N} d_j a_j \prod_{k=1}^{j} \left(m_k + \sum_{\mu \in \Pi_k'} m_k(\mu) e^{i\mu(x_k)} \right) \text{ for } x_k \in T^{G_k}.$$

Viewing f as a function on T_N, we can read off the Fourier coefficients:

$$\hat{f}(0, \ldots, 0) = \sum_{j=1}^{N} d_j a_j m_1 \cdots m_j;$$

$$\hat{f}(\mu_1, \ldots, \mu_k, 0, \ldots, 0) = \left(d_k a_k + \sum_{j=k+1}^{N} d_j a_j m_{k+1} \cdots m_j \right) m_1(\mu_1) \cdots m_k(\mu_k)$$

$$\text{if } \mu_i \in \Pi(\sigma_i) \text{ for } i = 1, \ldots, k-1,\ \mu_k \in \Pi_k';$$

and $\hat{f}(\mu_1, \ldots, \mu_N) = 0$ otherwise.

For $x_j = (x_{j1}, \ldots, x_{j\ell(j)}) \in T^{G_j}$ (here $\ell(j) = \operatorname{rank} G_j$), and M very large, let

$$H_j(x_j) = \prod_{k=1}^{\ell(j)} \sum_{n=-M}^{M} \left(1 - \frac{|n|}{M+1}\right) e^{in(x_{jk})},$$

and for $n \leq N$ let

$$K_n(x_1, \ldots, x_N) = \prod_{j=1}^{n-1} H_j(x_j)(H_n(x_n) - 1).$$

Observe that if $\mu_j \in \widehat{T^{G_j}}$ for $j = 1, \ldots, N$, and $\widehat{K}_n(\mu_1, \ldots, \mu_N) \neq 0$, then $\mu_j = 0$ for $j > n$ and $\mu_n \neq 0$. Consider the convolution of K_n and $f|_{T_N}$. If M is chosen sufficiently large then

$$2 \geq |f * K_n(0)| \geq \frac{1}{2} \left| d_n a_n + \sum_{j=n+1}^{N} d_j a_j m_{n+1} \cdots m_j \right| \sum_n{}' m_1(\mu_1) \cdots m_n(\mu_n)$$

where $\sum'_n$ denotes the sum over all $\mu_j \in \Pi(\sigma_j)$ for $j = 1, \dots, n-1$, and $\mu_n \in \Pi'_n$. Clearly

$$\begin{aligned}\sum_n{}' m_1(\mu_1)\cdots m_n(\mu_n) &= d_{\sigma_1}\cdots d_{\sigma_{n-1}}(d_{\sigma_n} - m_n)\\ &= d_{n-1}(d_{\sigma_n} - m_n).\end{aligned}$$

Thus

$$|d_n a_n| \le \frac{4}{d_{n-1}(d_{\sigma_n} - m_n)} + \left|\sum_{j=n+1}^{N} d_j a_j m_{n+1}\cdots m_j\right|.$$

Furthermore,

$$\begin{aligned}\left|\sum_{j=n+1}^{N} d_j a_j m_{n+1}\cdots m_j\right| &= \left|d_{n+1}a_{n+1} + \sum_{j=n+2}^{N} d_j a_j m_{n+2}\cdots m_j\right| m_{n+1}\\ &\le \frac{4m_{n+1}}{d_n(d_{\sigma_{n+1}} - m_{n+1})}\end{aligned}$$

(where the empty sum and m_{N+1} equal 0). Thus

$$|d_n a_n| \le \frac{4}{d_{n-1}(d_{\sigma_n} - m_n)} + \frac{4m_{n+1}}{d_n(d_{\sigma_{n+1}} - m_{n+1})}.$$

In [4] Gallagher proves that if σ is any non-trivial representation of a compact, simply-connected, semisimple Lie group then $\operatorname{Tr}\sigma$ has a root, say x, in the maximal torus. Evaluating $\operatorname{Tr}\sigma$ at x we derive the formula

$$m_\sigma(0) = -\sum_{\mu\in\Pi(\sigma)\setminus\{0\}} m_\sigma(\mu)e^{i\mu(x)},$$

from which one readily sees that $m_\sigma(0) \le d_\sigma/2$. Hence $|d_n a_n| \le 12/d_n$, and so

$$\|\hat{f}\|_{(a,1)} = \sum_{j=1}^{N} d_n^{1+a}|a_n| \le \sum_{j=1}^{N}\frac{12}{d_n^{1-a}}.$$

