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Regularity properties of commutators and BMO-Triebel-Lizorkin spaces

REGULARITY PROPERTIES OF COMMUTATORS
AND BMO-TRIEBEL-LIZORKIN SPACES

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1. Introduction.

Let b be a locally integrable function and $1 < p < +\infty$. It is a well known fact that the commutators $([b, R_k])_{1 \leq k \leq n}$ are bounded on $L^p(\mathbb{R}^n)$ if and only if $b \in BMO$, where $(R_k)_{1 \leq k \leq n}$ are the Riesz transforms in the n-dimensional euclidean space $\mathbb{R}^n$. This result is due to Coifman–Rochberg–Weiss [5], and extends to the Hardy space $\mathcal{H}^1$ in several variables certain well known factorization theorems on the unit disk. In [4] this result gives the regularity of various nonlinear quantities (like the jacobian, "div-curl"...) identified by the compensated compactness theory.

Another interest of commutators comes from the pseudo-differential calculus and the theory of singular integral operators (see [7]). In particular in the case when b is Lipschitz, the commutators $([b, R_k])_k$ are bounded from L^2 to the homogeneous Sobolev spaces $\dot{H}^1$. This result is due to Calderón [3] and has been generalized to the commutator $[b, T]$ when T is a reasonable Calderón-Zygmund operator.

Our purpose here is the study of the intermediate case, in particular we give necessary and sufficient conditions for the boundedness of $([b, R_k]_k$ from L^p into the Besov space $\dot{B}_p^{s,p}$ for $1 < p < +\infty$ and $0 < s < 1$.

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Notice that our result is similar to the Murray's theorem [8] when the Riesz transforms R_k is replaced by the Riesz potential operator of order s and $p = 2$.

The paper is organized as follows. In Section 2 we give the statements of our results. In section 3 we establish some results which are related to the boundedness of the paraproduct. In section 4 we prove our main results.

In the sequel, C will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters. The symbols $\hat{f}$ will stand for the Fourier transform of f and $\check{f}$ for inverse Fourier transform of f. We also use the notations :

$\mathcal{D}(\mathbb{R}^n)$ = space of C^∞-functions with compact support, $\mathcal{D}'(\mathbb{R}^n)$ its dual.

$\mathcal{S}(\mathbb{R}^n)$ = the space of Schwartz test functions.

$\mathcal{S}'(\mathbb{R}^n)$ = space of tempered distributions.

$[s]$ = the greatest integer smaller than or equal to s and $s^* = s - [s]$.

2. Definitions and statements of the main results.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be supported in the ball $|\xi| \leq 1$ and satisfy $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. The function

$$\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$$

is C^∞, supported in

$$\left\{ \frac{1}{2} \leq |\xi| \leq 2 \right\}$$

and satisfies the identity

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1$$

for $\xi \neq 0$. We denote by Δ_j and S_j the convolution operators with symbols $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$ respectively.

For $s \in \mathbb{R}$, $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$ the homogeneous Besov space is defined by

$$\|f\|_{\dot{B}_p^{s,q}} = \left[\sum_{j \in \mathbb{Z}} 2^{sjq} \|\Delta_j f\|_p^q \right]^{\frac{1}{q}}$$

with the usual modification if $q = +\infty$.

For $s \in \mathbb{R}$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}$ is defined by

$$\|f\|_{\dot{F}_p^{s,q}} = \left\| \left[\sum_{j \in \mathbb{Z}} 2^{sqj} |\Delta_j f|^q \right]^{\frac{1}{q}} \right\|_p$$

with usual modification if $q = +\infty$. To give the definition of the BMO-Triebel-Lizorkin space $\dot{F}_\infty^{s,q}$, let us first recall the definition of Carleson measures. We shall say that a sequence of positive Borel measures $(\nu_j)_{j \in \mathbb{Z}}$ is a Carleson measure in $\mathbb{R}^n \times \mathbb{Z}$ if there exists a positive constante $C > 0$ such that

$$\sum_{j \geq k} \nu_j(B) \leq C|B|$$

for all $k \in \mathbb{Z}$ and all euclidean balls B with radius 2^{-k}, where $|B|$ is the Lebesgue measure of B. The norm of the Carleson measure $\nu = (\nu_j)_{j \in \mathbb{Z}}$ is given by

$$\||\nu\|| = \sup \left\{ \frac{1}{|B|} \sum_{j \geq k} \nu_j(B) \right\}$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k}.

