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MEAN PERIODIC FUNCTIONS ON PHASE SPACE AND THE POMPEIU PROBLEM WITH A TWIST

by Sundaram THANGAVELU

1. Introduction.

A continuous function f on $\mathbb{R}^n$ is said to be mean periodic if the closed subspace $T(f)$ generated by f and all its translates is proper in $C(\mathbb{R}^n)$, the space of continuous functions. The fundamental theorem of mean periodic functions, due to L. Schwartz [7] says that if f is mean periodic on $\mathbb{R}$ then $T(f)$ contains an exponential $e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. An exact analogue of this result fails to be true in the case of $\mathbb{R}^n, n \geq 2$ (see [6]). Nevertheless, a weaker version of Schwartz theorem is true in many situations including $\mathbb{R}^n$.

In the case of $\mathbb{R}^n$, it was proved by Brown et al in [4] that if $V \subset C(\mathbb{R}^n)$ is a closed subspace invariant under translations and rotations then V contains a function

$$\varphi_\lambda(x) = (\lambda|x|)^{\frac{-n}{2}+1} J_{\frac{n}{2}-1}(\lambda|x|)$$

for some $\lambda \in \mathbb{C}$. Note that φ_λ are the elementary spherical functions on the Euclidean space. A similar result for non compact symmetric spaces was established by Bagchi and Sitaram [2]. The case of the motion group $M(2)$ was considered by Weit [13]. In all these cases it was proved that the appropriate subspace V contains an elementary spherical function φ_λ.

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For an excellent survey of these results with applications to the Pompeiu problem we refer to Bagchi and Sitaram [3].

Our aim in this paper is to study mean periodic functions on the Heisenberg group H^n. If f is a mean periodic function on H^n and if $V(f)$ is the closed subspace of $C(H^n)$ invariant under translations and rotations then we can ask if $V(f)$ contains an elementary spherical function. In the case of H^n there are two types of spherical functions: one family

$$e_k^\lambda(z,t) = e^{i\lambda t}\phi_k^\lambda(z), \quad \lambda \in \mathbb{R}, \lambda \neq 0$$

where ϕ_k are Laguerre functions comes from the infinite dimensional Schrodinger representations π_λ; the other family

$$j_\lambda(z) = (\lambda|z|)^{-n+1} J_{n-1}(\lambda|z|), \quad \lambda \in \mathbb{R}, \lambda \geq 0$$

comes from the one dimensional representations.

It was proved by Agronovsky et al in [1] that if f is a bounded mean periodic function then $V(f)$ contains an elementary spherical function. For the reduced Heisenberg group we will prove that a similar result is true for any mean periodic function of tempered growth. For the general case we conjecture that the subspace $V(f)$ contains either an e_k^λ or $j_\lambda(z)$ with $\lambda \in \mathbb{C}$.

The study of mean periodic functions on H^n is closely related to the study of twisted mean periodic functions on the phase space $\mathbb{C}^n$. If f is a continuous function on $\mathbb{C}^n$ and if $V(f)$ is the closed subspace of $C(\mathbb{C}^n)$ invariant under twisted translations and rotations then we say that f is twisted mean periodic whenever $V(f)$ is proper. When f is a tempered continuous function we will show that $V(f)$ contains a ϕ_k. In the general case we conjecture that $V(f)$ contains a function of the form $e^{i[z,\zeta]}\phi_k(z)$. An affirmative answer to this conjecture will solve our conjecture on H^n.

The classical Paley-Wiener theorem for the Euclidean Fourier transform is an indispensable tool in the study of mean periodic functions on $\mathbb{R}^n$. In the same way to study twisted mean periodic functions we need a Paley-Wiener theorem for the Fourier-Weyl transform, which we prove in section 3. Relevant facts about Weyl transform and twisted convolution are collected in section 3. In sections 4 and 5 we study twisted mean periodic functions. In section 6 we apply these results to the case of the Heisenberg group. Finally, in section 7 we make some remarks on the twisted version of the Pompeiu problem.

2. Weyl transform, twisted convolution and special Hermite expansions.

In this section we collect some relevant facts about the Weyl transfrom, twisted convolution and special Hermite functions. These are all closely related to analysis on the Heisenberg group. This group denoted by H^n is simply $\mathbb{C}^n \times \mathbb{R}$ with coordinates $(z,t), z \in \mathbb{C}^n, t \in \mathbb{R}$. The group law is given by

$$(z,t)(w,s) = \Big(z + w, t + s + \frac{1}{2}\,\mathrm{Im}(z \cdot \overline{w})\Big). \tag{2.1}$$

Here $z \cdot \overline{w} = \sum_1^n z_j \overline{w}_j$. Given two functions F and G on H^n their convolution is defined by

$$F * G(z,t) = \int_{H^n} F(z - w, t - s - \frac{1}{2}\,\mathrm{Im}(z \cdot \overline{w}))G(w,s)dwds. \tag{2.2}$$

If we define $f(z) = \int F(z,t)e^{it}dt$ and $g(z) = \int G(z,t)e^{it}dt$ then it follows that

$$\int F * G(z,t)dt = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\,\mathrm{Im}(z \cdot \overline{w})}dw. \tag{2.3}$$

The right hand side is called the twisted convolution of f and g denoted by $f \times g(z)$.

