TODOR GRAMCHEV
GEORGI POPOV

Nekhoroshev type estimates for billiard ball maps

<http://www.numdam.org/item?id=AIF_1995__45_3_859_0>
NEKHOROSHEV TYPE ESTIMATES
FOR BILLIARD BALL MAPS

by T. GRAMCHEV(1) and G. POPOV(2)

1. Introduction.

This paper is concerned with the effective stability of the billiard flow near the boundary of a strictly convex bounded domain $\Omega$ in $\mathbb{R}^{n+1}$, $n \geq 1$, with an analytic boundary $\partial \Omega$. The billiard flow in $\Omega$, called as well generalized geodesic flow, is described by a particle moving with unit velocity along straight lines inside $\Omega$ and reflecting at the boundary by the law of the geometric optics "angle of reflection equals angle of incidence". The boundary is invariant with respect to the billiard flow, and if the particle is on it with velocity tangent to $\partial \Omega$, then it travels with unit speed along the corresponding geodesic of $\partial \Omega$. We are interested in effective stability estimates for the billiard flow near the boundary, i.e. in stability for finite but exponentially long time intervals.

The billiard flow induces a discrete dynamical system at the boundary described by the billiard ball map

$$B : J \to J, \quad J = \{(x, \xi) \in T^*\mathbb{R}^{n+1} : x \in \partial \Omega, \, |\xi| = 1\},$$

(see Sect. 2) which is easier to deal with. The map $B$ is analytic in $J$ and it coincides with the identity mapping on the glancing manifold

(1) Partially supported by funds MURST 40 %, Italy, by GNAFA of the CNR, Italy and by grant MM-48/91 with MES, Bulgaria.
(2) Supported by the Alexander von Humboldt Foundation and partially supported by grant MM-8/91 with MES, Bulgaria.

Key words : Billiard ball map – Nekhoroshev type estimate – Glancing hypersurfaces.
Math. classification : 58F.
\[ K = \{(x, \xi) \in J : \langle \xi, n(x) \rangle = 0 \}, \]
\( n(x) \) being the inward unit normal to \( \partial \Omega \) at \( x \).

The main problem we are concerned with, is to find an upper bound for the rate of diffusion of the iterates

\[ B^j(\varrho), \quad j = 1, 2, \ldots, \]

from the glancing manifold \( K \) for \( \varrho \) close to \( K \). The plane case is trivial, because there is a large set of smooth invariant curves of \( B \) accumulating at \( K \) [16], [21], which trap any orbit starting close to \( K \) inside an annulus. The corresponding caustics (the envelope of the rays issued from the invariant circles) are smooth and strictly convex curves situated an a neighborhood of \( \partial \Omega \) inside \( \Omega \) [16]. The picture is completely different when \( n \geq 2 \). It was shown by Berger [4] that if \( n = 2 \) and \( \partial \Omega \) admits a smooth caustic inside \( \Omega \), then \( \Omega \) is an ellipsoid. Generically, even if there exist invariant tori of \( B \) in \( J \), they are at least of codimension two, and the orbits could escape any fixed neighborhood of \( K \).

The first effective stability result for perturbations of a completely integrable Hamiltonian was obtained by Nekhoroshev. The Nekhoroshev theorem [22], [23], states that the variation of the action of each orbit of an analytic Hamiltonian \( H_0 \), close to a completely integrable one \( H_0 \), remains stable in a finite but exponentially large time interval

\[ 0 \leq t \leq T \exp(t^{-\alpha}), \]

if \( H_0 \) satisfies certain generic steepness conditions. Since then a number of new results about effective stability have appeared. Sharper estimates on the stability exponent \( \alpha \) have been proved recently by Benettin, Gallavotti, Galgani, Giorgilli and others in the convex case (\( H_0 \) is convex) in order to investigate stability problems in Celestial and Statistical mechanics [2], [3], [9]. A new approach to the effective stability of convex Hamiltonians, based on an analysis near the worst resonances of the system, has been recently proposed by Lochak [17]. The best exponent \( \alpha = 1/2n \) in the quasi-convex case was found by Lochak and Neishtadt [18] and by Pöschel [25] who added new geometric ideas to the traditional proof. Exponential stability for time dependent potentials has been proved by Giorgilli and Zehnder [10].

Effective stability results for iterates of a symplectic mapping \( P_\varrho \) close to a completely integrable one \( P_0 \) have been proved by Kuksin and Pöschel
[15]. The main idea there, is to write $P_\varepsilon$ as a time-one-shift of the flow of a 1-periodic analytic Hamiltonian $H_\varepsilon$ which is $\varepsilon$-close to a completely integrable one. Bazzani, Marmi and Turchetti have obtained Nekhoroshev estimates for isochronous non resonant symplectic maps [1].

The main difference between this paper and the results cited above is that the billiard ball map is in general far from a completely integrable one for $n \geq 2$. Nevertheless, there exists a smooth "approximate" first integral $\zeta$ of $B$ defining $K$, which means that $\zeta \in C^\infty(J)$, $\zeta = 0$ and $d\zeta \neq 0$ on $K$, and

\begin{equation}
\zeta(B(x,\xi)) = \zeta(x,\xi) + r(x,\xi), \quad (x,\xi) \in J,
\end{equation}

where the function $r \in C^\infty(J)$ is flat at $K$. Hereafter, we say that a function $r \in C^\infty(J)$ is flat at $K$, if $r$ has a zero of infinite order at $K$, i.e. for any smooth vector field $Y$ in $J$ and any integer $k \geq 0$ the function $Y^k r$ vanishes at $K$. A natural candidate for $\zeta$ is the approximate interpolating Hamiltonian of $B$ which has been introduced in a little bit different context for the corresponding boundary maps $\delta^\pm$ by Marvizi and Melrose in [19] (see also [12]). In our case $J$ is equipped with an analytic two-form $\omega_0$ given by the pull-back of the canonical two-form in $T^*\mathbb{R}^{n+1}$ via the natural inclusion map $J \to T^*\mathbb{R}^{n+1}$. Note that the two-form $\omega_0$ is symplectic in $J \setminus K$ but it is degenerate at $K$. More precisely, taking $n-1$ times the exterior product of $\omega_0$ by itself we get

$$
\omega_0^n = \langle \xi, n(x) \rangle dv, \quad (x,\xi) \in J,
$$

where $dv$ is a volume form in $J$ and $d\langle \xi, n(x) \rangle \neq 0$ at $K$ since $\partial \Omega$ is strictly convex (see Sect. 2). In particular, for any $\phi \in C^\infty(J)$ satisfying $d\phi(\emptyset) = 0$, $\forall \emptyset \in K$, the Hamiltonian vector field $H_\phi$ of $\phi$ with respect to $\omega_0$ is well defined by the inner product $\iota(H_\phi)\omega_0 = -d\phi$, and $H_\phi$ is smooth in $J$. We will call $\zeta \in C^\infty(J)$ an approximate interpolating Hamiltonian of $B$ if $\zeta = 0$ and $d\zeta \neq 0$ on $K$, and for any $f \in C^\infty(J)$ the function

\begin{equation}
R(\emptyset) = f(B(\emptyset)) - f \left( \exp \left( \zeta(\emptyset) H_{\zeta^2} \right) (\emptyset) \right), \quad \emptyset \in J,
\end{equation}

is flat at $K$. Here $t \to \exp(tH_{\zeta^2})$ stands for the one-parameter group of diffeomorphisms of the smooth vector field $H_{\zeta^2}$ of $\zeta^2$ with respect to $\omega_0$. Making use of the normal forms of glancing hypersurfaces obtained by Melrose [20], one can find an approximate interpolating Hamiltonian $\zeta$ of $B$ as in [12] and [19].
The main goal in this paper is to prove the existence of an approximate interpolating Hamiltonian \( \zeta \) of \( B \) of Gevrey class \( G^2(J) \) (for a definition of Gevrey classes see Sect. 2) provided that the boundary \( \partial \Omega \) is analytic and strictly convex. Then the function \( R \) in (1.2), respectively \( r \) in (1.1), will be in \( G^2 \) as well, and since \( r \) is flat at \( K \) we will get the estimate

\[
|r(x, \xi)| \leq C_1 \exp \left( -\frac{C}{|\zeta(x, \xi)|} \right), \quad (x, \xi) \notin K,
\]

with some \( C, C_1 > 0 \), which leads to effective stability at the boundary. Note that \( \zeta(x, \xi) \) measures the "distance" in \( J \) from a given point \( (x, \xi) \in J \) to the glancing manifold \( K \).

For any \( \varrho \in J \) and any positive integer \( k \) we denote by \( T_k(\varrho) \) the length of the broken geodesic arc issuing from \( \varrho \) and having \( k \) points of reflection at \( \partial \Omega \), i.e.

\[
T_k(\varrho) = \sum_{j=0}^{k-1} |x_j^{j+1} - x_j^j|,
\]

where \( \varrho = (x^0, \xi^0), \quad x^j = (x_j, \xi_j) = B^j(\varrho), \quad j = 1, \ldots, k, \) and \( |x_j^{j+1} - x_j^j| \) stands for the usual distance in \( \mathbb{R}^{n+1} \).

Our main result is:

**Theorem 1.** — Let \( \Omega \) be a strictly convex bounded domain in \( \mathbb{R}^{n+1}, \quad n \geq 1, \) with an analytic boundary. Then there exists an approximate interpolating Hamiltonian \( \zeta \in G^2(J) \) of the billiard ball map \( B \). Moreover, there exist positive constants \( \delta \) and \( C \) such that for any \( 0 < \varepsilon \leq 1, \) and any \( \varrho = (x, \xi) \in J, \) \( 0 < |\zeta(\varrho)| < \delta, \) we have

\[
|\zeta(B^k(\varrho)) - \zeta(\varrho)| < \varepsilon \zeta(\varrho)^2
\]

provided that

\[
0 \leq T_k(\varrho) \leq \varepsilon |\zeta(\varrho)|^5 \exp(C|\zeta(\varrho)|^{-1}).
\]

More generally, we prove in Section 2 that Theorem 1 holds for the billiard ball map associated with any pair of analytic glancing hypersurfaces having a compact glancing manifold \( K \). In particular, we obtain effective stability estimates near \( K \) for the billiard ball map of any compact real-analytic Riemannian manifold whose boundary is strictly geodesically
convex. The main idea in the proof is to find simultaneously a local normal form

\( \sigma_0 = 2\xi_1 d\xi_1 \wedge dx_1 + \sum_{j=2}^{n} d\xi_j \wedge dx_j \) \hspace{1cm} (1.3)

for \( \omega_0 \) and an approximate local normal form for the billiard ball map in \( G^2 \)

\( B(x, \xi) = (x_1 + \xi_1, x_2, \ldots, x_n, \xi) + R(x, \xi), \) \hspace{1cm} (1.4)

where \( K = \{ \xi_1 = 0 \} \), the function \( R \) belongs to the Gevrey class \( G^2 \) and it is flat at \( K \). When the boundary is strictly convex and \( C^\infty \) smooth, Melrose [20] has found smooth local coordinates \((x, \xi)\) such that (1.3) and (1.4) hold with \( R \equiv 0 \). In particular, the billiard ball maps of any two strictly convex domains with smooth boundaries are locally equivalent to each other in the \( C^\infty \) category. More generally, Melrose proved that any two pairs of glancing hypersurfaces are locally symplectically equivalent in the \( C^\infty \) case. As it was observed by Oshima [24], this is not true in the analytic case. In fact, the example of Oshima shows even that there exist pairs of analytic glancing hypersurfaces which are not locally symplectically equivalent in the Gevrey classes \( G^s \), \( 1 < s < 2 \), (see Remark 2.4). Oshima’s example suggests that \( s = 2 \) is the best Gevrey regularity for the approximate interpolating Hamiltonian one can hope for.