The homogeneous BMO-Triebel-Lizorkin space $\dot{F}_\infty^{s,q}$ $(1 \leq q < +\infty)$ is the space of all distributions b for which the sequence $(2^{sjq}|\Delta_j(b)(x)|^q dx)_j$ is a Carleson measure (see [6]). The norm of b in $\dot{F}_\infty^{s,q}$ is given by

$$\|b\|_{\dot{F}_\infty^{s,q}} = \sup \left[\frac{1}{|B|} \sum_{j \geq k} \int_B 2^{sjq} |\Delta_j(b)(x)|^q dx \right]^{\frac{1}{q}}$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k}. For $q = +\infty$, we set $\dot{F}_\infty^{s,\infty} = \dot{B}_\infty^{s,\infty}$. In the inhomogeneous case, the BMO-Triebel-Lizorkin spaces were studied by different methods in [12].

When $q = 2$, the space $\dot{F}_p^{s,2}$ is the Sobolev space $(1 < p < +\infty)$ and the space $\dot{F}_\infty^{0,2}$ is, modulo polynomials, the BMO space. More generally, $\dot{F}_\infty^{s,2}$ is, modulo polynomials, the BMO-Sobolev space considered by Strichartz [11].

Note that the spaces $\dot{F}_p^{s,q}$ and $\dot{B}_p^{s,q}$ consist of distributions modulo polynomials. We consider now the operator

$$\mathcal{R}(f) = \sum_{j \in \mathbb{Z}} \Delta_j(f).$$

Notice that, when $f \in \mathcal{S}(\mathbb{R}^n)$, $f = \mathcal{R}(f)$ in $\mathcal{S}'(\mathbb{R}^n)$. But this equality does not hold for f is a polynomial.

Let $s \in \mathbb{R}$, $1 \le p, q \le +\infty$ and let $m = \left[s - \dfrac{n}{p}\right]$ and denote by $\dot{A}_p^{s,q}$ either $\dot{F}_p^{s,q}$ or $\dot{B}_p^{s,q}$. Then for all $f \in \dot{A}_p^{s,q}$ one can show that

$$f = \sum_{j \in \mathbb{Z}} \Delta_j(f)$$

modulo polynomials of degree less than or equal to m. In particular $\mathcal{R}(f) \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m$ where $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m$ is the space of distributions modulo polynomials of degre less than or equal to m. Moreover, $\mathcal{R}$ is an embedding from $\dot{A}_p^{s,q}$ to $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m$. For $f \in \dot{A}_p^{s,q}$, we set

$$\|f\|_{\mathcal{R}(\dot{A}_p^{s,q})} = \|f\|_{\dot{A}_p^{s,q}}.$$

The space $\mathcal{R}(\dot{A}_p^{s,q})$ is isomorphic to $\dot{A}_p^{s,q}$ and invariant by dilations and translations. Throughout the paper we identify $\dot{A}_p^{s,q}$ with $\mathcal{R}(\dot{A}_p^{s,q})$. For the study of these realizations the reader is referred to [2]. In the particular case where $0 < s < \dfrac{n}{p}$, $A = F$ and $q = 2$, we have

$$\mathcal{R}(\dot{F}_p^{s,2}) = I_s(L^p(\mathbb{R}^n))$$

where

$$I_s(f)(x) = \int \frac{f(y)}{|x-y|^{n-s}} dy$$

is the Riesz potential.