Since $d_n \ge 2^n$, this sum converges provided $a < 1$, and thus $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is a central $(a,1)$-Sidon set for all $a < 1$. □

This set of representations is independent in the sense that if

$$\int_G \prod_{j=1}^{N} (\operatorname{Tr}\sigma_1 \times \cdots \times \sigma_j)^{\varepsilon_j} \ne 0$$

for some $N \in \mathbf{N}$ and $\varepsilon_j = 0, \pm 1$ for $j = 1, \dots, N$, then necessarily all $\varepsilon_j = 0$. This independence condition is not sufficient to be Sidon [1]. It is not sufficient for central-Sidon either as the next example demonstrates.

Example 2.6. — Suppose $G_j = SU(2)$, $G = \prod_{j=1}^{\infty} G_j$ and $\sigma_j = 2\lambda_1$. The set $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is not central $(2p-1,p)$-Sidon for any $p \geq 1$.

Proof. — It is well known that the torus of $SU(2)$ is the circle group T and that $\operatorname{Tr} 2\lambda_1|_T = 1 + e^{ix} + e^{-ix}$ ([6], 29.25). Therefore $\operatorname{Tr} 2\lambda_1|_T$ takes on precisely the values in [-1,3]. Let

$$f_N = \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p} 3^j} \operatorname{Tr} \sigma_1 \times \cdots \times \sigma_j;$$

$\|\hat{f}_N\|_{(2p-1,p)} = \sum_{j=1}^{N} \frac{1}{j}$ which diverges as $N \to \infty$.

Being a central function, $\|f_N\|_\infty = \|f_N|_T\|_\infty$, and from the remark above the latter equals

$$\sup_{w_i \in [-\frac{1}{3},1]} \left| \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p}} w_1 \cdots w_j \right|.$$

We will now prove that this supremum is bounded over N which certainly suffices to prove $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is not central $(2p-1,p)$-Sidon.

Set $j_1 = 1$ and inductively define j_k to be the least integer greater than j_{k-1} with

$$(-1)^{j_k} w_1 \cdots w_{j_k} (-1)^{j_{k-1}} w_1 \cdots w_{j_{k-1}} \leq 0.$$

Consider first the alternating sum

$$\sum_k \frac{(-1)^{j_k}}{j_k^{1/p}} w_1 \cdots w_{j_k}.$$

Since $\frac{|w_1 \cdots w_{j_k}|}{j_k^{1/p}}$ decreases to zero, this sum is bounded in absolute value by $\frac{1}{j_1^{1/p}} = 1$.

If $j \notin \{j_i\}$ then $(-1)^j w_1 \cdots w_j$ and $(-1)^{j-1} w_1 \cdots w_{j-1}$ have the same sign. This can occur only if $w_j < 0$, but then $|w_1 \cdots w_j| \leq \frac{1}{3} |w_1 \cdots w_{j-1}|$. As $|w_i| \leq 1$ for all i, it follows that

$$\left| \sum_{j \notin j_i} (-1)^j \frac{w_1 \cdots w_j}{j^{1/p}} \right| \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2}.$$

These estimates clearly combine to give

$$\sup_{w_i \in [-\frac{1}{3},1]} \left| \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p}} w_1 \cdots w_j \right| \leq \frac{3}{2}. \qquad \square$$

THEOREM 2.7. — *Let $G = \prod_\alpha G_\alpha$ be a product (possibly finite) of compact, simply-connected, simple Lie groups. Then any infinite subset of $\hat{G}$ contains an infinite central (a,p)-Sidon set for all $p \geq 1$ and $a < 2p-1$.*

First we introduce some notation and prove a lemma.