The existence of the normal forms (1.3) and (1.4) of \( \omega_0 \) and \( B \) is influenced by the construction of the normal forms in [11] and [20]. The novelty in this paper is, that we analyse rather precisely the formal power series arising in the traditional proof. These series do not converge in the usual sense but they do converge in suitable “Gevrey” spaces of formal series. Similar idea (to estimate the rate of divergence of formal series arising in normal forms) has been used in [9] to study the effective stability for a hamiltonian system in the vicinity of an elliptic equilibrium point. Our approach is different from those in [9], it is based on certain techniques which come from the calculus in Gevrey classes (see [5], [6], [13], [26]). The relationship of the Gevrey classes with this type of problems was pointed out by Lochak [17] as well. We would like to mention that our method could be used to treat Gevrey normal forms for the billiard ball maps of pairs of non analytic Gevrey glancing hypersurfaces as well.

The paper is organized as follows: In Sect. 2 we consider pairs of analytic glancing hypersurfaces \( F, G \), in an analytic symplectic manifold
M and the corresponding analytic involutions $J_F$ and $J_G$ in $J = F \cap G$ having a common fixed point manifold $K$. The billiard ball map is given by $B = J_F \circ J_G$, and the analytic two-form $\omega_0$ is invariant with respect to both the involutions and it is degenerate at $K$. Theorem 1 follows from Theorem 2.3 which provides simultaneously a $G^2$ normal form for the two-form $\omega_0$ and the involution $J_F$ and gives at the same time an approximate normal form of $J_G$ modulo an error term of Gevrey class $G^2$ which is flat at $K$. Theorem 2.3 is proved in two steps. First we obtain an approximate normal form for the pair of involutions in $G^2$ (see Theorem 3.1) paying no attention to the form $\omega_0$. To put $\omega_0$ in a normal form keeping fixed the normal forms of the pair of involutions given by Theorem 3.1 one could adapt the proof of Theorem 21.4.4 in [11]. Instead, we give a new proof which is based on the deformation argument of Moser-Weinstein, exploring the invariance of the two-form $\omega_0$ under the involutions. Theorem 3.1 is proved in Sect. 4. The Appendix contains technical lemmas concerning some estimates in Gevrey classes.

Using the results of this paper we can show that the billiard ball maps of any two strictly convex domains with analytic boundaries are $G^3$ equivalent to each other. This represents a loss of Gevrey regularity with respect to the approximate interpolating Hamiltonian of $B$ which has $G^2$ Gevrey regularity. The corresponding result is a subject of another paper [8].

2. Glancing hypersurfaces.

The billiard ball map in a strictly convex bounded domain can be associated to a pair of transversally intersecting glancing hypersurfaces. First we recall certain facts about pairs of glancing hypersurfaces which can be found in [11], [20]. First we consider a strictly convex domain $\Omega$ in $\mathbb{R}^{n+1}$ with a real analytic boundary $\partial \Omega$, $n \geq 1$. Denote by

$$\omega = d\xi_1 \wedge dx_1 + \ldots + d\xi_{n+1} \wedge dx_{n+1}$$

the canonical two-form in $T^*\mathbb{R}^{n+1}$. Let $f$ be a real analytic function in $\mathbb{R}^{n+1}$ such that $f(x) = 0$, $df(x) \neq 0$, on $\partial \Omega$, and $f(x) > 0$ inside $\Omega$. Set $f(x, \xi) = f(x)$ and $g(x, \xi) = 1 - |\xi|^2$, and denote by $X_f$ and $X_g$ the corresponding Hamiltonian vector fields with respect to $\omega$. Consider the pair
of analytic hypersurfaces

\[ F = \{(x, \xi) \in T^*\mathbb{R}^{n+1} : f(x, \xi) = 0\}, \quad G = \{(x, \xi) \in T^*\mathbb{R}^{n+1} : g(x, \xi) = 0\}. \]

Since \( \partial \Omega \) is strictly convex, \( F \) and \( G \) form a pair of transversal glancing hypersurfaces \([11], [20]\), i.e. \( F \) and \( G \) intersect transversally at \( J = F \cap G \), and for any \( \varrho \in J \), either the integral curve of \( X_f \) \( (X_g) \) passing through \( \varrho \) intersects \( G \) \( (F) \) transversally or it is simply tangent to \( G \) \( (F) \). More precisely, the equality

\[ \{f, g\}(\varrho) = 0, \quad \varrho = (x, \xi) \in J, \]

(i.e. \( (\xi, n(x)) = 0 \)) implies

\[ \{f, \{f, g\}\}(\varrho) < 0, \quad \{g, \{g, f\}\} < 0. \]

Indeed, the first term in (2.2) is equal to \(-2|f_x(x)|^2 < 0\) and the second one is \(4(f_{xx}(x)\xi, \xi) < 0\). Here \( \{\cdot, \cdot\} \) stands for the Poisson brackets in \( T^*\mathbb{R}^{n+1} \) corresponding to \( \omega \) while \( \langle \cdot, \cdot \rangle \) denotes the Euclidean scalar product in \( \mathbb{R}^{n+1} \). Because of (2.2), the glancing set

\[ K = \{f = g = \{f, g\} = 0\} \]

is a submanifold of \( J \). Indeed, the differentials \( df, dg \), and \( d\{f, g\} \) are linearly independent at any \( \varrho \in K \), since \( df \) and \( dg \) are linearly independent at \( \varrho \) and

\[ df(X_f) = dg(X_f) = 0, \quad d\{f, g\}(X_f) \neq 0, \]

at \( \varrho \) according to (2.2). There are two analytic involutions

\[ J_F : J \rightarrow J, \quad J_G : J \rightarrow J, \]

defined as follows. For any \( \varrho \) in \( J \) outside \( K \), \( J_F(\varrho) \) \( (J_G(\varrho)) \) is the second point of intersection with \( J \) of the integral curve of \( X_f \) \( (X_g) \) passing through \( \varrho \), and \( J_F \) \( (J_G) \) coincides with the identity mapping on the glancing manifold \( K \). The billiard ball map \( B \) is defined by

\[ B = J_F \circ J_G. \]

Note that \( J \) is a manifold of dimension \( 2n \) equipped with a two-form \( \omega_0 \) which is the pull-back of \( \omega \) via the natural inclusion mapping \( J \rightarrow T^*\mathbb{R}^{n+1} \). The two-form \( \omega_0 \) is symplectic in \( J \setminus K \) but it is degenerate on \( K \), indeed, in view of Lemma 21.4.7 in \([11]\) we have

\[ \omega_0^n = m \nu, \]

where \( m \) is an integer and \( \nu \) is a volume form.
where \( v \) is a volume form in \( J \), \( m(x, \xi) = \{f, g\}(x, \xi) \), and the differential of \( m(\varrho) \) does not vanish on \( K \) according to (2.2). Moreover, \( \omega_0 \) is invariant with respect to the involutions \( J_F \) and \( J_G \). The set of fixed points of both \( J_F \) and \( J_G \) coincides with \( K \) and their differentials are linearly independent at any point of \( K \).

More generally, we consider an analytic symplectic manifold \((M, \omega)\) where \( M \) is an analytic manifold of dimension \( 2n + 2 \), \( n \geq 1 \), and \( \omega \) is an analytic symplectic two-form on it. Denote by \( \{\cdot, \cdot\} \) the Poisson brackets in \( M \) corresponding to \( \omega \). We consider a pair of analytic transversal glancing hypersurfaces \( F, G, \) in \( M \) of the form

\[
(2.5) \quad F = \{\varrho \in M : f(\varrho) = 0\}, \quad G = \{\varrho \in M : g(\varrho) = 0\},
\]

where \( f \) and \( g \) are smooth functions in \( M \) and analytic in a neighborhood of \( J = F \cap G \), such that \( df \neq 0 \) on \( F \), \( dg \neq 0 \) on \( G \), and the differentials \( df \) and \( dg \) are linearly independent at \( J \). As above we suppose that equality (2.1) implies (2.2) and we denote by \( \omega_0 \) the pull-back of \( \omega \) via the inclusion map \( J \to M \). Moreover, we assume that the glancing manifold \( K = \{\varrho \in J : \{f, g\}(\varrho) = 0\} \) is compact. It is easy to see that the involutions \( J_F \) and \( J_G \) are well-defined and analytic in a neighborhood \( J_0 \) of \( K \) in \( J \), and we define \( B : J_0 \to J \) by (2.3). Moreover, the function \( t(\varrho) \), \( \varrho \in J_0 \), defined by

\[
\exp \left( t(\varrho)X_{g} \right) (\varrho) = J_G(\varrho)
\]

is analytic in \( J_0 \) in view of the implicit function theorem (see (2.10)), and we have \( \pm t(\varrho) > 0 \) in

\[
J_\pm = \{\varrho \in J_0 : \pm\{g, f\}(\varrho) > 0\}.
\]

On the other hand, for any positive integer \( k \) there is a neighborhood \( U \) of \( K \) in \( J_0 \) such that \( B^j(J_\pm \cap U) \subset J_\pm \) for every \( 0 \leq j \leq k \). Then we set \( T_0(\varrho) = 0 \), and define

\[
T_j(\varrho) = \sum_{p=0}^{j-1} t(B^p(\varrho)), \quad \varrho \in J_\pm \cap U, \quad 1 \leq j \leq k.
\]

The broken bicharacteristic \( \Phi^t(\varrho) \) of \( G \) issuing from a point \( \varrho \) in \( J_\pm \cap U \) and propagating for \( \pm \varrho \in [0, T_k+1(\varrho)) \) in the domain

\[
G_0 = \{(x, \xi) \in G : f(x, \xi) \geq 0\}
\]
will be defined as follows:

$$\Phi^t(\phi) = \exp(tX_\phi(B^j(\phi))), \quad T_j(\phi) \leq \pm t < T_{j+1}(\phi), \quad 0 \leq j \leq k.$$ 

Let us take for example $M = T^*\mathbb{R}^{n+1}$ equipped with the standard symplect two-form $\omega$. Let $f = f(x)$ be independent of $\xi$, and assume that the domain

$$\Omega = \{x \in \mathbb{R}^{n+1} : f(x) \geq 0\}$$

has a compact boundary. Suppose that $g$ has the form

$$g(x, \xi) = E - H(x, \xi) - V(x)$$

in a neighborhood of $J$ where $H(x, \xi) = \sum_{i,j=1}^{n+1} g^{ij}(x)\xi_i\xi_j$ is the Hamiltonian corresponding to an analytic Riemannian metric in a neighborhood of $\partial\Omega$. Then the first inequality in (2.2) is equivalent to

$$\sum_{i,j=1}^{n+1} g^{ij}(x)N_i(x)N_j(x) > 0, \quad \forall x \in \partial\Omega,$$

where $N(x) = \text{grad} f(x)$. On the other hand, if $V = 0$ and $E > 0$, the second inequality in (2.2) means that $\Omega$ is strictly geodesically convex with respect to the geodesic flow associated to the Hamiltonian $g$.