The paraproduct of J.-M. Bony between two functions f, g is defined by

$$\pi(g, f) = \pi_g(f) = \sum_{j \in \mathbb{Z}} \Delta_j(g) S_{j-3}(f).$$

It is a well known fact that, for $b \in \dot{B}_\infty^{0,\infty}$, π_b is bounded on L^2 if and only if $b \in \dot{F}_\infty^{0,2} = BMO$.

Our main results are the following.

THEOREM 1. — Let $s \in \mathbb{R}$, $b \in \dot{B}_\infty^{s,\infty}$ and $1 < p < +\infty$.

1) If $b \in \dot{F}_\infty^{s,p}$, then π_b is bounded from L^p into $\dot{B}_p^{s,p}$.

2) If π_b is bounded from $\dot{B}_p^{0,1}$ into $\dot{B}_p^{s,p}$, then $b \in \dot{F}_\infty^{s,p}$.

Notice that $\dot{B}_p^{0,1} \subset \dot{F}_p^{0,q} \subset L^p$ for $1 \le q \le 2$, then we obtain the following.

COROLLARY 1. — Let $s \in \mathbb{R}$, $b \in \dot{B}_\infty^{s,\infty}$, $1 < p < +\infty$ and $1 \leq q \leq 2$. Then $b \in \dot{F}_\infty^{s,p}$ if and only if the operator π_b is bounded from $\dot{F}_p^{0,q}$ into $\dot{B}_p^{s,p}$.

THEOREM 2. — Let $0 < s < 1$, $1 < p < +\infty$, $q = \inf(p, 2)$ and b be a locally integrable function.

1) If $b \in \dot{F}_\infty^{s,p}$, then the commutators $([b, R_k])_{1 \leq k \leq n}$ are bounded from $\dot{F}_p^{0,q}$ into $\dot{B}_p^{s,p}$.

2) If the commutators $([b, R_k])_{1 \leq k \leq n}$ are bounded from $\dot{B}_p^{0,1}$ into $\dot{B}_p^{s,p}$, then $b \in \dot{F}_\infty^{s,p}$.

COROLLARY 2. — Let $0 < s < 1$, $1 < p < +\infty$, $1 \leq q \leq \inf(p, 2)$ and b be a locally integrable function. Then $b \in \dot{F}_\infty^{s,p}$ if and only if the commutators $([b, R_k])_{1 \leq k \leq n}$ are bounded from $\dot{F}_p^{0,q}$ into $\dot{B}_p^{s,p}$.

3. Preparatory results.

The connection betwen the spaces $\dot{F}_p^{s,q}$ and $\dot{B}_p^{s,q}$ is given by the following.

PROPOSITION 1. — Let $1 \leq p, q \leq +\infty$ and $s \in \mathbb{R}$. Then

1) $\dot{F}_p^{s,q} \subset \dot{B}_p^{s,q}$ if $p \leq q$;

2) $\dot{B}_p^{s,q} \subset \dot{F}_p^{s,q}$ if $q \leq p$.

The spaces $\dot{F}_p^{s,q}$ and $\dot{B}_p^{s,q}$ are independent of the choice of ψ. This is due to the following lemma [13], [14], which will be needed herein later.

LEMMA 1. — Let $s \in \mathbb{R}$, $1 \leq p, q \leq +\infty$ and $\gamma > 1$. For any sequence $(f_j)_{j \in \mathbb{Z}}$ of functions such that for each j, $\hat{f}_j$ is supported in $\{\gamma^{-1} 2^j \leq |\xi| \leq \gamma 2^j\}$, we have

$$\left\| \sum_{j \in \mathbb{Z}} f_j \right\|_{\dot{B}_p^{s,q}} \leq C \left[\sum_{j \in \mathbb{Z}} 2^{sjq} \|f_j\|_p^q \right]^{\frac{1}{q}}$$

and for $1 \leq p < +\infty$,

$$\left\| \sum_{j \in \mathbb{Z}} f_j \right\|_{\dot{F}_p^{s,q}} \leq C \left[\left\| \sum_{j \in \mathbb{Z}} 2^{sjq} |f_j|^q \right]^{\frac{1}{q}} \right\|_p$$

where $C = C(n, p, q, s, \gamma)$.