Notation. — Let $\sigma_j \in \widehat{G}$. Then $\sigma_j = \times \sigma_{j\alpha}$ where $\sigma_{j\alpha} \in \widehat{G}_\alpha$ and only finitely many $\sigma_{j\alpha}$ are non-trivial. Denote by $\operatorname{supp} \sigma_j$ the set $\{\alpha : \sigma_{j\alpha} \neq 1\}$. We will say σ_j is orthogonal to σ_k, and write $\sigma_j \perp \sigma_k$, if $\operatorname{supp} \sigma_j \cap \operatorname{supp} \sigma_k$ is empty. Recall that Parker [9] has shown that if $\{\sigma_j\}$ consists of mutually orthogonal, non-trivial representations then $\{\sigma_j\}$ is a central Sidon set.

LEMMA 2.8. — *Let $a \leq 1$ and suppose $\{\sigma_j\}$ is a central $(a,1)$-Sidon set in $\widehat{G}$. Suppose $\{\tau_j\} \subseteq \widehat{G}$ and $\tau_j \perp \sigma_k$ for all j,k. Then $\{\tau_j \times \sigma_j\}$ is another central $(a,1)$-Sidon set.*

Proof. — This is an easy consequence of the fact that

$$\|f\|_\infty \geq \sup \left\{ |f(x)| : x = (x_\alpha) \text{ and } x_\alpha = 1 \text{ if } \alpha \in \bigcup_j \operatorname{supp} \tau_j \right\}. \qquad \square$$

Proof of Theorem 2.7. — It suffices to show that any countably infinite set, $E = \{\sigma_j\}_{j=1}^\infty$, contains an infinite subset which is central $(a,1)$-Sidon for all $a < 1$.

Suppose first $\{\sigma_{j\alpha} : \sigma_j \in E\}$ is infinite for some α. By Corollary 2.4 we can find an infinite subset of $\{\sigma_{j\alpha}\}_{j=1}^\infty$ which is a central $(a,1)$-Sidon subset of $\widehat{G}_\alpha$, for all $a < 1$. The corresponding subset of E has the same property.

So we may assume $\{\sigma_{j\alpha} : \sigma_j \in E\}$ is finite for each α.

Case 1. — For each index $\alpha, \{\sigma_j : \sigma_{j\alpha} \neq 1\}$ is finite.

Set $j_1 = 1$ and inductively assume mutually orthogonal representations $\sigma_{j_1}, \dots, \sigma_{j_n} \subseteq E$ have been picked. Since there are only finitely many representations σ_j with $\sigma_{j\alpha} \neq 1$ for $\alpha \in \bigcup_{k=1}^n \operatorname{supp} \sigma_k$, we can choose $\sigma_{j_{n+1}}$ orthogonal to each of $\sigma_{j_1}, \dots, \sigma_{j_n}$. By Parker [9] $\{\sigma_{j_k}\}_k$ is central Sidon.

Case 2. — $\{\sigma_j : \sigma_{j\alpha} \neq 1\}$ is infinite for some α, say $\alpha = \alpha_1$.

Since $\{\sigma_{j\alpha_1} : \sigma_j \in E\}$ is finite there must be a non-trivial representation ϕ_1 of G_{α_1}, with $\phi_1 = \sigma_{j\alpha_1}$ for all $\sigma_j \in F_1$, an infinite subset of E. Select $\sigma_{j_1} \in F_1$.

If $\{\sigma_j \in F_1 : \sigma_{j\alpha} \neq 1\}$ is finite for all $\alpha \notin \operatorname{supp} \sigma_{j_1}$, then by arguments similar to case 1 we can obtain an infinite subset of F_1 of the form $\{\tau_k \times w_k\}_{k=2}^{\infty}$ where $\operatorname{supp} \tau_k \subseteq \operatorname{supp} \sigma_{j_1}$ and the representations w_k are non-trivial, mutually orthogonal, and all orthogonal to σ_{j_1}. By [9] and the lemma this set is central Sidon.

Otherwise we repeat the argument to produce infinite sets $F_n \subset F_{n-1}$ ($F_0 = E$), representations $\sigma_{j_n} \in F_n$ and ϕ_n orthogonal to σ_{j_k} for $k \leq n-1$, and an index α_n with the property that $\sigma_{j\alpha_n} = \phi_n$ for all $\sigma_j \in F_n$. If $\{\sigma_j \in F_n : \sigma_{j\alpha} \neq 1\}$ is finite for all $i \notin \bigcup_{k=1}^{n} \operatorname{supp} \sigma_{j_k}$ we quit this process and produce an infinite central Sidon set in F_n by standard arguments. Otherwise, as in the first step of case 2, we choose F_{n+1}, α_{n+1}, ϕ_{n+1} and $\sigma_{j_{n+1}}$ with the properties above.