Before formulating the main results we recall certain basic facts about the Gevrey classes. If $X$ is an open domain in $\mathbb{R}^n$ and $\sigma \geq 1$, we denote by $G^\sigma(X)$ the space of all Gevrey functions in $X$ of index $\sigma$, namely $f \in G^\sigma(X)$ iff $f \in C^\infty(X)$ and for every compact subset $Y$ of $X$ there exists $C > 0$ such that

$$\sup_{x \in Y} |\partial^\alpha_x f(x)| \leq C^{(|\alpha|+1)}(\alpha!)^\sigma, \quad \alpha \in \mathbb{Z}_+^n.$$ 

Evidently $G^1(X)$ coincides with the space of all real analytic functions in $X$, while for $\sigma > 1$ there are nonzero compactly supported $G^\sigma$ functions, namely $G^\sigma(X) \cap C^\infty_0(X) \neq \{0\}$. We point out that that for any $G^\sigma$ function, $\sigma > 1$, which is flat at the hypersurface $x_1 = 0$, and any compact $Y$, there exist two constants $C$ and $c$ such that for every $\alpha \in \mathbb{Z}_+^n$ the following estimate holds:

$$|\partial_2^\alpha f(x)| \leq C^{(|\alpha|+1)}(\alpha!)^\sigma \exp \left(-c|x_1|^{-1/(\sigma-1)}\right), \quad x_1 \neq 0, x \in Y.$$ 

For more details on Gevrey classes we refer to [17] and [26].
We formulate now a general result which yields in particular effective stability estimates for the billiard ball map of a real-analytic manifold with a strictly geodesically convex boundary.

**Theorem 2.1.** — Let $F, G$ be a pair of transversal glancing hypersurfaces in an analytic symplectic manifold $(M, \omega)$. Suppose that $F$ and $G$ are analytic in a neighborhood of the glancing manifold $K$ and assume that $K$ is compact. Then there exists an approximate interpolating Hamiltonian $\zeta$ of Gevrey class $G^2$ for the billiard ball map $B$ such that $\pm \zeta > 0$ in $J_{\pm}$. Moreover, there are positive constants $\delta$ and $C$ such that for any $0 < \varepsilon \leq 1$ and any $\varrho \in J$ with $0 < |\zeta(\varrho)| < \delta$, we have

$$|\zeta(B^k(\varrho)) - \zeta(\varrho)| < \varepsilon \zeta(\varrho)^2,$$

provided that

$$0 \leq T_k(\varrho) \leq \varepsilon |\zeta(\varrho)|^5 \exp(C|\zeta(\varrho)|^{-1}).$$

For any $\varrho \in G$ we define the "distance" to the gliding manifold $K$ by

$$d(\varrho) = |f(\varrho)| + |\{f, g\}(\varrho)|.$$

As a consequence of Theorem 2.1 we obtain

**Theorem 2.2.** — Suppose that the assumptions of Theorem 2.1 hold. Then there exist positive constants $\delta$ and $C$, and some $0 < C_1 < 1 < C_2$ such that for any $\varrho \in J_{\pm}$ with $d(\varrho) < \delta$ we have

$$C_1 d(\varrho) \leq d(\Phi^t(\varrho)) \leq C_2 d(\varrho),$$

provided that

$$0 \leq \pm t \leq d(\varrho)^5 \exp \left( \frac{C}{d(\varrho)} \right).$$

The results formulated above are based on an approximate normal form for a pair of symplectic involutions.

**Theorem 2.3.** — Let $J$ be an analytic manifold of dimension $2n$ and let $\omega_0$ be a closed analytic two-form on it satisfying (2.4) where $dm \neq 0$ on $K = \{ \varrho \in J : m(\varrho) = 0 \}$. Let $J_j, j = 1, 2$, be a pair of analytic involutions such that $J_j^* \omega_0 = \omega_0$. Suppose that the sets of fixed points of
$J_j, \ j = 1, 2,$ coincide with $K$, and that $dJ_j$ are linearly independent over $K$. Then for any $\rho_0 \in K$ there exists a diffeomorphism $\chi$ of Gevrey class $G^2$ mapping a neighborhood of the origin $W$ in $\mathbb{R}^{2n}$ to a neighborhood of $\phi_0$ in $J$, $\chi(0) = \phi_0$, such that

$$\chi^*(\omega_0) = \sigma_0 \equiv 2\xi_1 d\xi_1 \wedge dx_1 + \sum_{j=2}^n d\xi_j \wedge dx_j,$$

while the involutions $J^0_j = \chi^{-1} \circ J_j \circ \chi, \ j = 1, 2,$ become

$$J^0_1(x, \xi) = (x, -\xi_1, \xi'),$$

$$J^0_2(x, \xi) = (x_1 + \xi_1, x', -\xi_1, \xi') + R(x, \xi), \ (x, \xi) \in W,$$

where $R$ belongs to $G^2(W)$ and it is flat at $\{(x, \xi) \in W : \xi_1 = 0\}$.

**Remark 2.4.** — Making use of the example of Oshima of analytic glancing hypersurfaces which are not analytically equivalent (see (7), [24]), one can show that the Gevrey regularity $G^2$ in the theorem above is optimal. Indeed, the estimates in [24], p. 57, can be used to prove that the pair of analytic involutions associated to the glancing hypersurfaces (7) are not locally symplectically equivalent to the pair $J^0_1, J^0_2$, in any Gevrey class $G^s, 1 \leq s < 2$.

**Proof of Theorem 2.1.** — The first statement in Theorem 2.1 follows from Theorem 2.3 taking $J_1 = J_F, J_2 = J_G$, and then choosing local coordinates $(x(\rho), \xi(\rho)) = \chi^{-1}(\rho) \in W$ where $\rho \in U = \chi(W)$. Then we have

$$(\chi^{-1} \circ B \circ \chi)(x, \xi) = (x_1 + \xi_1, x', \xi') + R(x, \xi)$$

$$= \exp \left(\xi_1 H_{\xi_1^2}\right)(x, \xi) + R(x, \xi), \ (x, \xi) \in W,$$

where $H_{\xi_1^2}$ is the Hamiltonian vector field of $\xi_1^2$ with respect to $\sigma_0$, $R$ belongs to $G^2(W)$, and $R \approx 0$. Hereafter, we say that $R \approx 0$ if $R$ is flat at $\{\xi_1 = 0\}$. Since $\chi^*(\omega_0) = \sigma_0$, the $G^2$ function $\zeta(\rho) = \xi_1(\rho)$ is an approximate interpolating Hamiltonian of $B$ in $U$. Moreover, changing eventually $\xi_1$ with $-\xi_1$ we can assume that $\zeta > 0$ in $J_+$. We are going to show that $\zeta$ is uniquely determined in $U$ modulo a flat function.

Let $\chi_0 : W_0 \rightarrow U$ be another $G^2$ diffeomorphism given by Theorem 2.3. Denote by $(y(\rho), \eta(\rho)) = \chi_0^{-1}(\rho) \in W_0$ the corresponding $G^2$ local coordinates in $U$, and set $\zeta_0(\rho) = \eta_1(\rho)$ where $\zeta_0 > 0$ in $J_+$. 

LEMMA 2.5. — The function $\zeta_0(\varphi) - \zeta(\varphi)$ is flat at $K$.

Proof. Set $\psi = \chi_0^{-1} \circ \chi : W \to W_0$ and denote $\phi(x, \xi) = \zeta_0(\chi(x, \xi)) = (\eta_1 \circ \psi)(x, \xi)$. Then $\pm \phi(x, \xi) > 0$ in $\{(x, \xi) \in W : \pm \xi_1 > 0\}$. We are going to show that $\phi(x, \xi) - \xi_1 \approx 0$. Set
$$B_0(x, \xi) = (x_1 + \xi_1, x', \xi).$$
Then we have $\psi^{-1} \circ B_0 \circ \psi - B_0 \approx 0$, hence,
$$\exp(\phi H_{\varphi^2}) - B_0 = \psi^{-1} \circ \exp(\eta_1 H_{\eta_1^2}) \circ \psi - B_0 \approx 0,$$
and for any smooth function $f$ and any integer $N \geq 1$ we obtain the equality
$$f(x_1 + \xi_1, x', \xi) \approx f(\exp(\phi H_{\varphi^2})(x, \xi))$$
$$= f(x, \xi) + \sum_{k=1}^{N} \frac{\phi(x, \xi)^k}{k!} H_{\varphi^2} f(x, \xi) + O(|\phi(x, \xi)|^{N+1}).$$
Set
$$H_{\varphi^2} = \sum_{j=1}^{n} \left( \alpha_j(x, \xi) \frac{\partial}{\partial x_j} + \beta_j(x, \xi) \frac{\partial}{\partial \xi_j} \right).$$
Taking $f(x, \xi) = \xi_j$, $1 \leq j \leq n$, we get $\beta_j \approx 0$. In the same way we obtain $\alpha_1 \approx 1$ and $\alpha_j \approx 0$ for $2 \leq j \leq n$. Hence, $d\phi^2 = -\imath(H_{\varphi^2})\sigma_0 \approx d\xi_1^2$, and we obtain $\phi \approx \xi_1$ since $\phi$ and $\xi_1$ have the same sign. \hfill \Box

Patching together in $G^2$ the local approximate interpolating Hamiltonians obtained above, we get an approximate interpolating Hamiltonian $\zeta$ of $B$ in a neighborhood of $K$ of the Gevrey class $G^2$.

We are going to prove the second statement in Theorem 2.1.

LEMMA 2.6. — There exist positive constants $C$, $C_1$, and $\delta$ such that
$$|\zeta(B^j(\varphi)) - \zeta(\varphi)| \leq 2C_1 j \exp(-C|\zeta(\varphi)|^{-1}) \leq 2\varepsilon C_1 |\zeta(\varphi)|^3$$
for any $0 < |\zeta(\varphi)| \leq \delta$ and $0 < \varepsilon \leq 1$ provided that
$$0 \leq j \leq \varepsilon |\zeta(\varphi)|^3 \exp(C|\zeta(\varphi)|^{-1}).$$
Proof. — Taking \( f = \zeta \) in (1.2), and using the equality
\[
\zeta(\exp(\zeta H^2)(\vartheta)) = \zeta(\vartheta),
\]
we show that the \( C^2 \) function \( \zeta(B(\vartheta)) - \zeta(\vartheta) \) is flat at \( K \). Hence, there exist positive constants \( C, C_1, \) and \( \delta \) such that the estimate
\[
(2.8) \quad |\zeta(B(\vartheta)) - \zeta(\vartheta)| \leq C_1 \exp(-C|\zeta(\vartheta)|^{-1})
\]
holds if \( 0 < |\zeta(\vartheta)| \leq 3\delta \). We have shown that (2.8) is valid for \( j = 1 \). Fix \( j \) such that (2.7) holds. Suppose that (2.6) is valid for some \( 1 \leq k < j \). Then the inductive assumption implies
\[
(2.9) \quad |\zeta(B^k(\vartheta)) - \zeta(\vartheta)| \leq 2C_1 k \exp(-C|\zeta(\vartheta)|^{-1}) \leq 2\varepsilon C_1 |\zeta(\vartheta)|^3 < \delta,
\]
if \( \delta^2 < (2C_1)^{-1} \). Now using (2.8) we get
\[
|\zeta(B^{k+1}(\vartheta)) - \zeta(\vartheta)| \leq |\zeta(B^{k+1}(\vartheta)) - \zeta(B^k(\vartheta))| + 2C_1 k \exp(-C|\zeta(\vartheta)|^{-1}) \\
\leq C_1 \exp(-C|\zeta(B^k(\vartheta))|^{-1}) + 2C_1 k \exp(-C|\zeta(\vartheta)|^{-1}).
\]
On the other hand, for \( 0 < \delta < (4C_1)^{-1} \) estimate (2.9) implies
\[
|\zeta(B^k(\vartheta))|^{-1} \geq |\zeta(\vartheta)|^{-1} - 4\varepsilon C_1 |\zeta(\vartheta)|,
\]
which yields for \( \delta < (4CC_1)^{-1} \ln 2 \) the inequality
\[
\exp(-C|\zeta(B^k(\vartheta))|^{-1}) \leq 2\exp(-C|\zeta(\vartheta)|^{-1}),
\]
and using (2.7) we get (2.6) for \( B^{k+1}(\vartheta) \). The proof of the lemma is complete.