For $s > 0$ we can replace $\{\gamma^{-1}2^j \leq |\xi| \leq \gamma 2^j\}$ by the ball $\{|\xi| \leq \gamma 2^j\}$.

LEMMA 2. — *Let $s > 0$, $1 \leq p, q \leq +\infty$ and $\gamma > 1$. For any sequence $(f_j)_{j \in \mathbb{Z}}$ of functions such that for each j, $\hat{f}_j$ is supported in $\{|\xi| \leq \gamma 2^j\}$, we have*

$$\Big\| \sum_{j \in \mathbb{Z}} f_j \Big\|_{\dot{B}_p^{s,q}} \leq C \Big[\sum_{j \in \mathbb{Z}} 2^{sjq} \|f_j\|_p^q \Big]^{\frac{1}{q}}$$

where $C = C(n, p, q, s, \gamma)$.

Now we consider the boundedness of the paraproduct operator on Besov spaces. One of the most useful characterizations of the boundedness of the paraproduct on Besov spaces comes from the following.

PROPOSITION 2. — *Let $b \in \dot{B}_\infty^{s,\infty}$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ and $1 \leq p, q_1, q_2 \leq +\infty$, $q_1 \leq q_2$. Then*

1) *π_b is bounded from $\dot{B}_p^{s_1,q_1}$ to $\dot{B}_p^{s+s_1,q_2}$ if and only if there exists $C > 0$ such that*

$$\Big[\sum_{j \in \mathbb{Z}} 2^{(s+s_1)jq_2} \|\Delta_j(b) S_{j-3}(f)\|_p^{q_2} \Big]^{\frac{1}{q_2}} \leq C \|f\|_{\dot{B}_p^{s_1,q_1}},$$

for all $f \in \dot{B}_p^{s_1,q_1}$.

2) *The commutator $[\pi_b, R_k]$ is bounded from $\dot{B}_p^{s_1,q_1}$ to $\dot{B}_p^{s+s_1,q_2}$ if and only if there exists a constant $C > 0$ such that*

$$\Big[\sum_{j \in \mathbb{Z}} 2^{(s+s_1)jq_2} \|[\Delta_j(b), R_k](S_{j-3}(f))\|_p^{q_2} \Big]^{\frac{1}{q_2}} \leq C \|f\|_{\dot{B}_p^{s_1,q_1}},$$

for all $f \in \dot{B}_p^{s_1,q_1}$.

Proof. — Proposition 2 is due to the author [15] for the case $s = 0$ and $s_1 \geq 0$. To obtain extension for the general case we observe that

$$b' = \sum_{j \in \mathbb{Z}} 2^{sj} \Delta_j(b) \in \dot{B}_\infty^{0,\infty}.$$

Using this remark we are led to establish Proposition 2 for $s = 0$ and $s_1 < 0$. Indeed, by Peetre's Theorem ([9], p. 155-158) one has that, for $s_1 < 0$,

$$\Big[\sum_{j \in \mathbb{Z}} 2^{s_1 jq} \|S_{j-3}(f)\|_p^q \Big]^{\frac{1}{q}} \leq C \|f\|_{\dot{B}_p^{s_1,q}}.$$

Hence we obtain the *"only if part"*; for the *"if part"* we use the Lemma 1.

Remarks. — 1) Notice that for $s_1 < 0$ the operator π_b is bounded from $\dot{B}_p^{s_1,q_1}$ to $\dot{B}_p^{s+s_1,q_2}$ if and only if $b \in \dot{B}_\infty^{s,\infty}$.

2) In Proposition 2, one can replace R_k by polynomials of the Riesz transforms.

The first approach in dealing with the commutators consists of Lemmas 3 and 5 below. The Lemma 3 is classical and due to Calderón [3].