In order to define the Weyl transform let us recall that all the infinite dimensional irreducible unitary representations of H^n are given by $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z)$ where $\lambda \in \mathbb{R}\backslash\{0\}$ and $\pi_\lambda(z)$ acts on $L^2(\mathbb{R}^n)$ as

$$\pi_\lambda(z)\varphi(\xi) = e^{i\lambda(x.\xi + \frac{1}{2}x.y)}\varphi(\xi + y) \tag{2.4}$$

where $x + iy = z, \xi \in \mathbb{R}^n$. Then $\pi(z) = \pi_1(z)$ defines a projective representation of $\mathbb{C}^n$. The Weyl transform $W(f)$ of a function f, say in $L^1(\mathbb{C}^n)$ is defined to be the operator

$$W(f) = \int_{\mathbb{C}^n} f(z)\pi(z)dz. \tag{2.5}$$

It can be shown that for f in $L^1 \cap L^2(\mathbb{C}^n), W(f)$ is a Hilbert-Schmidt operator and one has the Plancherel formula

$$\|f\|_2^2 = (2\pi)^{-n}\|W(f)\|_{HS}^2. \tag{2.6}$$

For all these facts and more we refer to the monograph of Folland [5].

If we take φ, ψ in $L^2(\mathbb{R}^n)$ it follows from (2.5) that

$$(W(f)\varphi, \psi) = \int_{\mathbb{C}^n} f(z)(\pi(z)\varphi, \psi)dz. \tag{2.7}$$

From this it is clear that if we want to extend the definition of $W(f)$ for f a distribution then we need to study functions of the form $(\pi(z)\varphi, \psi)$ which are the entry functions or matrix coefficients of the projective representation $\pi(z)$. The Fourier-Wigner transform of φ, ψ in $L^2(\mathbb{R}^n)$ is defined by

$$V_\varphi(\psi, z) = (2\pi)^{-\frac{n}{2}}(\pi(z)\varphi, \psi). \tag{2.8}$$

Using the properties of the Euclidean Fourier transform it is not difficult to prove the following proposition (see Folland [5] and Thangavelu [9]):

PROPOSITION 2.1. — *Let $\varphi, \psi \in L^2(\mathbb{R}^n)$ and let $V_\varphi(\psi)$ be their Fourier-Wigner transform. Then $V_\varphi(\psi) \in L^p(\mathbb{C}^n)$ for $p \geq 2$ and we have*

$$\|V_\varphi(\psi)\|_p \leq \|\varphi\|_2\|\psi\|_2.$$

When $p = 2$ we actually have equality.

From this proposition and (2.7) it follows that when $f \in L^p(\mathbb{C}^n)$, $1 \leq p \leq 2$ we have

$$|(W(f)\varphi, \psi)| \leq \|f\|_p\|\varphi\|_2\|\psi\|_2. \tag{2.9}$$

Hence $W(f)$ defines a bounded operator on $L^2(\mathbb{R}^n)$. From the equality $\|V_\varphi(\psi)\|_2 = \|\varphi\|_2\|\psi\|_2$ we get by polarisation

$$(V_\varphi(\psi), V_{\varphi'}(\psi')) = (\varphi, \varphi')(\psi', \varphi). \tag{2.10}$$

If we take $f = \overline{V}_\varphi(\psi)$ then from (2.10) we get the formula

$$W(f)\varphi' = (\varphi, \varphi')\psi. \tag{2.11}$$

Let us now take an orthonormal basis $\{\psi_j : j = 0, 1, 2, \cdots\}$ of $L^2(\mathbb{R}^n)$ and define functions ψ_{jk} on $\mathbb{C}^n$ by $\psi_{jk}(z) = V_{\psi_j}(\psi_k, z)$. Then from (2.10) it follows that $\{\psi_{jk}\}$ is an orthonormal system for $L^2(\mathbb{C}^n)$. From the formula (2.7) we also infer that $\{\psi_{jk}\}$ is actually an orthonormal basis for $L^2(\mathbb{C}^n)$.

The relation between the Weyl transform and the twisted convolution is given by

$$W(f \times g) = W(f)W(g). \tag{2.12}$$

Using this relation and (2.10) we can easily prove the following useful formula.

PROPOSITION 2.2. — *For f in $L^2(C^n)$*

$$f \times \psi_{kk}(z) = (2\pi)^n \sum_{j=0}^{\infty} (f, \psi_{jk})\psi_{jk}(z).$$

Proof. — Since $\{\psi_{jk}\}$ is an orthonormal basis we can expand f in terms of ψ_{jk} :

$$f = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(f, \psi_{jl})\psi_{jl}. \tag{2.13}$$

From (2.10) and (2.12) we have the formula

$$\psi_{jl} \times \psi_{kk} = \delta_{kl}(2\pi)^n \psi_{jk}. \tag{2.14}$$

If we use this in (2.13) we immediately get the proposition.