Lemma 2.7. — There exists \( a \in C^\infty(J), \ a > 0, \) such that
\[
t(\vartheta) = a(\vartheta)\zeta(\vartheta)
\]
in a neighborhood of \( K \) in \( J_0 \).

Proof. — The manifold \( K \) is defined in \( J \) by \( \{ g, f \}(\vartheta) = 0, \ (d\{ g, f \} \neq 0 \) on \( K \) and we have \( \zeta(\vartheta) = 0, \ d\zeta(\vartheta) \neq 0, \ \vartheta \in K \). Hence,
\[
\{ g, f \}(\vartheta) = a_1(\vartheta)\zeta(\vartheta), \ \vartheta \in J_0,
\]
and we can suppose that \( a_1 > 0 \) since \( \zeta > 0 \) in \( J_+ \). Applying Taylor’s formula to
\[
f(\exp(t(\vartheta)X_g)(\vartheta)) = f(\vartheta) = 0, \ \vartheta \in J_0,
\]
we obtain

\[(2.10) \quad \{g, f\}(q) + \frac{t(q)}{2} \{g, \{g, f\}\}(q) + O(t(q)^2) = 0,\]

which proves the assertion.

**Lemma 2.8.** — There exists \( \delta > 0 \) such that (2.7) holds if \( 0 < \varepsilon \leq 1, \)

\[0 < |\zeta(q)| \leq \delta, \text{ and}\]

\[(2.11) \quad T_j(q) \leq \varepsilon |\zeta(q)|^5 \exp(C|\zeta(q)|^{-1}).\]

**Proof.** — Let \( 0 < k < j \) and suppose that \( T_j \) satisfies (2.11) and

\[k \leq \varepsilon |\zeta(q)|^3 \exp(C|\zeta(q)|^{-1}).\]

Clearly Lemma 2.6 and Lemma 2.7 lead to

\[T_{k+1}(q) = \sum_{p=0}^{k} |t(B^p(q))| \geq C_0 \sum_{p=0}^{k} |\zeta(B^p(q))| \geq (k + 1)C_0 |\zeta(q)| \left(1 - 2C_1\delta^2\right) \geq (k + 1) \frac{C_0}{2} |\zeta(q)|,\]

if \( \delta^2 < 1/4C_1 \). Now (2.11) yields

\[k + 1 \leq \varepsilon \frac{2}{C_0} \zeta(q)^4 \exp(C|\zeta(q)|^{-1}) \leq \varepsilon |\zeta(q)|^3 \exp(C|\zeta(q)|^{-1})\]

for \( \delta < C_0/2 \), which proves the assertion.

Making use of Lemma 2.6 and Lemma 2.8 we complete the proof of

Theorem 2.1.

To prove Theorem 2.2 we note that

\[d(q) = |\{f, g\}(q)| = a_1(q)|\zeta(q)|, \quad q \in J_0,\]

and use Theorem 2.1.

**3. Normal forms of pairs of analytic involutions.**

We are going to find a \( G^2 \) normal form of the pair of analytic

involutions \( J_1 \) and \( J_2 \) given in Theorem 2.3. If \( n \geq 3 \) and \( y \in \mathbb{R}^n \), we

set \( y = (y_1, y') = (y_1, y'', y_n) \), \( y'' = (y_2, \ldots, y_{n-1}) \). We have:
Theorem 3.1 — Let \( f \) and \( g \) be two real analytic involutions in a neighborhood of the origin in \( \mathbb{R}^n, \ n \geq 2 \). Suppose that \( f \) and \( g \) coincide with the identity on \( K = \{ y_n = 0 \} \) and that the differentials \( df \) and \( dg \) are linearly independent at the origin. Then there exists a diffeomorphism \( u \) of Gevrey class \( G^2 \) in a neighborhood \( U \) of the origin such that the maps \( f^0 = u \circ f \circ u^{-1} \) and \( g^0 = u \circ g \circ u^{-1} \) have the form

\[
    f^0(y) = (y_1, y''_n, -y_n), \quad g^0(y) = (y_1 + y_n, y''_n, -y_n) + R(y),
\]

where \( R \) belongs to the Gevrey class \( G^2(U) \) and it is flat at \( K \).

The proof of Theorem 3.1 will be given in Section 4.

Proof of Theorem 2.3. — According to Theorem 3.1 there exist \( G^2 \) coordinates \((x, \xi)\) in a neighborhood \( U \) of \((0,0)\) in \( \mathbb{R}^n \times \mathbb{R}^n \) such that

\[
    \mathcal{J}_1(x, \xi) = (x, -\xi_1, \xi'), \quad \mathcal{J}_2(x, \xi) = \mathcal{J}_0(x, \xi) + R(x, \xi), \quad (x, \xi) \in U,
\]

where \( \mathcal{J}_0 \) stands for the involution

\[
    \mathcal{J}_0(x, \xi) = (x_1 + \xi_1, x', -\xi_1, \xi'),
\]

\( R \in G^2(U) \), and \( R \) is flat at \( K = \{ (x, \xi) \in U : \xi_1 = 0 \} \). Using the invariance of \( \omega_0 \) with respect to the involutions given above, we are going to show that it has a quite simple form which allows us to use the deformation argument of Moser - Weinstein.

First we introduce the following notations: We set \( \xi_1 = \xi_1^2 \) and \( \xi' = \xi' \), \( \xi = (\xi_1, \xi') \in \mathbb{R}^n \). Next, for given \( r \in C^\infty(U) \) we say as above that \( r \approx 0 \) if \( r \) is flat at \( K \). More generally, for a given \( k \)-form \( \omega \) in \( T^*(\mathbb{R}^n) \) we say that \( \omega \approx 0 \), if all the coefficients of \( \omega \) are flat functions at \( K \).

Now, using the equality \( \mathcal{J}_1^* \omega_0 = \omega_0 \) we write the two-form \( \omega_0 \) as follows

\[
    (3.1) \quad \omega_0 = \sum_{1 \leq i, j \leq n} a_{ij}(x, \xi)d\xi_i \wedge dx_j + \sum_{1 \leq i < j \leq n} b_{ij}(x, \xi)dx_i \wedge dx_j + \sum_{1 \leq i < j \leq n} c_{ij}(x, \xi)d\xi_i \wedge d\xi_j,
\]

where the functions \( a_{ij}(x, \xi) \), \( b_{ij}(x, \xi) \), and \( c_{ij}(x, \xi) \) belong to \( G^2(U) \) and they are even with respect to \( \xi_1 \), i.e.

\[
    (3.2) \quad \mathcal{J}_1^*a_{ij} = a_{ij}, \quad \mathcal{J}_1^*b_{ij} = b_{ij}, \quad \mathcal{J}_1^*c_{ij} = c_{ij}.
\]
Now, condition (2.4) reads

\begin{equation}
\omega^n = \xi_1 dv,
\end{equation}

where \(dv\) is a volume form in \(U\) and we have taken \(n\) times the exterior product of \(\omega_0\).

**Proposition 3.2.** — Let \(\omega_0\) be a closed two-form in a neighborhood of the origin in \(T^*(\mathbb{R}^n)\) with \(G^2\) coefficients satisfying (3.3). Suppose that \(\mathcal{J}_j^* \omega_0 = \omega_0, \ j = 1, 2\). Then there exists a local diffeomorphism \(\psi \in G^2(U)\) in a neighborhood \(U\) of \(\varphi^0 = (0,0)\) such that

\[ \psi \circ \mathcal{J}_1 = \mathcal{J}_1 \circ \psi, \ \psi \circ \mathcal{J}_0 = \mathcal{J}_0 \circ \psi, \ \text{in} \ U, \]

and

\begin{equation}
\psi^* \omega_0 = \sum_{1 \leq i,j \leq n} a_{ij}^0(y', \eta) d\tilde{\eta}_i \wedge dy_j + \sum_{1 \leq i < j \leq n} b_{ij}^0(y', \eta) dy_i \wedge dy_j
\end{equation}

\[ + \sum_{1 \leq i < j \leq n} c_{ij}^0(y', \eta) d\tilde{\eta}_i \wedge d\tilde{\eta}_j + \tilde{\omega}, \]

where \(\tilde{\omega} \approx 0, \ \tilde{\eta} = (\eta_1^2, \eta'), \) and

\begin{equation}
a_{11}^0(y', \eta) \equiv 1, \ a_{1i}^0(y', \eta) \equiv 0, \ b_{1i}^0(y', \eta) \equiv 0, \ 2 \leq i \leq n.
\end{equation}

We postpone for a while the proof of the proposition. Now we can apply the deformation argument of Moser - Weinstein to the form \(\sigma_1 = \psi^* \omega_0\).

**Lemma 3.3.** — Let \(\sigma_1\) be a closed two-form in a neighborhood of the origin in \(\mathbb{R}^{2n}\) with \(G^2\) coefficients given by (3.4) and satisfying (3.3) and (3.5). Suppose that \(\mathcal{J}_j^* \omega_0 = \omega_0, \ j = 1, 2\). Then there exists a diffeomorphism \(\phi \in G^2(U)\) in a neighborhood \(U\) of \(\varphi^0 = (0,0)\) such that

\[ \phi^* \sigma_1 = \sigma_0 \equiv 2\eta_1 d\eta_1 \wedge dy_1 + \sum_{j=2}^{n} d\eta_j \wedge dy_j, \]

and

\[ \phi \circ \mathcal{J}_1 = \mathcal{J}_1 \circ \phi, \ \phi \circ \mathcal{J}_2 \approx \mathcal{J}_2 \circ \phi \ \text{in} \ U. \]
Proof. — After performing a linear change of the variables

\[ z_i = y_1 + \sum_{j=2}^{n} (a_{ij}(0)y_j + c_{ij}(0)\eta_j), \quad \zeta_i = \eta_i, \]

\[ (z', \zeta') = A(y', \eta'), \quad \det(A) \neq 0, \]

which commutes with both \( J_1 \) and \( J_0 \) we can suppose that \( \sigma_1 = \sigma_0 + \sigma_2 \), where \( \sigma_2 \) has the form (3.4), its coefficients satisfy (3.2) (\( J_1^* \sigma_2 = \sigma_2 \)), and

\[ a_{i1}^0(y', \eta) \equiv 0, \quad b_{i1}^0(y', \eta) \equiv 0, \quad 1 \leq i \leq n, \]

\[ a_{ij}^0(0) = b_{ij}^0(0) = c_{ij}^0(0) = 0, \quad 1 \leq i, j \leq n. \]

We interpolate \( \sigma_1 \) and \( \sigma_0 \) by the family

\[ \sigma_t = \sigma_0 + t(\sigma_1 - \sigma_0), \quad 0 \leq t \leq 1. \]

We are looking for a time dependent vector field \( X_t \) such that

\[ (\phi^t)^*\sigma_t = \sigma_0, \]

where \( \phi^t \) is defined by

\[ \frac{d}{dt} \phi^t = X_t(\phi^t), \quad \phi^0 = \text{Id}. \]

Differentiating (3.7) with respect to \( t \) we get

\[ L_{X_t}\sigma_t + \sigma_1 - \sigma_0 = 0, \]

where \( L_{X_t} = \iota_{X_t} \circ d + d \circ \iota_{X_t} \) is the Lie derivative, \( \iota_{X_t} \) being the inner product by \( X_t \). Since \( \sigma_t \) is closed, we arrive at the equality

\[ d(\iota_{X_t}\sigma_t) = \sigma_0 - \sigma_1. \]

We have \( \sigma_0 - \sigma_1 = d\alpha, \quad \alpha(\phi^0) = 0 \), where \( \alpha \) can be chosen in the form

\[ \alpha \approx \sum_{j=2}^{n} \alpha_j(y', \eta)dy_j + \sum_{j=1}^{n} \beta_j(y', \eta)d\eta_j, \]

in view of (3.4) and (3.6). We can suppose that \( \alpha_j \) and \( \beta_j \) are \( G^2 \) functions. Moreover, taking \( (J_1^* \alpha + \alpha)/2 \) instead of \( \alpha \) we can arrange \( J_1^* \alpha = \alpha \) still keeping (3.9), which implies \( \beta_1(y', \eta) = \eta_1\gamma(y', \eta) \) with \( \gamma \) in \( G^2 \). Set
In view of (3.9) we have $B_0 \alpha = \alpha$ which implies $J_0^* \alpha \approx \alpha$ since $J_0 = J_1 \circ B_0$.