LEMMA 3. — *Let h be a differentiable function and suppose that $\dfrac{\partial}{\partial x_i} h \in L^\infty$ for $i = 1, ..., n$. Then the commutator $[h, R_k]$ is bounded from L^p into the Sobolev space $\dot{F}_p^{1,2}$ $(1 < p < +\infty)$ and*

$$\|[h, R_k](f)\|_{\dot{F}_p^{1,2}} \le C\Big[\sum_{i=1}^n \Big\|\frac{\partial}{\partial x_i} h\Big\|_\infty\Big] \cdot \|f\|_p.$$

The following lemma related to the Bernstein's inequality, is classical and will be used in the proof of Lemma 5 below.

LEMMA 4. — *Let $\gamma > 1$ and $1 \le p \le +\infty$, then for all $R > 0$ and each function f such that $\hat{f}$ is supported in*

$$\{\xi, R\gamma^{-1} \le |\xi| \le R\gamma\}$$

we have

$$C_1 R\|f\|_p \le \sum_{i=1}^n \Big\|\frac{\partial}{\partial x_i} f\Big\|_p \le C_2 R\|f\|_p$$

where $C_i = C(\gamma, p)$, $i = 1, 2$.

LEMMA 5. — *Let $b \in \dot{B}_\infty^{s,\infty}$, $1 < p < +\infty$, $1 \le q \le +\infty$, $0 < s < 1$, $N \in \mathbb{N}$ and $s_1 \in]-s, +\infty[$, then the operator*

$$T(f) = \sum_j [S_{j+N}(b), R_k](\Delta_j(f))$$

is bounded from $\dot{B}_p^{s_1,q}$ to $\dot{B}_p^{s+s_1,q}$.

Proof. — Since the Fourier transform of $[S_{j+N}(b), R_k](\Delta_j(f))$ is supported in the ball $\{\xi, |\xi| \le 2^{j+N+2}\}$ and $s + s_1 > 0$, then by Lemma 2, it suffices to show that

$$\Big[\sum_j 2^{(s+s_1)jq}\|[S_{j+N}(b), R_k](\Delta_j(f))\|_p^q\Big]^{\frac{1}{q}} \le C\|f\|_{\dot{B}_p^{s_1,q}}$$

and it is enough to prove that

$$\|[S_{j+N}(b), R_k](\Delta_j(f))\|_p \leq C2^{-sj}\|\Delta_j(f)\|_p.$$

But

$$S_{j+N}(b) = S_{j-3}(b) + \sum_{\ell=j-2}^{j+N} \Delta_\ell(b)$$

and

$$\|[\Delta_\ell(b), R_k](\Delta_j(f))\|_p \leq C2^{-s\ell}\|b\|_{\dot{B}_\infty^{s,\infty}}\|\Delta_j(f)\|_p.$$

Thus we are led to show that

$$\|[S_{j-3}(b), R_k](\Delta_j(f))\|_p \leq C2^{-sj}\|\Delta_j(f)\|_p.$$

To do so, we observe that the Fourier transform of $[S_{j-3}(b), R_k](\Delta_j(f))$ is supported in

$$\{2^{j-2} \leq |\xi| \leq 2^{j+2}\}.$$

By virtue of Lemma 4, we obtain

$$\left\|[S_{j-3}(b), R_k](\Delta_j(f))\right\|_p \leq C2^{-j} \sum_{i=1}^n \left\|\frac{\partial}{\partial x_i}([S_{j-3}(b), R_k](\Delta_j(f)))\right\|_p.$$

Indeed

$$\left\|\frac{\partial}{\partial x_i}(S_{j-3}(b))\right\|_\infty \leq C2^{j(1-s)}$$

so that by Lemma 3 we see that

$$\left\|\frac{\partial}{\partial x_i}([S_{j-3}(b), R_k](\Delta_j(f)))\right\|_p \leq C2^{j(1-s)}\|\Delta_j(f)\|_p.$$

This completes the proof.