We can specialise the above discussion to the Hermite basis of $L^2(\mathbb{R}^n)$ to get special Hermite expansions. Let $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ be the set of all normalised Hermite functions on $\mathbb{R}^n$ which are eigenfunctions of the Hermite operator $(-\Delta + |x|^2)$. If we take $\varphi = \Phi_\alpha$ and $\psi = \Phi_\beta$ then $V_\varphi(\psi) = \Phi_{\alpha\beta}$ are called the special Hermite functions. Thus

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi}\Phi_\alpha\Big(\xi + \frac{1}{2}y\Big)\Phi_\beta\Big(\xi - \frac{1}{2}y\Big)d\xi. \tag{2.15}$$

The system $\{\Phi_{\alpha\beta}\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. The functions $\Phi_{\alpha\beta}$ can be expressed in terms of Laguerre functions. In particular when $n = 1$ we have the following formulas. Let $L_k^\alpha(t)$ be the k^{th} Laguerre polynomial of type $\alpha > -1$. Then

$$\Phi_{j,j+m}(z) = (2\pi)^{-\frac{1}{2}}\left(\frac{j!}{(j+m)!}\right)^{\frac{1}{2}}\left(\frac{-i}{\sqrt{2}}\right)^m z^m L_j^m\Big(\frac{1}{2}|z|^2\Big)e^{-\frac{1}{4}|z|^2}, \tag{2.16}$$

$$\Phi_{j+m,j}(z) = (2\pi)^{-\frac{1}{2}}\left(\frac{j!}{(j+m)!}\right)^{\frac{1}{2}}\left(\frac{i}{\sqrt{2}}\right)^m \bar{z}^m L_j^m\Big(\frac{1}{2}|z|^2\Big)e^{-\frac{1}{4}|z|^2}. \tag{2.17}$$

For the orthonormal basis of special Hermite functions Proposition 4.2 reads

$$f \times \varphi_k(z) = (2\pi)^n \sum_{|\beta|=k}\sum_{\alpha}(f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}. \tag{2.18}$$

Consequently, the special Hermite expansion of a function f can be put in the compact form

$$f = (2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k. \tag{2.19}$$

In these formulas φ_k stand for the Laguerre function

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)e^{-\frac{1}{4}|z|^2}. \tag{2.20}$$

For a variety of results concerning special Hermite series we refer to the monograph Thangavelu [9].

When f is in $L^2(\mathbb{C}^n)$ the series (2.18) converges in the L^2 norm. We can also prove much stronger convergence properties of the series when f is in the Schwartz class. Indeed, the functions φ_k are eigen functions of the elliptic operator

$$L = -\Delta + \frac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right) \tag{2.21}$$

with eigenvalue $(2k + n)$; that is $L\varphi_k = (2k + n)\varphi_k$. We also have $L(f \times \varphi_k) = (2k+n)f \times \varphi_k$. If we use this property then it is not difficult to show that (2.19) converges to f uniformly when f is in the Schwartz class. For applications we need to know when the series will converge in the topology of the Schwartz space $\mathcal{S}(\mathbb{C}^n)$.

We say that a function g defined on $\mathbb{C}^n$ is homogeneous of degree $m = (m_1, m_2, \cdots, m_n)$ where m_j are integers if

$$g(e^{i\theta}z) = e^{im\cdot\theta}g(z). \tag{2.22}$$

Here $\theta = (\theta_1, \cdots, \theta_n), e^{i\theta}z = (e^{i\theta_1}z_1, \cdots, e^{i\theta_n}z_n)$ and $m \cdot \theta = m_1\theta_1 + \cdots + m_n\theta_n$. When g is homogeneous of some degree m then it can be shown that $g \times \varphi_k$ reduces to a finite sum. We have the following proposition which will be used in the next section.

PROPOSITION 2.3. — *Let g be a Schwartz class function. Then the following are true:*

(i) *Finite linear combinations of homogeneous Schwartz class functions are dense in $\mathcal{S}(\mathbb{C}^n)$.*

(ii) *If g is homogeneous then it can be approximated by finite linear combinations of $\Phi_{\alpha\beta}$ in the topology of $\mathcal{S}(\mathbb{C}^n)$.*

For a proof of this proposition we refer to Thangavelu [12].

We conclude this section by collecting some facts about a class of Sobolev spaces defined using the Hermite operator $H = (-\Delta + |x|^2)$. Recall that the Hermite functions $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$. These are eigenfunctions of the Hermite operator, namely

$$H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{2.23}$$

On the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ introduce the norms

$$\|f\|_{(s)}^2 = \sum(2|\alpha| + n)^{2s}|(f, \Phi_\alpha)|^2. \tag{2.24}$$

Let $W_H^s(\mathbb{R}^n)$, called the Hermite Sobolev space, be the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the above norm. Then $W_H^s(\mathbb{R}^n)$ becomes a Hilbert space whose dual can be identified with $W_H^{-s}(\mathbb{R}^n)$. When $s = m$ is a nonnegative integer it can be shown that $f \in W_H^m(\mathbb{R}^n)$ if and only if

$$x^\alpha \partial_x^\beta f \in L^2(\mathbb{R}^n), |\alpha| + |\beta| \leq 2m. \tag{2.25}$$

For all these and more properties of $W_H^s(\mathbb{R}^n)$ we refer to Thangavelu [10].