Now (3.8) leads to the equation $\iota X_t \sigma_t = \alpha$ which has a unique solution

$$X_t = \sum_{j=1}^{n} p_j(y, \eta) \frac{\partial}{\partial y_j} + \sum_{j=1}^{n} q_j(y, \eta) \frac{\partial}{\partial \eta_j}.$$ 

Indeed, since $\beta_1(y', \eta) = \eta_1 \gamma(y', \eta)$ and the coefficient of $dy_1$ in $\alpha$ is flat at $\eta_1 = 0$, taking into account (3.6) we obtain an (algebraic) non-degenerate linear system for $p_j$ and $q_j$ with $G^2$ coefficients which has an unique solution in $G^2$.

We are going to show that $\phi = \phi^1$ is the desired diffeomorphism. We have to prove that $\phi$ commutes with $J_1$. Setting

$$Y_t = J_1^* X_t \equiv (dJ_1)^{-1} X_t \circ J_1$$

we get

$$\alpha = J_1^* (\iota X_t \sigma_t) = \iota_{Y_t} (J_1^* \sigma_t) = \iota_{Y_t} \sigma_t,$$

which implies $J_1^* X_t = X_t$. Hence, $J_1 \circ \phi^t = \phi^t \circ J_1$. In the same way we show the relation $J_0 \circ \phi^t \approx \phi^t \circ J_0$ which implies $J_2 \circ \phi^t \approx \phi^t \circ J_2$. The proof of Lemma 3.3 is complete.

Now, Theorem 2.3 follows immediately from Theorem 3.1, Proposition 3.2 and Lemma 3.3.

Proof of Proposition 3.2. — To put $\omega_0$ into a normal form, keeping the involutions $J_1$ and $J_0$ fixed, we use the following

LEMMA 3.4. — Let $p \in C^\infty(U)$, $J_1^* p = p$, and $J_0^* p \approx p$. Then $\partial p / \partial x_1 \approx 0$.

Proof. — Consider the Taylor expansion at $(x, 0, \xi')$ of the smooth function

$$p(x_1 + \xi_1, x', \xi) - p(x, \xi) = B_0^* p(x, \xi) - p(x, \xi) \approx 0,$$

where $B_0 = J_1 \circ J_0$. For any integer $\beta \geq 0$ the coefficient of $\xi_1^{\beta+1}$ is equal to

$$\sum_{\gamma=0}^{\beta} \sum_{\alpha=1}^{\beta+1-\gamma} \frac{1}{\alpha! \gamma!} \frac{\partial^\alpha}{\partial y_1} \frac{\partial^\gamma}{\partial \xi_1} p(x, 0, \xi') = 0.$$
By inductive arguments with respect to $\beta$ we get

$$\partial_{x_1} \partial_{\xi_1}^2 p(x, 0, \xi') = 0,$$

which proves the assertion.

The equality $B_0^* \omega_0 \approx \omega_0$ yields the identities

$$B_0^* a_{11} \approx a_{11}, \quad B_0^* a_{ij} \approx a_{ij}, \quad 2 \leq i \leq n, \quad 1 \leq j \leq n,$$

$$B_0^* b_{ij} \approx b_{ij}, \quad 1 \leq i < j \leq n,$$

$$B_0^* c_{ij} \approx c_{ij}, \quad 2 \leq i < j \leq n.$$

Now Lemma 3.4 implies

$$a_{11}(x, \xi) \approx a_{11}^0(x', \xi'),$$

$$a_{ij}(x, \xi) \approx a_{ij}^0(x', \xi'), \quad 2 \leq i \leq n, \quad 1 \leq j \leq n;$$

$$b_{ij}(x, \xi) \approx b_{ij}^0(x', \xi'), \quad 1 \leq i < j \leq n,$$

$$c_{ij}(x, \xi) \approx c_{ij}^0(x', \xi'), \quad 2 \leq i < j \leq n.$$

A similar argument has been used by Melrose [20] in a little bit different context. Applying Lemma 3.4 to $\partial_{x_1} a_{11}$ and $\partial_{x_1} c_{11}$ we obtain

$$a_{11}(x, \xi) \approx a_{11}^0(x', \xi'), \quad 2 \leq i \leq n$$

$$c_{11}(x, \xi) \approx c_{11}^0(x', \xi'), \quad 2 \leq j \leq n$$

where

$$a_{1j}^1(x', \xi) = -\frac{1}{2\xi_1} b_{1j}(x', \xi) \in G^2 \quad \text{since} \quad b_{1j}(x', 0, \xi') = 0, \quad 2 \leq j \leq n,$$

$$c_{1j}^1(x', \xi) = \frac{1}{2\xi_1} a_{j1}(x', \xi) \in G^2 \quad \text{since} \quad a_{j1}(x', 0, \xi') = 0, \quad 2 \leq j \leq n.$$
Moreover, we can suppose that

\[ a_{11}^0(x', 0, \xi') = 1. \]

Indeed, we have

**Lemma 3.5.** — There exists a diffeomorphism \((x, \xi) = \psi(y, \eta)\) in a neighborhood of \(q^0 = (0, 0)\) having the form

\[ x_1 = Y(y', \eta)y_1, \quad x' = y', \quad \xi_1 = X(y', \eta), \quad \xi' = \eta', \]

where \(X\) and \(Y\) are \(G^2\) functions,

\[ Y(q^0) > 0, \quad X(y', 0, \eta') = 0, \quad dX(q^0) \neq 0, \]

such that \(J_1 \circ \psi = \psi \circ J_1, \quad B_0 \circ \psi = \psi \circ B_0, \) and the coefficient of \(d\eta_1^2 \wedge dy_1\) in \(\sigma = \psi^*\omega_0\) is equal to 1.

**Proof.** — Taking into account (3.3), (3.10), and (3.11), we can suppose that \(a_{11}^0(0) > 0\). The coefficient of \(d\eta_1^2 \wedge dy_1\) in \(\sigma\) is equal to

\[ a_{11}^0(\psi(y, \eta)) Y(y', \eta) \frac{\partial X}{\partial \eta_1}(y', \eta) = 1. \]

Moreover, \(\psi\) commutes with \(B_0\) iff \(X(y', \eta) = Y(y', \eta)\eta_1\), and we obtain the equation

\[ a_{11}^0(y', X, \eta') X \frac{\partial X}{\partial \eta_1}(y', \eta) = \eta_1, \quad X(y', 0, \eta') = 0. \]

The solution of this equation is given by

\[ 2 \int_0^X a_{11}^0(y', t, \eta')tdt = \eta_1^2. \]

Set

\[ p(y', u, \eta') = 2 \int_0^u a_{11}^0(y', t, \eta')tdt. \]

Then \(p(y', u, \eta') = u^2q(y', u, \eta')^2, \quad q(0) = \sqrt{a_{11}^0(0)}\), where \(q\) is a \(G^2\) function, and we obtain the equation \(Xq(y', X, \eta') = \eta_1\), which has a unique solution \(X(y', \eta) = \eta_1T(y', \eta)\), where \(T\) is a \(G^2\) function and \(T(y', 0, \eta') > 0, \) in view of the implicit function theorem in Gevrey classes. Moreover, \(J_1^*p = p, \) and we obtain \(J_1^*T = T\) which implies \(J_1 \circ \psi = \psi \circ J_1. \)

\[ \square \]
From now on we suppose that (3.12) holds for the form $\sigma = \psi^* \omega_0$. We have $\sigma = \sigma_1 + y_1 \sigma_2 + \sigma_3$ in $U$, where

$$\sigma_1 = \sum_{1 \leq i, j \leq n} a^0_{ij} d\tilde{\eta}_i \wedge dy_j + \sum_{1 \leq i < j \leq n} b^0_{ij} dy_i \wedge dy_j + \sum_{1 \leq i < j \leq n} c^0_{ij} d\tilde{\eta}_i \wedge d\tilde{\eta}_j,$$

(3.14)

$$\sigma_2 = \sum_{j=2}^{n} a^1_{ij} d\eta_1^2 \wedge dy_j + \sum_{j=2}^{n} c^1_{ij} d\eta_1^2 \wedge d\eta_j,$$

$\tilde{\eta} = (\eta^2_1, \eta')$, and $\sigma_3 \approx 0$. We are going to explore the two-form $\sigma_2$. Since $\sigma$ is closed and the coefficients of $\sigma_1$ do not depend on $y_1$ we have $d\sigma_2 \approx 0$. Then $\sigma_2 = d\alpha + a(y', \eta) d\eta_1$, where the coefficients of the one-form $\alpha$ do not depend on $y_1$, and $\alpha$ does not contain multiples of $dy_1$ and $d\eta_1$. In other words, one can consider $\alpha$ as one-form in $(y', \eta')$ depending on the parameter $\eta_1$. Now (3.14) implies $dy_{12} \alpha = 0$, hence $\alpha = d\gamma + p(y', \eta) d\eta_1$, where $\gamma, p \in G^2(U)$, and we get

$$\sigma_2 = dp \wedge d\eta_1.$$

Next we write $\sigma = \sigma_1^0 + \sigma_2^0 + \sigma_3$, where $\sigma_3 \approx 0$,

$$\sigma_2^0 = \left( \sum_{i=1}^{n} a^0_{i1}(y', \eta) d\eta_i - \sum_{j=2}^{n} b^0_{ij}(y', \eta) dy_j \right) \wedge dy_1 + y_1 dp(y', \eta) \wedge d\eta_1,$$

while the two-form $\sigma_1^0$ does not contain multiples of $dy_1$, and the coefficients of $\sigma_1^0$ are functions of $(y', \eta)$ only. Our aim is to show that $\sigma_2^0 \approx d\eta_1^2 \wedge dy_1$.