We recall now the definition of the maximal function related to a sequence of measurable functions. For a sequence of measurable functions $F = (f_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^n$, the maximal function F^* is defined by

$$F^*(x) = \sup\{|f_j(y)| / |x - y| \leq 2^{-j}, j \in \mathbb{Z}\}.$$

The Carleson measure and the maximal function are related by the following classical lemma (see [1]).

LEMMA 6. — If $(\nu_j)_{j\in\mathbb{Z}}$ is a Carleson measure in $\mathbb{R}^n \times \mathbb{Z}$ and $p \in]0, +\infty[$, then

$$\sum_{j\in\mathbb{Z}} \int |f_j(x)|^p d\nu_j(x) \leq C\|\|\nu\|\| \cdot \|F^*\|_p^p.$$

4. Proof of the main results.

4.1. Proof of Theorem 1.

To prove "*part 1)*", notice that $\dot{F}_\infty^{s,p} \subset \dot{B}_\infty^{s,\infty}$. By Lemma 1 we only need show that

$$\Big[\sum_{j\in\mathbb{Z}} 2^{sjp}\|\Delta_j(b)S_{j-3}(f)\|_p^p\Big]^{\frac{1}{p}} \leq C\|f\|_p$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Denote by $F = (S_j f)_j$ and observe that $F^*(x) \leq Cf^*(x)$ where $f^*(x)$ is the Hardy-littlewood maximal function. By virtue of Lemma 6 we obtain that

$$\Big[\sum_{j\in\mathbb{Z}} 2^{sjp}\|\Delta_j(b)S_{j-3}(f)\|_p^p\Big]^{\frac{1}{p}} \leq C\|f^*\|_p \leq C\|f\|_p.$$

Next we prove "*part 2)*". Proposition 2 implies that

$$\Big[\sum_{j\in\mathbb{Z}} 2^{sjp}\|\Delta_j(b)S_{j-3}(f)\|_p^p\Big]^{\frac{1}{p}} \leq C\|f\|_{\dot{B}_p^{0,1}}.$$

Let $B = (x_0, 2^{-\ell})$ $(\ell \in \mathbb{Z})$ and let $f \in \mathcal{S}(\mathbb{R}^n)$ be such that $f(x) = 1$ for $|x| \leq 1$. Set

$$f_B(x) = f(2^\ell(x - x_0)).$$

For $j \geq \ell$ one has that

$$S_j(f_B)(x) \geq C > 0 \quad \text{for all} \quad x \in B.$$

Thus

$$\sum_{j\geq\ell} 2^{sjp}\int_B |\Delta_j(b)(x)|^p dx \leq C\|f_B\|_{\dot{B}_p^{0,1}}^p \leq C|B|,$$

from which the theorem follows.

4.2. Proof of Theorem 2.

To prove "*part 1)*", we write

$$[b, R_k](f) = [\pi_b, R_k](f) + A_k(f)$$

where

$$A_k(f) = \sum_{j\in\mathbb{Z}} [S_{j+2}(b), R_k](\Delta_j f).$$

By virtue of Lemma 5, A_k is bounded from $\dot{B}_p^{0,p}$ to $\dot{B}_p^{s,p}$; in particular, A_k is bounded from $\dot{F}_p^{0,\inf(p,2)}$ to $\dot{B}_p^{s,p}$. On the other hand, Theorem 1 guarantees that π_b is bounded from $\dot{F}_p^{0,\inf(p,2)}$ to $\dot{B}_p^{s,p}$. Since the Riesz transforms are bounded on $\dot{B}_p^{s,q}$ and on $\dot{F}_p^{s,q}$, it follows that the commutators $([b, R_k])_k$ are bounded from $\dot{F}_p^{0,\inf(p,2)}$ into $\dot{B}_p^{s,p}$.

To prove "*part 2)*", we use the following lemma.

LEMMA 7. — *Let* $0 < s < 1$, $1 < p < +\infty$ *and* b *be a locally integrable function. If the commutators* $([b, R_k])_{1 \le k \le n}$ *are bounded from* $\dot{B}_p^{0,1}$ *to* $\dot{B}_p^{s,p}$, *then* $b \in \dot{B}_\infty^{s,\infty}$.