3. Weyl transform of distributions and a Paley-Wiener theorem.

In this section we define Weyl transform of distributions and prove a Paley-Wiener theorem for compactly supported L^2 functions. We propose to use the relation $(W(f)\varphi, \psi) = (2\pi)^{\frac{n}{2}}(f, V_\varphi(\psi))$ in defining the Weyl transform of a distribution. As we have already seen this relation in conjunction with Proposition 2.1 allows us to define the Weyl transform of L^p functions as long as $1 \leq p \leq 2$. But if f is only a distribution then $(f, V_\varphi(\psi))$ need not make sense. In what follows we will show that when φ and ψ are in $\mathcal{S}(\mathbb{R}^n)$ then $V_\varphi(\psi)$ and its derivatives can be estimated in terms of the W_H^s norms of φ and ψ. Using this we will extend $W(f)$ as a bounded operator between $W_H^{s'}$ and W_H^s for suitable s and s'.

We prove the following extension of Proposition 2.1.

PROPOSITION 3.1. — *Let φ and ψ be Schwartz class functions on $\mathbb{R}^n$.*

(i) $\sum_{|\alpha|+|\beta|\leq 2m} \|x^\alpha y^\beta V_\varphi(\psi)\|_\infty \leq C\|\varphi\|_{(m)}\|\psi\|_{(m)}$.

(ii) $\sum_{|\alpha|+|\beta|\leq 2m} \|\partial_x^\alpha \partial_y^\beta V_\varphi(\psi)\|_\infty \leq C\|\varphi\|_{(m)}\|\psi\|_{(m)}$.

Proof. — The proof is easy. We only prove (ii), leaving the proof of (i) to the reader. Writing out the definition

$$V_\varphi(\psi, z) = (2\pi)^{-\frac{n}{2}} \int e^{ix\xi} \varphi\Big(\xi + \frac{1}{2}y\Big)\overline{\psi}\Big(\xi - \frac{1}{2}y\Big) d\xi \tag{3.1}$$

and differentiating we get

$$\frac{\partial}{\partial x_j} V_\varphi(\psi, z) = (2\pi)^{-\frac{n}{2}} \int e^{ix\xi} \; i\xi_j \varphi\Big(\xi + \frac{1}{2}y\Big)\overline{\psi}\Big(\xi - \frac{1}{2}y\Big) d\xi, \tag{3.2}$$

(3.3) $\frac{\partial}{\partial y_j} V_\varphi(\psi, z)$

$$= (2\pi)^{-\frac{n}{2}} \int e^{ix\xi} \left\{ \partial_j \varphi\left(\xi + \frac{1}{2}y\right) \overline{\psi}\left(\xi - \frac{1}{2}y\right) + \varphi\left(\xi + \frac{1}{2}y\right) \partial_j \overline{\psi}\left(\xi - \frac{1}{2}y\right) \right\} d\xi.$$

Writing $\xi_j = \frac{1}{2}\left(\xi_j + \frac{1}{2}y_j + \xi_j - \frac{1}{2}y_j\right)$ we have

$$\frac{\partial}{\partial x_j} V_\varphi(\psi) = -\frac{i}{2} V_{M_j\varphi}(\psi) + \frac{i}{2} V_\psi(M_j\psi) \tag{3.4}$$

where $M_j\varphi(\xi) = \xi_j\varphi(\xi)$. We also have

$$\frac{\partial}{\partial y_j} V_\varphi(\psi) = V_{\partial_j\varphi}(\psi) + V_\varphi(\partial_j\psi). \tag{3.5}$$

Iteration shows that $\partial_x^\alpha \partial_y^\beta V_\varphi(\psi)$ is a finite linear combination of terms of the form

$$\int e^{ix\xi} (M^{\alpha'} D^{\beta'} \varphi) \left(\xi + \frac{1}{2}y\right) (M^\alpha D^\delta \overline{\psi}) \left(\xi - \frac{1}{2}y\right) d\xi \tag{3.6}$$

with obvious notations. By Cauchy-Schwarz we get

$$\|\partial_x^\alpha \partial_y^\beta V_\varphi\|_\infty \leq \|M^{\alpha'} D^{\beta'} \varphi\|_2 \|M^\alpha D^\delta \overline{\psi}\|_2. \tag{3.7}$$

Since $|\alpha'| + |\beta'| \leq |\alpha| + |\beta|, |\gamma| + |\delta| \leq |\alpha| + |\beta|$ we get the proposition.

Using the proposition we can now define $W(f)$ when f is a tempered distribution.