Notice that $d\sigma_2^0 \approx 0$ which implies

$$\sum_{i=1}^{n} da^0_{i1} \wedge d\tilde{\eta}_i - \sum_{j=2}^{n} db^0_{ij} \wedge dy_j + dp \wedge d\eta_1 \approx 0.$$

Therefore,

$$\sum_{i=1}^{n} a^0_{i1} d\tilde{\eta}_i - \sum_{j=2}^{n} b^0_{ij} dy_j \approx dr - pd\eta_1,$$

where $r \in G^2(U)$, and we have

$$\sigma_2^0 = dr \wedge dy_1 - pd\eta_1 \wedge dy_1 - y_1 d\eta_1 \wedge dp.$$

On the other hand, since $\sigma_1^0$ is invariant with respect to $B_0$, we get $B_0^* \sigma_2^0 \approx \sigma_2^0$, which gives

$$r(y', \eta) \approx -p(y', \eta) \eta_1 + q(\eta_1), \ q \in G^2(\mathbb{R}),$$
while (3.12) yields
\[ \frac{\partial r}{\partial \eta_1}(y', \eta) - p(y', \eta) = 2\eta_1. \]

Hence, \( r \) satisfies the equation
\[ \eta_1 \frac{\partial r}{\partial \eta_1} + r \approx q(\eta_1) + 2\eta_1^2. \]

Then \( v(y', \eta) = r(y', \eta) - r(0, \eta_1, 0) \) should satisfy the equation
\[ \eta_1 \frac{\partial v}{\partial \eta_1} + v \approx 0, \]
and we obtain \( v \approx 0 \). Therefore,
\[ \sigma_2^0 \approx d\eta_1^2 \land dy_1, \]
which completes the proof of the proposition. \( \square \)

4. Proof of Theorem 3.1.

Consider the real analytic involutions \( f \) and \( g \). As in [11] we can suppose after a local analytic change of the variables that
\[ f(y) = (y', -y_n) \text{ for } y \in B^n(\delta), \ 0 < \delta \ll 1, \]
where \( B^n(\delta) = \{ z \in \mathbb{R}^n : |z_j| < \delta, \ j = 1, \ldots, n \} \).

Then as \( g(y) \) is analytic in \( B^n(\delta) \) we can write \( g(y) = (g'(y), g_n(y)), \ y \in B^n(\delta) \) in the following form:

\[ g'(y) = y' + \sum_{j=1}^{\infty} y_j^j a^j(y'), \ a^j(y') = (a_1^j(y'), \ldots, a_{n-1}^j(y')), \]

\[ g_n(y) = -y_n + \sum_{j=1}^{\infty} y_{n+1}^j A_j(y'). \]

Moreover, there is \( C > 0 \) such that
\[ \sup_{y' \in B^{n-1}(\delta)} |\partial_{y'} q^j(y')| \leq C^{j+|\alpha|} \alpha!, \ \alpha \in \mathbb{Z}_+^{n-1}, \ j = 1, 2, \ldots, \]
where \( q^j \) stands either for \( a_1^j \), \( r = 1, \ldots, n - 1 \), or for \( A_j \). The requirement that the differentials of \( f \) and \( g \) are linearly independent on their fixed set yields

\[
a^1(y') \neq 0 \quad \text{for} \quad y' \in B^{n-1}(\delta).
\]

Under the assumptions above Theorem C.4.6 in [11] provides a \( C^\infty \) smooth change of the variables \( u = u(y) \), \( u(0) = 0 \), which preserves \( f \) and transforms \( g \) into \( u^{-1} \circ g \circ u = g_0 \), where

\[
g_0(y_1, y'', y_n) = (y_1 + y_n, y'', -y_n).
\]

We are concerned here with the problem whether \( u \) can be found in a suitable Gevrey class \( G^\alpha \) provided that the involutions are analytic. Fix an integer \( p \geq 2 \). As in the proof of Theorem C.4.6 in [11], making an analytic change of the variables, we can assume from the very beginning that

\[
(4.3) \quad a^1(y') = (1, 0, \ldots, 0), \quad A_j(y') = 0, \quad 1 \leq j \leq p - 1.
\]

First we consider \( u(y) \) as formal power series in \( y_n \)

\[
u(y', y_n) = \sum_{j=0}^{\infty} y_n^j (u^j(y'), y_n u_n^j(y')) , \quad u^0(y') = y', \quad u_n^0(y') = 1,
\]

with the subsequent compositions obeying the rules of the calculus for formal power series expansions.

**Lemma 4.1.** — If \( u(y) \) satisfies \( f \circ u = u \circ f \) then \( u(y) \) has only even powers in \( y_n \), namely

\[
u(y', y_n) = \sum_{j=0}^{\infty} y_n^{2j} (u^{2j}(y'), y_n u_n^{2j}(y')).
\]

**Proof.** — We have

\[
f(u(y)) = (u'(y), -u_n(y)) = \sum_{j=0}^{\infty} (y_n^j u_n^{j'}(y'), -y_n^{j+1} u_n^j(y')) ,
\]

\[
u(f(y)) = u(y', -y_n) = \sum_{j=0}^{\infty} ((-y_n)^j u^{j'}(y'), (-y_n)^{j+1} u_n^j(y')).
\]
Evidently the last two identities imply that $f \circ u = u \circ f$ iff $u^j(y') = (u^{j'}(y'), u^j_n(y')) \equiv 0$ for all $j$ odd.

So we look for $u(y)$ given by the formal power series in Lemma 4.1 and satisfying

$$u(g(y)) = g_0(u(y)). \tag{4.4}$$

First we write explicitly the left and the right hand side of (4.4) as formal power series in $y_n$. We check easily that formally

$$g_0(u(y)) = \sum_{j=0}^{\infty} y_n^{2j} \left( u'_1(y') + y_n u'_n(y'), u^{j''}(y'), -y_n u'_n(y') \right). \tag{4.5}$$

Further we obtain in view of (4.1), (4.2) and Lemma 4.1 that

$$u(g(y)) = \sum_{j=0}^{\infty} (-y_n + \sum_{r=1}^{\infty} y_n^{r+1} A_r(y'))^{2j} \times \left( u^{j'}(y') + \sum_{s=1}^{\infty} y_n^s a^s(y'), (-y_n + \sum_{r=1}^{\infty} y_n^{r+1} A_r(y')) u'_n(y') + \sum_{s=1}^{\infty} y_n^s a^s(y') \right).$$

Next we recall the power series expansions

$$(-y_n + \sum_{r=p}^{\infty} y_n^{r+1} A^r(y'))^j = y_n^j \left( -1 \right)^j + \sum_{\nu=p}^{\infty} A_{j, \nu}(y') y_n^\nu,$$

$$A_{j, \nu}(y') = \sum_{j_0, j_p, j_{p+1}, \ldots, j_\nu} \frac{j!}{j_0! j_p! \cdots j_\nu!} (-1)^{j_0} (A_p(y'))^{j_p} \cdots (A_\nu(y'))^{j_\nu}, \tag{4.6}$$

where the latter sum is taken over the set $Z_0(j, \nu)$ of all

$$(j_0, j_p, j_{p+1}, \ldots, j_\nu) \in \mathbb{Z}^{\nu-p+2}$$

such that $j_0 + j_p + j_{p+1} + \ldots + j_\nu = j, p j_p + (p+1) j_{p+1} + \ldots + \nu j_\nu = \nu$.

We denote by $\mathbb{Z}_+$ the set of the non-negative integers and by $\mathbb{N}$ the set of the positive ones. We set by convention

$$A_{j, 0} = (-1)^j, \ A_{j, \nu} = 0, \text{ for } 1 \leq \nu \leq p - 1.$$
Moreover,

\[ h(y') + \sum_{s=1}^{\infty} y_s^s a^s(y') = \sum_{\mu=0}^{\infty} y_\mu^\mu \sum_{|\beta| \leq \mu} a_{\mu, \beta}(y') \frac{\partial^\beta y_h(y')}{\beta!}, \]

(4.7) \[ a_{\mu, \beta}(y') = \sum_{\gamma} \frac{\beta!}{\gamma_1! \cdots \gamma_\mu!} (a_1(y'))^{\gamma_1} \cdots (a_\mu(y'))^{\gamma_\mu}, \]

where the latter sum is over the set of multiindices \( \mathbf{Z}(\beta, \mu) \) of all \( (\gamma^1, \ldots, \gamma^\mu) \in (\mathbf{Z}_+^{n-1})^\mu \), where \( \gamma^s \in \mathbf{Z}_+^{n-1}, 0 \leq s \leq \mu \), and

\[ \gamma^1 + \gamma^2 + \cdots + \gamma^\mu = \beta, \quad |\gamma^1| + 2|\gamma^2| + \cdots + \mu|\gamma^\mu| = \mu. \]

We have used the notations

\[ |\gamma^s| = \gamma^1 + \cdots + \gamma^{s-1}, \quad \gamma^s! = \gamma^1! \cdots \gamma^{s-1}!, \quad \gamma^s = (\gamma^1, \ldots, \gamma^{s-1}), \]

and

\[ (a^s(y'))^{\gamma^s} = a_1^s(y')^{\gamma^1} \cdots a_{n-1}^s(y')^{\gamma^{n-1}}. \]

We set by convention \( a_0,0 = 1 \), and \( a_{\mu, \beta} = 0 \) if \( |\beta| \leq \mu \) and \( \mathbf{Z}(\mu, \beta) \) is empty which happens for example when \( \mu \geq 1 \) and \( \beta = 0 \).

**Lemma 4.2.** — Let \( \beta = (\beta_1, \ldots, \beta_{n-1}) \in \mathbf{Z}_+^{n-1}, \beta \neq 0, \) and let \( \mu = |\beta| \). Then, \( a_{\mu, \beta} = 0 \) if \( \beta \neq (\mu, 0, \ldots, 0) \), and \( a_{\mu, \beta} = 1 \) if \( \beta = (\mu, 0, \ldots, 0) \).

**Proof.** — Let \( (\gamma^1, \ldots, \gamma^\mu) \in \mathbf{Z}(\mu, \beta) \). Then

\[ |\gamma^1| + |\gamma^2| + \cdots + |\gamma^\mu| = |\beta| = \mu = |\gamma^1| + 2|\gamma^2| + \cdots + \mu|\gamma^\mu|, \]

and we obtain \( \gamma^2 = \cdots = \gamma^\mu = 0 \) and \( \gamma^1 = \beta \). Now, using (4.3) and (4.7) we prove the assertion. \( \Box \)

The power series expansions given above yield

\[ w(y) = u(g(y)) = \sum_{k=0}^{\infty} y_n^k w^k(y'), y_n w^k_n(y')). \]

with

\[ w^k(y') = \sum_{0 \leq j \leq k/2} \sum_{\nu=0}^{k-2j} \sum_{|\beta| \leq k-2j-\nu} A_{2j, \nu}(y') a_{k-2j-\nu, \beta}(y') \frac{\partial^\beta y_u(y')}{\beta!}, \]
Now, taking into account (4.5), we obtain from (4.4) the equalities

\[ w^{2k}(y') = (u^{k}(y'), -u^{k}_{n}(y')), \quad k \in \mathbb{Z}_{+}, \]

(4.8)

\[ w^{2k+1}(y') = (u^{k}_{n}(y'), 0), \quad k \in \mathbb{Z}_{+}, \]

The resolution of (4.4) in the \( C^{\infty} \) category shows that one has only to solve (4.8) for each \( k \), since the other sequence of identities involving \( w^{2k} \) follows from (4.8) taking into account the fact that \( g \) is an involution (cf. case (a) in (C.4.3), [11]).