Let us first assume Lemma 7 and prove "*part 2)*". As in [5] we shall make use of spherical harmonics. Let $(Y_m)_m$ be an orthogonal basis for the space of spherical harmonics of degree n. There exists a finite sequence $(Y_m)_m$ (see [10], p. 137–145) such that $(Y_m)_m$ are homogeneous polynomials of degree n; *i.e.* $Y_m(x)$ are of the form

$$Y_m(x) = \sum_{|\gamma|=n} C_{\gamma,m} x^\gamma$$

and satisfy

$$\sum_m \frac{(Y_m(x))^2}{|x|^{2n}} = C_n.$$

In particular the operator T_m of kernel

$$\frac{Y_m(x-y)}{|x-y|^{2n}}$$

is a polynomial in the Riesz transforms and

$$Y_m(x-y) = \sum_{|\alpha|+|\beta|=n} C_{\alpha,\beta,m} x^\alpha y^\beta.$$

Suppose that the commutators $([b, R_k])_k$ are bounded from $\dot{B}_p^{0,1}$ to $\dot{B}_p^{s,p}$. By virtue of Lemma 7, we see that $b \in \dot{B}_\infty^{s,\infty}$. Lemma 5 implies that the operator

$$A_k(f) = \sum_{j \in \mathbb{Z}} [S_{j+2}(b), R_k](\Delta_j f)$$

is bounded from $\dot{B}_p^{0,1}$ to $\dot{B}_p^{s,p}$. Hence the commutators $([\pi_b, R_k])_k$ are bounded from $\dot{B}_p^{0,1}$ to $\dot{B}_p^{s,p}$. Indeed, a little computing shows that

$$[\pi_b, R_k R_{k'}] = [\pi_b, R_k] R_{k'} + R_k [\pi_b, R_{k'}].$$

Hence $[\pi_b, T_m]$ is bounded from $\dot{B}_p^{0,1}$ to $\dot{B}_p^{s,p}$.

For $\ell \in \mathbb{Z}$ and a ball B with radius $2^{-\ell}$, we prove that b satisfies

$$\sum_{j \geq \ell} 2^{sjp} \int_B |\Delta_j(b)(x)|^p dx \leq C|B|.$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be such that $f(x) \geq C > 0$ for $|x| \leq 1$, $\hat{f}$ supported in $\left\{ |\xi| \leq \dfrac{1}{16} \right\}$ and

$$\int f(x)dx = 1.$$

For any ball $B = B(x_0, 2^{-\ell})$ ($\ell \in \mathbb{Z}$), we set

$$f_B(x) = 2^{n\ell} f(2^\ell(x - x_0)).$$

Since, for $j \geq \ell$, the Fourier transform of $\Delta_j(b)f_B$ is supported in

$$\{2^{j-2} \leq |\xi| \leq 2^{j+2}\},$$

it follows that

$$\int \Delta_j(b)(y) f_B(y) dy = 0.$$

We write

$$\Delta_j(b)(x) f_B(x) = \int (f_B(x)\Delta_j(b)(x)f_B(y) - f_B(x)\Delta_j(b)(y)f_B(y))dy$$

$$= \sum_m \int \frac{|Y_m(x-y)|^2}{|x-y|^{2n}} (f_B(x)\Delta_j(b)(x)f_B(y) - f_B(x)\Delta_j(b)(y)f_B(y))dy$$

$$= \sum_m \sum_{|\alpha|+|\beta|=n} C_{\alpha,\beta,m} \int \frac{Y_m(x-y)}{|x-y|^{2n}} (f_B^\alpha(x)\Delta_j(b)(x)f_B^\beta(y)$$

$$- f_B^\alpha(x)\Delta_j(b)(y)f_B^\beta(y))dy,$$

where $f_B^\alpha(x) = (x - x_0)^\alpha f_B(x)$. Then we get

$$\Delta_j(b)(x)f_B(x) = \sum_m \sum_{|\alpha|+|\beta|=n} C_{\alpha,\beta,m} f_B^\alpha(x) \big[\Delta_j(b)(x) T_m(f_B^\beta)(x)$$

$$- T_m(\Delta_j(b)f_B^\beta)(x) \big].$$

But

$$\int_B |\Delta_j(b)(x)|^p dx \leq C 2^{-n\ell p} \int_B |\Delta_j(b)(x) f_B(x)|^p dx.$$