THEOREM 3.1. — *Let f be a tempered distribution. Then $W(f)$ is a bounded operator from $W_H^m(\mathbb{R}^n)$ into $W_H^{-m}(\mathbb{R}^n)$ for some m.*

Proof. — Let g be in $\mathcal{S}(\mathbb{C}^n)$. Then for some m

$$|(f, g)| \leq C \sum_{|\alpha|+|\beta|\leq m} \sum_{|\gamma|+|\delta|\leq m} \|x^\alpha y^\beta \partial_x^\gamma \partial_y^\delta g\|_\infty \tag{3.8}$$

as f is a tempered distribution. If $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ then it follows from the proposition and the above observation that

$$|(W(f)\varphi, \psi)| = |(f, V_\varphi(\psi))| \leq C\|\varphi\|_{(m)} \|\psi\|_{(m)}. \tag{3.9}$$

This shows that $W(f)\varphi \in W_H^{-m}(\mathbb{R}^n)$ and also

$$\|W(f)\varphi\|_{(-m)} \leq C\|\varphi\|_{(m)}, \tag{3.10}$$

in view of the duality between W_H^m and W_H^{-m}. Hence $W(f)$ can be extended as a bounded operator mapping W_H^m into W_H^{-m}.

Thus we are able to define Weyl transform of tempered distributions as bounded operators between certain Sobolev spaces. We next investigate whether the Weyl transform is injective on the space of tempered distributions. An affirmative answer is given in the next theorem.

THEOREM 3.2. — *The Weyl transform is injective on $\mathcal{S}'(\mathbb{C}^n)$. That is $W(f) = 0$ implies $f = 0$ for f in $\mathcal{S}'(\mathbb{C}^n)$.*

Proof. — Observe that $(f, \Phi_{\alpha,\beta}) = (2\pi)^{\frac{n}{2}}(W(f)\Phi_\alpha, \Phi_\beta) = 0$ if $W(f) = 0$ for all α and β. In view of (ii) of Proposition 2.3 this means that $(f, g) = 0$ whenever $g \in \mathcal{S}(\mathbb{C}^n)$ is homogeneous. Again by (i) of the same proposition we get $(f, g) = 0$ for any Schwartz class function. Hence $f = 0$.

In the case of compactly supported distributions we can prove the following improvement of Theorem 3.2

THEOREM 3.3. — *Let f be a compactly supported distribution. Then for some $m, W(f)$ maps $W_H^m(\mathbb{R}^n)$ continuously into $L^2(\mathbb{R}^n)$.*

Proof. — Let K be the support of f. Then for some m we have

$$|(f, g)| \le C \sup_K \sum_{|\alpha|+|\beta|\le 2m} |\partial_x^\alpha \partial_y^\beta g|. \tag{3.11}$$

Let us estimate $\partial_x^\alpha \partial_y^\beta V_\varphi(\psi)$ on K. Write

$$V_\varphi(\psi) = (2\pi)^{-\frac{n}{2}} \int e^{i(x,\xi+\frac{1}{2}x,y)} \varphi(\xi + y)\overline{\psi}(\xi) d\xi. \tag{3.12}$$

This gives

$$\begin{aligned} \frac{\partial}{\partial x_j} V_\varphi(\psi, z) &= i V_{M_j\varphi}(\psi, z) - \frac{i}{2} y_j V_\varphi(\psi, z), \\ \frac{\partial}{\partial y_j} V_\varphi(\psi, z) &= V_{\partial_j\varphi}(\psi, z) + \frac{i}{2} x_j V_\varphi(\psi, z). \end{aligned} \tag{3.13}$$

Iteration shows that $\partial_x^\alpha \partial_y^\beta(\psi)$ is a linear combination of terms of the form

$$P_{\gamma\delta}(x, y) = \int e^{i(x.\xi+\frac{1}{2}x.y)} (M^\gamma D^\delta \varphi)(\xi + y)\overline{\psi}(\xi) d\xi \tag{3.14}$$

which gives the estimate

$$\sup_K \sum_{|\alpha|+|\beta|\le 2m} |\partial_x^\alpha \partial_y^\beta V_\varphi(\psi)| \le C(K) \|\varphi\|_{(m)} \|\psi\|_2. \tag{3.15}$$

From this we get the estimate

$$|(W(f)\varphi, \psi)| = |(f, V_\varphi(\psi))| \leq C(K)\|\varphi\|_{(m)}\|\varphi\|_2 \tag{3.16}$$

which shows that $W(f)\varphi \in L^2(\mathbb{R}^n)$ and $\|W(f)\varphi\| \leq C\|\varphi\|_{(m)}$. Hence the theorem.

We now turn our attention towards proving a Paley Wiener theorem for the Weyl transform. The study of mean periodic functions on $\mathbb{R}^n$ depends heavily on Paley-Wiener theorems for the Euclidean Fourier transform. In the same way we need an analogous theorem for the Weyl transform for the study of mean periodic functions on phase space. We now define the Fourier-Weyl transform and prove a Paley-Wiener theorem for compactly supported L^2 functions.

Let K stand for the space of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. This is a Hilbert space with the inner product $(T, S) = \operatorname{tr}(TS^*)$ and norm $\|T\|_{HS}^2 = \operatorname{tr}(TT^*)$. If $f \in L^2(\mathbb{C}^n)$ we know that $W(f) \in K$. We now embed $W(f)$ in a family of Hilbert-Schmidt operators in the following way. For $\xi \in \mathbb{R}^{2n}$ let us write $U(\xi) = \pi(\xi' + i\xi''), \xi = (\xi', \xi'')$ and π is the projective representation of $\mathbb{C}^n$ used to define $W(f)$. To each f in $L^2(\mathbb{C}^n)$ we now define the Fourier-Weyl transform by

$$\tilde{f}(\xi) = U(\xi)W(f)U(-\xi). \tag{3.17}$$

As $U(\xi)$ is unitary, $\tilde{f}(\xi) \in K$ for each $\xi \in \mathbb{R}^{2n}$.