If this process never stops we produce an infinite set $\{\sigma_{j_n}\} \subseteq E$. By construction $\sigma_{j_n} = \phi_1 \times \cdots \times \phi_n \times \tau_n$ where $\phi_n \perp \phi_1 \times \cdots \times \phi_j \times \tau_j$ for all $n > j$. From Theorem 2.5 $\{\phi_1 \times \cdots \times \phi_n\}_{n=1}^{\infty}$ is central $(a,1)$-Sidon for all $a < 1$ and hence so is $\{\sigma_{j_n}\}$.

In either case we can find an infinite central $(a,1)$-Sidon subset of E and thus the proof of the theorem is complete. □

The main result will now be seen to follow from the structure theorem ([10], 6.5.6): If G is a compact, connected group then there is a continuous epimorphism $\phi : T \times \prod_{\alpha} G_\alpha \to G$ where T is a compact abelian group and each G_α is a compact, simply-connected, simple Lie group.

THEOREM 2.9. — *If G is a compact, connected group then any infinite subset of $\widehat{G}$ contains an infinite central (a,p)-Sidon set for all $p \geq 1$ and $a < 2p-1$.*

We need only one additional lemma whose proof is obvious.

LEMMA 2.10. — *If $\phi : H \to G$ is a continuous epimorphism of compact groups then $E \subseteq \widehat{G}$ is a (central) (a,p)-Sidon set if and only if the same is true for $E \circ \phi = \{\sigma \circ \phi : \sigma \in E\} \subseteq \widehat{H}$.*

Proof of Theorem 2.9. — Let $E \subset \widehat{G}$ be an infinite set and let $\phi : T \times \prod_{\alpha} G_\alpha \to G$ be the structure theorem epimorphism. Since ϕ is onto $E \circ \phi$ is also infinite. For $\sigma \circ \phi \in E \circ \phi$, write $\sigma \circ \phi = \tau_\sigma \times \psi_\sigma$ where $\tau_\sigma \in \widehat{T}$ and $\psi_\sigma \in \prod \widehat{G_\alpha}$. If $\{\tau_\sigma : \sigma \in E\}$ is infinite, then since T is an abelian group there is an infinite Sidon subset of $\{\tau_\sigma\}$, and by Lemma 2.8 the corresponding subset of $E \circ \phi$ is central Sidon. If $\{\psi_\alpha\}$ is infinite we appeal to Theorem 2.7 and Lemma 2.8 to obtain an infinite central $(a,1)$-Sidon set for all $a < 1$. In either case the corresponding infinite subset of E has the required property. □

COROLLARY 2.11. — *Suppose G is a compact, connected group. Any infinite subset of $\widehat{G}$ contains an infinite set which is central p-Sidon for all $p > 1$.*

Remark. — This answers the open problem left in [3].

3. Central $(0,1)$-Sidon sets.

In this section we investigate the relationship between weighted central Sidonicity for a Lie group G and Sidonicity for its abelian torus. This investigation is motivated in part by the fact that both Dooley's examples [3] of central p-Sidon sets and our examples from Theorem 2.1 correspond to Sidon sets in $\mathbf{Z}^{\mathrm{rank}G}$.

THEOREM 3.1. — *Let G be a compact, simply-connected, semisimple Lie group of rank ℓ, with torus T^ℓ. If $E \subseteq (\mathbf{Z}^+)^\ell$ is a Sidon set for T^ℓ, then E viewed as a subset of $\widehat{G}$ is central $(0,1)$-Sidon.*

Proof. — Let $f = \sum_{\sigma\in E} d_\sigma a_\sigma \operatorname{Tr}\sigma$ be a central trigonometric polynomial on G. Since $|q(x)| \le |W|$, the Weyl character formula implies

$$\|f\|_\infty \ge \frac{1}{|W|} \sup_{x\in T^\ell} \left| \sum_{\sigma\in E} d_\sigma a_\sigma \sum_{w\in W} \det(w) e^{iw(\sigma+\rho)(x)} \right|.$$