Therefore,

$$\left[\int_B |\Delta_j(b)(x)|^p dx \right]^{\frac{1}{p}} \leq C 2^{-n\ell} \sum_m \sum_{|\alpha|+|\beta|=n} \|f_B^\alpha\|_\infty \|\Delta_j(b) T_m(f_B^\beta)$$

$$- T_m(\Delta_j(b)f_B^\beta)\|_p.$$

On the other hand, $S_{j-3}(f_B) = f_B$ for $j \geq \ell$, so that by Proposition 2 we see that

$$\left[\sum_{j \geq \ell} 2^{sjp} ||\Delta_j(b)T_m(f_B^\beta) - T_m(\Delta_j(b)f_B^\beta)||_p^p\right]^{\frac{1}{p}} \leq C||f_B^\beta||_{\dot{B}_p^{0,1}}.$$

Since $|\alpha| + |\beta| = n$, $||f_B^\alpha||_\infty \leq C2^{(n-|\alpha|)\ell}$ and

$$||f_B^\beta||_{\dot{B}_p^{0,1}} \leq C2^{\ell(n-|\beta|-\frac{n}{p})},$$

it follows that

$$\sum_{j \geq \ell} 2^{sjp} \int_B |\Delta_j(b)(x)|^p dx \leq C|B|.$$

This completes the proof.

4.3. Proof of Lemma 7. — We need to prove that

$$||\Delta_j(b)||_\infty \leq C2^{-sj}.$$

For $t > 0$ and $u \in \mathbb{R}^n$, we set

$$b_{t,u}(x) = b\left(u - \frac{x}{t}\right)$$

and denote by $||[b_{t,u}, R_k]||_{s,p}$ the norm of the bounded operator $[b_{t,u}, R_k]$ from $\dot{B}_p^{0,1}$ into $\dot{B}_p^{s,p}$. Next we write

$$\Delta_j(b)(x) = \; < b_{2^j,x}, \check{\psi} > .$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int f(x)dx = 1$ and write

$$\check{\psi}(y) = \int (\check{\psi}(y)f(z) - (-1)^n \check{\psi}(z)f(y))dz$$

because

$$\int \check{\psi}(z)dz = 0.$$

Using now the spherical harmonics, we obtain

$$\check{\psi} = \sum_m \sum_{|\alpha|+|\beta|=n} C_{\alpha,\beta,m} T_m(\psi^\alpha)f^\beta - (-1)^n \psi^\alpha T_m(f^\beta)$$

where $\psi^\alpha(x) = x^\alpha \check{\psi}(x)$ and $f^\beta(x) = x^\beta f(x)$. Hence

$$< b_{2^j,x}, \check{\psi} > = -(-1)^n \sum_m \sum_{|\alpha|+|\beta|=n} C_{\alpha,\beta,m} < [b_{2^j,x}, T_m](f^\beta), \psi^\alpha >,$$

showing that

$$|< b_{2^j,x}, \check{\psi} >| \leq C \sum_m \sum_{|\alpha|+|\beta|=n} |C_{\alpha,\beta,m}| ||[b_{2^j,x}, T_m]||_{s,p} ||f^\beta||_{\dot{B}_p^{0,1}} ||\psi^\alpha||_{\dot{B}_{p'}^{-s,p'}}.$$

To complete the proof it is enough to notice that

$$||[b_{2^j,x}, T_m]||_{s,p} \leq 2^{-sj} ||[b, T_m]||_{s,p}.$$

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