We write the equations (4.8) explicitly, replacing \( w^{k} \) by the corresponding expressions given above. Using (4.3) we obtain

\[ \partial y_{1} u^{k}(y') = u^{k}_{n}(y') e_{1} - G^{k}(u), \]

where \( e_{1} = (1, 0, \ldots, 0) \), \( G^{0} = 0 \), and the operator \( G^{k} \), \( k \geq 1 \), is given by

\[ G^{k}(u)(y') = \sum_{j=0}^{k-1} \sum_{|\beta| \leq 2k+1-2j} \Gamma^{k}_{j,\beta}(y') \frac{\partial^{\beta} u^{j}(y')}{\beta!}. \]

Here

\[ \Gamma^{k}_{j,\beta}(y') = \begin{pmatrix} \Gamma^{k}_{j,\beta}(y') & 0 \\ 0 & \Gamma^{k,n}_{j,\beta}(y') \end{pmatrix} \]

is \( n \times n \) matrix with

\[ \Gamma^{k}_{j,\beta}(y') = \sum_{\nu=0}^{2k+1-2j-|\beta|} A_{2j+1,\nu}(y') a_{2k+1-2j-\nu,\beta}(y') E_{n-1}, \]

(4.10)

\[ \Gamma^{k,n}_{j,\beta}(y') = \sum_{\nu=0}^{2k+1-2j-|\beta|} A_{2j+1,\nu}(y') a_{2k+1-2j-\nu,\beta}(y'), \]

(4.11)

\( E_{n-1} \) being the unit matrix in \( \mathbb{R}^{n-1} \).

Before solving equations (4.9) with suitable initial conditions, we rewrite \( G^{k} \) as a sum of two terms, grouping in one of them all \( \Gamma^{k}_{j,\beta} \) with \( |\beta| = 2k - 2j + 1 \). Let \( \beta = (\beta_{1}, \ldots, \beta_{n-1}) \in \mathbb{Z}_{+}^{n-1} \), \( \beta \neq 0 \), and let
\(|\beta| = 2k - 2j + 1\) for some positive integers \(k\) and \(j\). Then \(\nu = 0\) in (4.10) and (4.11), and Lemma 4.2 implies that either

\[ \Gamma_{j,\beta}^k = 0 \]

if \(\beta \neq (2k - 2j + 1, 0, \ldots, 0)\), or

\[ \Gamma_{j,\beta}^k = \begin{pmatrix} E & 0 \\ 0 & -1 \end{pmatrix} \]

if \(\beta = (2k - 2j + 1, 0, \ldots, 0)\). Hence,

\[ C^k = P^k + R^k, \]

where

\[
(4.12) \quad P^k(u)(y') = \sum_{j=0}^{k-1} \sum_{|\beta| \leq 2k-2j} \Gamma_{j,\beta}^k(y') \frac{\partial^{2j} u^{j}(y')}{\beta!},
\]

and

\[
(4.13) \quad R^k(u)(y') = \begin{pmatrix} E & 0 \\ 0 & -1 \end{pmatrix} \sum_{j=0}^{k-1} \frac{\partial^{2j} u^{j}(y')}{(2k - 2j + 1)!}.
\]

Denote by \(S\) the real analytic hypersurface in \(B^{n-1}(\delta)\)

\[ S := \{ y' \in B^{n-1}(\delta) : y_1 = 0 \}, \]

and consider the system of equations (4.9) with initial conditions

\[
(4.14) \quad u^0|_S = (0, y'', 1), \quad u^k|_S = 0, \; k = 1, 2, \ldots.
\]

They can be solved successively for every \(k \in \mathbb{Z}_+\), and the corresponding solution \(u^k(y')\) is analytic in \(B^{n-1}(\delta)\).

To estimate \(\sup_{y' \in B^{n-1}(\delta)} |u^k(y')|\) as \(k \to \infty\), we introduce suitable Banach spaces of formal power series which are adapted for the calculus with the Taylor series of Gevrey functions. Similar spaces have been used in [6] to study the calculus of classical formal pseudo-differential operators in Gevrey classes.

Let \(FA(\delta, n - 1)\) be the set of all sequences

\[ h := h(y') = \{h^0(y'), h^1(y'), \ldots\} \]
of analytic functions \( h^k \) in \( B^{n-1}(\delta) \). For any \( T > 0, \ k \in \mathbb{Z}_+, \) and \( \alpha \in \mathbb{Z}_+^{n-1} \) we set

\[
(4.15) \quad M_k^\alpha(h : T) = \frac{T^{3k + |\alpha|}}{(2k + |\alpha|)!} \sup_{y' \in B^{n-1}(\delta)} |\partial_{y'}^\alpha h^k(y')|
\]

and denote

\[
(4.16) \quad \|h\|_T = \sum_k M_k^\alpha(h : T)
\]

where the sum is taken over all \( k \in \mathbb{Z}_+ \) and \( \alpha \in \mathbb{Z}_+^{n-1} \). Let

\[
E(T) = \{ h \in FA(\delta, n - 1) : \|h\|_T < \infty \}
\]

be the corresponding Banach space.

**Proposition 4.3.** — There exist constants \( \delta > 0 \) and \( C > 0 \) such that

\[
\sup_{y' \in B^{n-1}(\delta)} |\partial_{y'}^\alpha u^k(y')| \leq C^{k+1+|\alpha|} (2k)! \alpha!, \quad \forall k \in \mathbb{Z}_+, \quad \forall \alpha \in \mathbb{Z}_+^{n-1}.
\]

**Proof.** — To prove Proposition 4.3 we have to show that

\[
(4.17) \quad u(y') = \{u^0(y'), u^1(y'), \ldots\} \in E(T) \text{ for some } T > 0.
\]

Indeed, we observe that (4.16) and (4.17) imply

\[
\sup_{y' \in B^{n-1}(\delta)} |\partial_{y'}^\alpha u^k(y')| \leq C_1 \frac{(2k + |\alpha|)!}{T^{3k + |\alpha|}} \leq C^{k+1+|\alpha|} (2k)! \alpha!,
\]

where \( C_1 > 1, \ 0 < T < 1, \) and \( C = C(C_1, n, T) \).

We are going to prove (4.17). To do this, we rewrite the Cauchy problem (4.9) and (4.14) into a system of \( n \) integral equations. In this way we obtain an equation

\[
u(y') = H(\mathcal{P}(u))(y') + H(\mathcal{R}(u))(y') + U^0(y'), \quad u \in FA(\delta, n - 1),
\]

where \( U^0(y') = \{(O, y''', 1), 0, \ldots, 0, \ldots \} \) and the operator \( H : FA(\delta, n - 1) \rightarrow FA(\delta, n - 1) \) is defined by \( H(v)^0 = 0 \) and

\[
H(v)^k = - \left( \int_0^{y_1} \int_0^t v_n^k(s, y'') \, ds \, dt \right) e_1 - \int_0^{y_2} v^k(t, y'') \, dt, \quad k \in \mathbb{N}.
\]

The idea is to prove that the linear operator \( H \circ \mathcal{P} + H \circ \mathcal{R} \) is a contraction in \( E(T) \), provided that \( 0 < T \ll 1 \) and \( 0 < \delta \ll 1 \), and then to
use the fixed point theorem in the Banach space $E(T)$. First we show that $H$ is a contraction in $E(T)$.

**Lemma 4.4.** — There exist positive constants $\delta_0$, $T_0$ and $C_0$, such that

\[
H : E(T) \ni v \mapsto H(v) \in E(T)
\]

is continuous and

\[
\|H(v)\|_T \leq C_0(\delta + T)\|v\|_T, \quad \forall v \in E(T), \ T \in (0, T_0], \ \delta \in (0, \delta_0].
\]

**Proof.** — For any fixed $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_{+}^{n-1} \text{ and } k \in \mathbb{N}$, we consider the semi-norm $M_k^\alpha(v : T)$.

There are two possibilities:

i) $\alpha_1 = 0$. In that case $\partial_y \alpha = \partial_y \alpha''$ and the definition of $H(u)$ implies that

\[
M_k^\alpha(H(v) : T) \leq \delta M_k^\alpha(v : T).
\]

The second possibility is

ii) $\alpha_1 \geq 1$. In that case

\[
\partial_y \alpha \partial_y \alpha'' \int_0^{y_{n-1}} v^k(t, y'')dt = \partial_y \alpha \partial_y (v^k(y')).
\]

Hence, setting $\tilde{\alpha} = (\alpha_1 - 1, \alpha'')$, we obtain

\[
M_k^\alpha(H(v) : T) \leq \frac{T}{2k + |\alpha|} M_k^\alpha(v : T).
\]

The estimates above show that

\[
\|H(v)\|_T = \sum_k M_k^\alpha(H : T) \leq (\delta + T)\|v\|_T, \ v \in E(T),
\]

where the sum is taken over all $k \in \mathbb{Z}_{+}$, $\alpha \in \mathbb{Z}_{+}^{n-1}$.

**Lemma 4.5.** — There exist positive constants $\delta_0$ and $T_0$, such that

\[
H \circ R : E(T) \ni u \mapsto E(T)
\]

is continuous, and

\[
\|H(R(v))\|_T \leq 3T \|v\|_T, \quad \forall v \in E(T), \ T \in (0, T_0], \ \delta \in (0, \delta_0].
\]
Proof. — Let \( k \in \mathbb{N}, \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1} \), and \( v \in E(T) \). According to (4.13) we have

\[
H(\mathcal{R}(v))^k(y') = - \left( \begin{array}{cc} E & 0 \\ 0 & -1 \end{array} \right) \sum_{j=0}^{k-1} \frac{\partial_{y_1}^{2k-2j} v_j^j(y') - \partial_{y_1}^{2k-2j} v_j^j(0, y'')}{(2k-2j+1)!} + e_1 \sum_{j=0}^{k-1} \frac{\partial_{y_1}^{2k-2j-1} v_j^j(y') - \partial_{y_1}^{2k-2j-1} v_j^j(0, y'') - y_1 \partial_{y_1}^{2k-2j} v_j^j(0, y'')}{(2k-2j+1)!}.
\]

Hence,

\[
M_k^\alpha (H(\mathcal{R}) : T) \leq 3 \sum_{j=0}^{k-1} \frac{T^{k-j}}{(2k-2j+1)!} \left( M_j^{(2k-2j+\alpha_1, \alpha'')} (v : T) + T M_j^{(2k-2j+\alpha_1-1, \alpha'')} (v : T) \right).
\]

Summing up with respect to \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{Z}_+^{n-1} \), and setting \( p = k - j \) we obtain

\[
\|H(\mathcal{R}(v))\|_T \leq 6 \sum_{p=1}^{\infty} \sum_{j=0}^{\infty} \sum_{\alpha} \frac{T^p}{(2p+1)!} M_j^{(2p+\alpha_1-1, \alpha'')} (v : T)
\]

\[
\leq 6 \sum_{p=1}^{\infty} \frac{T^p}{(2p+1)!} \|v\|_T \leq 3T \|v\|_T,
\]

for all \( T \in (0, T_0] \), and \( T_0 > 0 \) sufficiently small. \( \square \)

It remains to show that the operator \( u \rightarrow \mathcal{P}(u) \), given by (4.12) is continuous in \( E(T) \).

**Proposition 4.6.** — There exist \( T_0 > 0 \) and \( C > 0 \) such that the linear map

\[
\mathcal{P} : E(T) \ni v \rightarrow \mathcal{P}(v) \in E(T)
\]

is uniformly continuous for \( 0 < T \leq T_0 \), and

\[
\|\mathcal{P}(v)\|_T \leq C \|v\|_T, \quad \forall T \in (0, T_0).
\]

**Proof.** — For any \( k \geq 1 \) and \( \alpha \in \mathbb{Z}^{n-1} \) we have

\[
M_k^\alpha (\mathcal{P}(v) : T) \leq \sum_{j=0}^{k-1} \sum_{|\gamma| \leq k-2j} \sum_{\rho + \theta = \alpha} \sup_{y' \in B^{n-1}(\delta)} |\partial_{y_1}^\rho \Gamma_j \gamma (y')|.
\]
To estimate the right hand side of (4.19) we are going to use the following

**Lemma 4.7.** — There exists a constant $C_1 > 0$ such that

$$
\sup_{v' \in B_{n-1}} |\partial^{\theta} R_k^{j, \gamma}(v')| \leq C_1^{k-j+|\theta|+1} \frac{(2k - |\gamma|)!}{(2j)!} \theta!,
$$

for any $\gamma, \theta \in \mathbb{Z}_+^{n-1}$, and $0 \leq j \leq k - 1$, $0 \leq |\gamma| \leq 2k - 2j$.