Because the representations $\sigma+\rho$, $\sigma\in\widehat{G}$, belong to the fundamental Weyl chamber, the weights $w(\sigma+\rho)$ are distinct as w varies over W and σ over E ([7], ch. 10). Furthermore, the family of Sidon sets in an abelian group is closed under linear transformations and finite unions ([8], p. 44) so this set of distinct elements, $\bigcup_{w\in W}\{w(\sigma+\rho) : \sigma\in E\}$, forms a Sidon set in $\mathbf{Z}^\ell$ (with the natural identification). With these observations it is straight forward to check that E is central $(0,1)$-Sidon. □

Our next result shows that Theorem 3.1 cannot be improved. Recall that $SU(2)$ has one fundamental weight so its dual can be identified with $\mathbf{Z}^+$. The degree of the representation indexed by n is $n+1$.

PROPOSITION 3.2. — *There is a Sidon set in* $\mathbf{Z}$*, which is contained in* $\mathbf{Z}^+$*, and is not a central* $(a,1)$*-Sidon set in* $\widehat{SU(2)}$ *for any* $a>0$.

Proof. — Let E be any infinite Sidon subset of $\mathbf{Z}$ contained in $\{2,3,4,\ldots\}$ and disjoint from $E-2$. Certainly $E \cup E-2$ is a Sidon set in $\mathbf{Z}$. If it was a central $(a,1)$-Sidon set in $\widehat{SU(2)}$ for some $a>0$, by Proposition 1.1 there would be a measure μ on $SU(2)$ satisfying

$$\hat{\mu}(n) = \begin{cases} \dfrac{1}{(n+1)^{1-a}} & \text{for } n \in E \\ 0 & \text{for } n \in E-2. \end{cases}$$

Coifman and Weiss [2] have shown that μ is a measure on $SU(2)$ if and only if

$$\sum_{n\geq 2} ((n+1)\hat{\mu}(n) - (n-1)\hat{\mu}(n-2)) \cos n\theta$$

represents a measure v on T. But for $n \in E$, $\hat{v}(n) = (n+1)^a$ which tends to infinity, so this is an impossibility. □

It is natural to ask if the converse to Theorem 3.1 is true. It is not.

THEOREM 3.3. — *There are subsets of* $\mathbf{Z}^+$ *containing arbitrarily long arithmetic progressions which are central* $(a,1)$*-Sidon sets in* $\widehat{SU(2)}$*, for all* $a<1$*; consequently a central* $(a,1)$*-Sidon set need not be a Sidon set in* $\mathbf{Z}$.

Proof. — The second statement follows from the first since sets containing arbitrarily long arithmetic progressions are never Sidon sets in $\mathbf{Z}$ ([8], p. 77). We follow the strategy of [3] to produce examples of central $(a,1)$-Sidon sets with this property.

Let $\{n_j\}_{j=1}^{\infty}$ be a sequence of positive integers, σ_j the representation of $SU(2)$ indexed by $2n_j$, and let $f = \sum_{j=1}^{N} (2n_j+1)a_j \operatorname{Tr}\sigma_j$ be a central trigonometric polynomial on $SU(2)$. It is well known ([6], 29.25) that for $t_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \in T$,

$$\operatorname{Tr}\sigma_j(t_\theta) = \sin\Big(n_j + \frac{1}{2}\Big)\theta / \sin\frac{\theta}{2} = D_{n_j}(\theta),$$

where D_n is the n'th Dirichlet kernel. Thus

$$\|f\|_\infty = \|f|_T\|_\infty = \sup_{\theta\in[0,2\pi)} \left|\sum_{j=1}^{N} (2n_j+1)a_j D_{n_j}(\theta)\right|.$$

For even integers $a < b$, let F_{ab} denote the translated Fejér kernel with transform supported on (a,b). One easily sees that

(i) if $k < a$ then $F_{ab} * D_k = 0$; while

(ii) if $b \le k$ then $F_{ab} * D_k(0) = F_{ab}(0) = \dfrac{b-a}{2}$.