The image of $L^2(\mathbb{C}^n)$ under the Fourier-Weyl transform thus consists of functions $F(\xi)$ taking values in K which verify the relation

$$F(0) = U(-\xi)F(\xi)U(\xi). \tag{3.18}$$

We now let E_0 stand for the subspace of this image whose elements are restrictions to $\mathbb{R}^{2n}$ of entire functions of exponential type taking values in K. In other words, $F \in E_0$ if and only if

(i) $F(\zeta)$ is an entire function of ζ in $\mathbb{C}^{2n}$ taking values in K and satisfies $\|F(\zeta)\|_{HS} \leq Ce^{B|\operatorname{Im}\zeta|}$ for some constant $B > 0$.

(ii) $F(0) = U(-\xi)F(\xi)U(\xi), \xi \in \mathbb{R}^{2n}$.

The space E_0 can be equipped with a topology as follows. Let $E'(\mathbb{R}^{2n}, K)$ stand for the space of compactly supported distributions in $\mathbb{R}^{2n}$ taking values in the Hilbert space K. That is any T in $E'(\mathbb{R}^{2n}, K)$ is a continuous linear operator from $C^\infty(\mathbb{R}^{2n})$ into K where $C^\infty(\mathbb{R}^{2n})$ is equipped with the topology of uniform convergence of all derivatives on compacta. To each T in $E'(\mathbb{R}^{2n}, K)$ we can define its Fourier transform

$\hat{T}(\xi) = (T, e_\xi)$ where $e_\xi(x) = \exp(ix\xi), x \in \mathbb{R}^{2n}$ is the exponential function. The classical Paley-Wiener theorem for compactly supported Hilbert space valued distributions says that $T \in E'(\mathbb{R}^{2n}, K)$ if and only if $\hat{T}(\xi)$ extends to an entire function of exponential type satisfying

$$\|\hat{T}(\zeta)\|_{HS} \leq C(1 + |\zeta|)^N e^{B|\operatorname{Im}\zeta|} \tag{3.19}$$

for some constants N and B. Let E stand for the image of $E'(\mathbb{R}^{2n}, K)$ under the Fourier transform.

The space E is equipped with the strong topology which makes the Fourier transform a topological isomorphism between $E'(\mathbb{R}^{2n}, K)$ and E. In this topology a sequence F_j converges to F iff the following two things happen:

(i) $F_j(\zeta) \to F(\zeta)$ uniformly on compact sets

(ii) F_j and F verify (3.19) with constants N and B independent of j.

The space E_0 inherits a topology from E . We claim that E_0 is a closed subspace of E.

PROPOSITION 3.2. — *E_0 is a closed subspace of E.*

Proof. — Suppose $F_j \in E_0$ and F_j converges to F in E. From the definition it is clear that $F_j(0)$ converges to $F(0)$ in K and hence $F(0) = W(f)$ for some f in $L^2(\mathbb{C}^n)$. We need to show that $F(\xi) = U(\xi)W(f)U(-\xi)$. To see that this is true we observe that

$$U(-\xi)F_j(\xi)U(\xi) - F(0) = U(-\xi)\left(F_j(\xi) - U(\xi)F(0)U(-\xi)\right)U(\xi). \tag{3.20}$$

This means that as $F_j \in E_0$

$$F_j(\xi) - U(\xi)F(0)U(-\xi) = U(\xi)\left(F_j(0) - F(0)\right)U(-\xi) \tag{3.21}$$

and consequently $F_j(\xi) - U(\xi)F(0)U(-\xi)$ converges to zero in K. But then we should have $F(\xi) = U(\xi)F(0)U(-\xi)$ which proves our claim.

We now state a Paley-Wiener theorem for the Fourier-Weyl transform. Let $L_0^2(\mathbb{C}^n)$ stand for the subspace of $L^2(\mathbb{C}^n)$ consisting of compactly supported functions.

THEOREM 3.4. — *The Fourier-Weyl transform sets up a topological isomorphism between $L_0^2(\mathbb{C}^n)$ and E_0.*

The proof of one implication is easy. In fact, one easily verifies that

$$U(\xi)\pi(z)U(-\xi) = e^{-i[z,\xi]}\pi(z) \tag{3.22}$$

where $[z,\xi] = x\cdot\xi'' - y\cdot\xi'$ is the symplectic form on $\mathbb{R}^{2n}$. This means that

$$\tilde{f}(\xi) = \int_{\mathbb{R}^{2n}} e^{-i(x\cdot\xi''-y\cdot\xi')} f(z)\pi(z)dxdy. \tag{3.23}$$

If now f is supported in $|z| \le B$ then $\tilde{f}(\xi)$ can be extended as an entire function of ζ in $\mathbb{C}^{2n}$ and we also have

$$\|\tilde{f}(\zeta)\|_{HS} \le Ce^{B|\operatorname{Im}\zeta|}. \tag{3.24}$$

The proof of the converse is a bit involved. We refer to the papers [8] and [11] from where a proof can be read out.