Lemma 4.7 will be proved in the Appendix. Assume for the moment that (4.20) is valid. Plugging it into (4.19) we arrive at the quantity

$$
(2k + |\rho|)! (2j + |\gamma| + |\rho|)! \theta! (2k + |\rho| + |\theta|)! \rho! \theta!.
$$

First we suppose that $|\gamma| = 2k - 2j$. Then (4.21) is estimated above by

$$
\frac{(2k + |\rho|)! (2j + |\gamma| + |\rho|)! (\theta + \rho)!}{(2k + |\rho| + |\theta|)! \rho! \theta!} = 1.
$$

We remind, that for any $z = (z_1, \ldots, z_{n-1}) \in \mathbb{Z}_+^{n-1}$ we have $|z| = z_1 + \cdots + z_{n-1}$, and that, by convention $0! = 1$.

For $|\gamma| < 2k - 2j$, we estimate (4.21) as above by

$$
\frac{(2j + |\gamma| + |\rho| + |\theta|)! (2j + 1) \ldots (2j + (2k - 2j - |\gamma|))}{(2k + |\rho| + |\theta|)! \theta!} \leq 1.
$$

Hence, (4.21) is estimated by $\frac{1}{\theta!}$, and we obtain

$$
M_k^\alpha(\mathcal{P}(v) : T) \leq C_1 \sum_{j=0}^{k-1} \sum_{|\gamma| \leq 2k-2j} \sum_{|\rho+\theta|=\alpha} \frac{(C_1 T)^{k-j+|\theta|}}{\theta!} M_j^{p+\gamma}(v : T).
$$

On the other hand, we have

$$
\sum_{k=j+1}^{\infty} \sum_{|\gamma| \leq 2k-2j} \frac{(C_1 T)^{k-j+|\theta|}}{\theta!} \leq \sum_{p=1}^{\infty} \sum_{\theta \in \mathbb{Z}_+^{n-1}} (2p)^{2n}(C_1 T)^p \frac{(C_1 T)^{|\theta|}}{\theta!} \leq 1,
$$
for any $T \in (0,T_0]$, and $T_0$ sufficiently small. Therefore,

$$
\|P(v)\|_T = \sum_{k \in \mathbb{Z}_+} \sum_{\alpha \in \mathbb{Z}_+^{n-1}} M_k^\alpha (P(v) : T) \leq \|v\|_T,
$$

which completes the proof of the proposition. \(\square\)

Now we pass to the Gevrey realization of the formal change of the variables $u(y)$ and to the proof of the theorem.

**Proposition 4.8.** — Let $u^k(y) = (u^k(y'), u^k_n(y'))$, $k \in \mathbb{Z}_+$ satisfy the estimates in Proposition 4.3. Then there exists a vector-function $u(y) = (u(y), u_n(y)) \in G^2(B^n(\delta))$ such that

$$
\partial_{y_n}^k u'(y', 0) = (2k)! u^{k'}(y'), \quad \partial_{y_n}^{2k+1} u'(y', 0) = 0, \quad k \in \mathbb{Z}_+
$$

$$
\partial_{y_n}^{2k+1} u_n(y', 0) = (2k+1)! u^k_n(y'), \quad \partial_{y_n}^{2k} u_n(y', 0) = 0, \quad k \in \mathbb{Z}_+,
$$

i.e. $u(y)$ is a $G^2$ realization of the formal solution from Proposition 4.3.

**Proof.** — We observe that in view of Lemma 4.1, $u'(y)$ must be an even function with respect to $y_n$.

We put $f_{2j+1}(y') = 0$, $f_{2j}(y') = (2j)! u^{j'}(y')$, $j \in \mathbb{Z}_+$. Then Proposition 4.3 yields after straightforward calculations that $\{f_j(y')\}_{j=1}^\infty$ satisfy

$$
\sup_{y' \in B^{n-1}(\delta)} |\partial_{y'}^\alpha f_k(y')| \leq C^{k+|\alpha|+1} \alpha! (k!)^2, \quad k \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^{n-1},
$$

where $C > 0$.

Now we can use the Whitney extension theorem in Gevrey classes which has been proved in different cases by many authors e.g. [7], [13], [14]. In our case, applying Lemma 1 in [13], or more generally Theorem 4.5 in [14], (see also Theorem 3.9 in [7]) we find a function $u'(y', y_n) \in G^2(B^{n-1}(\delta))$ having as a Taylor expansion in powers of $y_n$ the next formal sum

$$
\sum_{j=0}^\infty \frac{y_n^j}{j!} f_j.
$$

One deals similarly with $u_n(y)$ (which must be an odd function with respect to $y_n$) by defining $f_{2j}(y') = 0$, $f_{2j+1}(y') = (2j+1)! u^j_n(y')$, $j \in \mathbb{Z}_+$.

The proof of Theorem 3.1 is complete. \(\square\)
A.1. We are going to prove Lemma 4.7. Since $\Gamma_{\gamma}(y')$ is holomorphic in the polydisk $B^{n-1}(2\delta)$ in $\mathbb{C}^{n-1}$, it is enough to prove (4.16) for $\theta = 0$ (then one uses the Cauchy integral representation formulae in the polydisk $B^{n-1}(2\delta)$ with $y' \in B^{n-1}(\delta)$ for $0 < \delta \ll 1$). We are going to estimate separately $A_{j,\nu}$ and $a_{\mu,\gamma}$. We have

\begin{equation}
A_{j,0} = (-1)^j, \quad A_{j,\nu} = 0, \quad 1 \leq \nu \leq p - 1,
\end{equation}

according to (4.3) and (4.6), where $p \geq 2$ is fixed.

**Lemma A.1.** — Let $\nu \geq p \geq 2$. Then there exists a constant $C > 0$ such that

\begin{equation}
\sup_{y' \in B^{n-1}(\delta)} |A_{j,\nu}(y')| \leq C^{\nu+1} \frac{(j + \nu - 2)!}{(j - 1)!}
\end{equation}

for any $j \in \mathbb{Z}_+$, $j \geq 1$.

**Proof.** — Since $g(y)$ is analytic, there exists $C > 0$ such that

\[ \sup_{y' \in B^{n-1}(\delta)} |A_{r}(y')| \leq C^r, \quad r = 1, 2, \ldots \]

Hence, in view of (4.6) we obtain

\[ |A_{j,\nu}(y')| \leq \sum_{(j_0, j_p, \ldots, j_\nu) \in \mathbb{Z}_0(j, \nu)} \frac{j!}{j_0! j_p! \ldots j_\nu!} C^\nu, \]

for $y' \in B^{n-1}(\delta)$. Denote by $m$ the minimum of $j$ and the integer part of $\nu/p$. To estimate the right hand side of the inequality above, we consider for any $j \geq 1$ the function

\[ h(t) = \left(1 + \frac{t^p}{1 - t}\right)^j, \quad t \in (-1, 1), \quad p \geq 2. \]

Expanding $h(t)$ in power series, we get

\[ h(t) = \left(1 + \sum_{r=p}^{\infty} t^r \right)^j = \sum_{\nu=0}^{\infty} \sum_{(j_0, j_p, \ldots, j_\nu) \in \mathbb{Z}_0(j, \nu)} \frac{j!}{j_0! j_p! \ldots j_\nu!} t^\nu. \]

On the other hand,

\[ h(t) = 1 + \sum_{s=1}^{j} \binom{j}{s} \frac{t^s p}{(1 - t)^s}. \]
Hence, for any $\nu \geq p \geq 2$ we have

$$\sum_{(j_0, j_p, \ldots, j_\nu) \in \mathbb{Z}_0(j, \nu)} \frac{j!}{j_0! j_p! \ldots j_\nu!} = \frac{1}{\nu!} \frac{d^\nu h}{dt^\nu}(0)$$

$$= \frac{1}{\nu!} \sum_{s=1}^{\nu} \binom{j}{s} \left( \frac{d}{dt} \right)^s (t^{sp} (1-t)^{sp}) \big|_{t=0}$$

$$= \frac{1}{\nu!} \sum_{s=1}^{m} \binom{j}{s} \binom{\nu}{sp} (sp)! s(s+1) \ldots (s+\nu-sp-1)$$

$$\leq \sum_{s=1}^{m} j(j-1) \ldots (j-s+1) (s+\nu-sp-1)!$$

$$\leq \sum_{s=1}^{m} j(j+1) \ldots (j+s-1)(s+\nu-sp-1)!$$

$$\leq \sum_{s=1}^{m} \frac{(j+\nu-s(p-2)-2)!}{(j-1)!}$$

$$\leq \nu \frac{(j+\nu-2)!}{(j-1)!}.$$
which leads to

$$\tilde{a}_{\gamma, \mu}(C) = \frac{1}{\mu!} \left( \frac{d}{dt} \right)^\mu \left( (Ct)^{|\gamma|} (1 - Ct)^{-|\gamma|} \right)_{|t=0} = C^\mu \frac{(\mu - 1)!}{(\mu - |\gamma|)!(|\gamma| - 1)!} \leq (2C)^\mu,$$

provided $\gamma \in \mathbb{Z}_+^{n-1}$, $0 < |\gamma| \leq \mu$.

\begin{proof}
Fix $\gamma$, $k$, and $0 \leq j \leq k - 1$, so that $|\gamma| \leq 2k - 2j$. Using (4.10) - (4.11) as well as (A.1) - (A.3), we obtain

$$|\Gamma_{j, \gamma}^{k, n}(y')| \leq C^{2k+1-2j} \sum_{\nu=2}^{2k+1-2j-|\gamma|} C^{2k+1-2j} \frac{(2j + \nu - 1)!}{(2j)!} \leq C^{2k-2j+1}(2k - 2j + 1 - |\gamma|) \frac{(2k - |\gamma|)!}{(2j)!} \leq C_1^{k-j} \frac{(2k - |\gamma|)!}{(2j)!},$$

with $C_1 > 0$ satisfying $C^{2s+1}(2s + 1) \leq C_1^s$, $s \geq 1$. The proof of Lemma 4.5 is complete.
\end{proof}

\textbf{Acknowledgements.} Most of this paper was written while the second author was a fellow of the Alexander von Humboldt foundation at the Institute of Applied Mathematics, Bonn University. He would like to thank the Alexander von Humboldt foundation for the support as well as Rolf Leis for the hospitality. The authors are grateful to Vladimir Lazutkin who drew their attention on the problem of the effective stability for the billiard ball map some years ago.

\begin{thebibliography}{9}
\bibitem{1} A. BAZZANI, S. MARMI, G. TURCHETTI, Nekhoroshev estimate for isochronous non resonant symplectic maps, Celestial Mech. and Dynamical Astronomy, 47(1990), 333-359.
\end{thebibliography}


Manuscrit reçu le 24 février 1994,
accepté le 2 mars 1995.

T. GRAMCHEV,
Dipartimento di Matematica
Università di Cagliari
Via Ospedale 72
09124 Cagliari (Italie).

G. POPOV,
Institute of Mathematics
Bulgarian Academy of Sciences
1113 Sofia (Bulgarie).