To simplify notation we write $F'_j = F_{n_{j-1},n_j}$ (taking $n_0 = 0$), $B_j = (n_{N-j} - n_{N-j-1})/2$ and $X_j = (2n_{N-j}+1)|a_{N-j}|$. With this notation

$$(*) \qquad \|\hat{f}\|_{(a,1)} = \sum_{j=0}^{N-1} (2n_{N-j}+1)^a X_{N-j}.$$

Without loss of generality we may assume $\|f\|_\infty = 1$, so, for $m = N-k$,

$$\begin{aligned}
1 \ge |f * F'_m(0)| &= \left|\sum_{j=1}^{N} (2n_j+1)a_j F'_m * D_{n_j}(0)\right| \\
&\ge (2n_m+1)|a_m|\left(\frac{n_m - n_{m-1}}{2}\right) - \sum_{j>m} (2n_j+1)|a_j|\left(\frac{n_m - n_{m-1}}{2}\right) \\
&\ge X_k B_k - \sum_{j=0}^{k-1} X_j B_k.
\end{aligned}$$

Thus

$$X_k \le \frac{1}{B_k} + \sum_{j=0}^{k-1} X_j,$$

and simplifying this yields the estimate

$$X_k \le \frac{1}{B_k} + \sum_{j=0}^{k-1} \frac{2^j}{B_{k-1-j}}.$$

Obviously there are many ways to choose a sequence $\{n_j\}$ containing arbitrarily long arithmetic progressions, and yet have X_k sufficiently small so that $(*)$ bounded over all N and all $a < 1$. One choice, whose verification is routine, and is left for the reader, is to set $n_{2^k+i} = A^{A^k}(1+i)$ for $i = 0, 1, \ldots, 2^k - 1$, where A is sufficiently large. $\square$

There is however a partial converse to Theorem 3.1. We state it for $SU(2)$, the context in which we will apply it to show the failure of the union property, but similar results hold more generally for all compact, simply-connected, semisimple Lie groups.

PROPOSITION 3.4. — *Suppose E and $E-2$ are disjoint subsets of $\mathbf{Z}^+$ and that $E \cup E-2$ is a central $(0,1)$-Sidon set in $\widehat{SU(2)}$. Then E is a Sidon set in $\mathbf{Z}$.*

Proof. — Let $\phi \in \ell^\infty(E)$. Since $E \cup E-2$ is central $(0,1)$-Sidon, there exists a central measure μ on $SU(2)$ with $\hat{\mu}(n) = \phi(n)/(n+1)$ for $n \in E$ and $\hat{\mu} = 0$ on $E-2$. As in Proposition 3.2, [2] implies that there is a measure ν on T with $\hat{\nu}(\pm n) = (n+1)\hat{\mu}(n) - (n-1)\hat{\mu}(n-2)$ if $n \in \mathbf{Z}^+$. For $n \in E$, $\hat{\nu}(n) = \phi(n)$, and consequently E is a Sidon set in $\mathbf{Z}$. □

In contrast to the situation for abelian groups it is known that the union of two central Sidon sets need not be central Sidon [12]. This extends to central $(a,1)$-Sidon sets.

PROPOSITION 3.5. — *The union of two sets which are central $(a,1)$-Sidon for all $a < 1$, need not be a central $(0,1)$-Sidon set.*

Proof. — Consider the example $E = \{n_j\}$, where $n_{2^k+i} = A^{A^k}(1+i)$ for $i = 0, 1, \cdots, 2^k - 1$ and A sufficiently large. This example is seen in Theorem 3.3 to be a non-Sidon set in $\mathbf{Z}^+$ which is a central $(a,1)$-Sidon set for all $a < 1$. The set $E-2$ clearly has the same properties and is disjoint from E. By the previous proposition their union is not central $(0,1)$-Sidon. □

Remark. — Our understanding of weighted Sidon sets is much less satisfactory in the non-central case. It is known that any set of representations whose degrees tend to infinity sufficiently fast is $(-\varepsilon,1)$-Sidon for any given $\varepsilon > 0$, and that a compact Lie group admits no $(\varepsilon,1)$-Sidon set for $\varepsilon > 0$ [5], but we do not know if any of our examples of central $(a,1)$-Sidon sets, or any other infinite sets in the dual of a compact, simple-connected semisimple Lie group, are $(0,1)$-Sidon.

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