We conclude this section with the following remark. In view of (3.23) one has

$$(\tilde{f}(\xi)\varphi,\psi) = (2\pi)^{\frac{n}{2}}(f, V_\varphi^\xi(\psi)) \tag{3.25}$$

where $V_\varphi^\xi(\psi,z) = e^{-i[z,\xi]}V_\varphi^\xi(\psi,z)$. The equation (3.25) can be used to define $\tilde{u}(\xi)$ when u is a distribution of compact support. It is not difficult to show that $\tilde{u}(\xi)$ extends to an entire function of ζ taking values in the space of bounded operators from $W_H^m(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

4. Mean periodic functions of tempered growth.

In this section we consider mean periodic functions on the phase space $\mathbb{R}^{2n}$ which we identify with $\mathbb{C}^n$. If $z = x+iy$ and $w = u+iv$ are in $\mathbb{C}^n$ then

$$\operatorname{Im}(z\cdot\overline{w}) = u\cdot y - v\cdot x \tag{4.1}$$

is the symplectic form on $\mathbb{R}^{2n}$. We denote this by $[z,w]$. Let $\mathrm{Sp}(n)$ stand for the group of $2n\times 2n$ matrices that preserve the above symplectic form. It has been proved in Folland [5] that $\mathrm{Sp}(n)\cap O(2n) = U(n)$, the group of unitary matrices acting on $\mathbb{C}^n$. Therefore, if $\sigma\in U(n)$ then $[\sigma z,\sigma w] = [z,w]$. For $\sigma\in U(n)$ we define the operator R_σ acting on functions on $\mathbb{C}^n$ by

$$R_\sigma f(z) = f(\sigma z). \tag{4.2}$$

For each $w\in\mathbb{C}^n$ we also define the twisted translation operators $\tau(w)$ by

$$\tau(w)f(z) = f(z+w)e^{\frac{i}{2}[z,w]}. \tag{4.3}$$

Before taking up the study of mean periodic functions we consider subspaces of $L^p(\mathbb{C}^n)$ that are invariant under the action of R_σ and $\tau(w)$.

For $f \in L^p(\mathbb{C}^n), 1 \leq p < \infty$ we let $T_p(f)$ stand for the smallest closed subspace of $L^p(\mathbb{C}^n)$ which contains $\tau(w)f$ for all $w \in \mathbb{C}^n$. Likewise, we let $V_p(f)$ stand for the smallest closed subspace of $L^p(\mathbb{C}^n)$ which contains $\tau(w)f$ for all $w \in \mathbb{C}^n$ and $R_\sigma f$ for all $\sigma \in U(n)$. Note that $T_p(f)$ is invariant under the action of $\tau(w)$ and $V_p(f)$ under the actions of $\tau(w)$ and R_σ. An application of Hahn-Banach theorem shows that $T_p(f)$ is a proper subspace of $L^p(\mathbb{C}^n)$ if and only if there is a g in $L^{p'}(\mathbb{C}^n)$ such that $f \times g = 0$. In the case of $V_p(f)$ we can take g to be radial. To see this, note that $f \times g = 0$ means

$$\int h(z)g(-z)dz = 0 \tag{4.4}$$

for all $h \in V_p(f)$. Since $R_\sigma h \in V_p(f)$ whenever $h \in V_p(f)$ the above shows that

$$\int h(z)R_\sigma g(z)dz = 0 \tag{4.5}$$

for all $h \in V(f), \sigma \in U(n)$. If we let

$$Rg(z) = \int_{U(n)} R_\sigma g(z)d\sigma \tag{4.6}$$

stand for the radialisation of g then (4.5) says that Rg is orthogonal to all $h \in V_p(f)$ and consequently $f \times Rg = 0$.

Of course, we need to be assured that for some g with $f \times g = 0$ the radialisation Rg is nontrivial. If this were not true then for any g satisfying $f \times g = 0$ one would have $Rg = 0$. We claim this means all radial h belongs to $V_p(f)$. To see this if some radial h_0 were outside $V_p(f)$ then there will exist g with

$$\int h(z)g(z)dz = 0 \qquad \text{for all } \ h \in V_p(f) \tag{4.7}$$

but with

$$\int h_0(z)g(z)dz \neq 0. \tag{4.8}$$

But then as h_0 is radial (4.58) implies

$$\int h_0(z)Rg(z)dz \neq 0 \tag{4.9}$$

which in turn implies that Rg is nontrivial, a contradiction. Thus $Rg = 0$ for all g with $f \times g = 0$ implies $V_p(f)$ contains all radial functions. But this is possible only if $V_p(f) = L^p(\mathbb{C}^n)$ because $\varphi_k \in V_p(f)$ for all k implies $\varphi_k \times g = 0$ for all k whenever $f \times g = 0$ and hence $g = 0$. Thus in the case of $V_p(f)$ we are assured that there is a nontrivial radial g in $L^{p'}(\mathbb{C}^n)$ such that $f \times g = 0$. We now prove the following results concerning $T_p(f)$ and $V_p(f)$.

THEOREM 4.1. — (i) *Let $f \in L^p(\mathbb{C}^n), 1 \le p < \infty$ be radial. Then $T_p(f)$ is proper if and only if $f \times \varphi_k = 0$ for some k.*

(ii) *Let $f \in L^p(\mathbb{C}^n), 1 \le p < \infty$. Then $V_p(f)$ is proper if and only if $f \times \varphi_k = 0$ for some k.*

(iii) *Let $f \in L^2(\mathbb{C}^n)$. Then $T_2(f)$ is proper if and only if $f \times V_\varphi(\varphi) = 0$ for some $\varphi \in L^2(\mathbb{R}^n)$.*

Proof. — In proving (i) and (ii) we need the following fact about the special Hermite expansion of a radial function. Namely, if f is radial then $f \times \varphi_k = R_k(f)\varphi_k$ where

$$R_k(f) = C_n \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} f(z)\varphi_k(z)dz, \tag{4.10}$$

for some constant C_n. For a proof see [12].

If $T_p(f)$ is proper then $\exists\ g \neq 0$ in $L^{p'}(\mathbb{C}^n)$ such that $f \times g = 0$. Since g is tempered there is a k for which $\overline{g} \times \varphi_k \neq 0$ which also means $\varphi_k \times g \neq 0$. But then $f \times g = 0$ gives $0 = \varphi_k \times f \times g = R_k(f)\varphi_k \times g$ which implies $R_k(f) = 0$. This proves that $f \times \varphi_k = 0$. The converse is trivial. This proves (i). The proof of (ii) is similar to that of (i). So, we leave the details. In order to prove (iii) we make use of Proposition 2.2.

If $T_2(f)$ is proper then $\exists\ g$ in $L^2(\mathbb{C}^n)$ with $f \times g = 0$. If we let $f^v(z) = f(-z)$ and $g^v(z) = g(-z)$ then we also have $f^v \times g^v = 0$. Since $W(g^v) \neq 0$ there is a $\varphi \in L^2(\mathbb{R}^n)$ such that $\psi = W(g^v)\varphi \neq 0$ and $W(f^v)\psi = 0$. Let $\{\psi_j\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$ with $\psi_0 = \psi$ and define $\psi_{jk} = V_{\psi_j}(\psi_k)$ be their Fourier-Wigner transforms. Then one has $(W(f^v)\psi_0, \psi)_j) = 0$ for all j which means

$$\begin{aligned} \int f^v(z)(\pi(z)\psi_0, \psi_j)dz &= \int f(z)(\psi_0, \pi(z)\psi_j)dz \\ &= (2\pi)^{\frac{n}{2}} \int f(z)\overline{\psi}_{j0}(z)dz = 0. \end{aligned} \tag{4.11}$$

In view of Proposition 2.2 this means $f \times \psi_{oo} = 0$. This proves (iii).

We now consider the space $C(\mathbb{C}^n)$ of continuous functions on $\mathbb{C}^n$ equipped with the topology of uniform convergence on compact sets. Given a nontrivial continuous function f we let $T(f)$ (respectively $V(f)$) stand for the smallest closed subspace of $C(\mathbb{C}^n)$ containing f which is invariant under all $\tau(w)$ (resp. $\tau(w)$ and R_σ). We say that f is mean periodic if $T(f)$ is a proper subspace of $C(\mathbb{C}^n)$. When $V(f)$ is proper we call f spherically mean periodic. Equivalently, by Hahn-Banach, f is mean periodic if and only if there exists a compactly supported Radon measure μ such that $f \times \mu = 0$. Replacing μ by $\mu \times g$ with $g \in L_0^2$ we see that f is mean periodic if and only if $f \times g = 0$ for some $g \in L_0^2$. In the case of spherically mean periodic function we can choose μ and g to be radial.

The simplest example of a mean periodic function on $\mathbb{C}^n$ is given by the Laguerre function φ_k. Let μ_r stand for the normalised surface measure on the sphere $\{z \in \mathbb{C}^n : |z| = r\}$. Then it has been proved in [12] that

$$\varphi_k \times \mu_r(z) = \frac{k!(n-1)!}{(k+n-1)!}\varphi_k(r)\varphi_k(z) \tag{4.12}$$

where $\varphi_k(r) = L_k^{n-1}(\frac{1}{2}r^2)e^{-\frac{1}{4}r^2}$. If $r > 0$ is a zero of $\varphi_k(t)$ then it is clear that $\varphi_k \times \mu_r = 0$ and hence φ_k is spherically mean periodic. Note that φ_k is a Schwartz class function. This is in sharp contrast with the case of ordinary mean periodic functions on $\mathbb{R}^n$. As is well known no mean periodic function on $\mathbb{R}^n$ can be integrable. Thus though integrable mean periodic functions on $\mathbb{R}^n$ doesn't make sense we can study such mean periodic functions on the phase space $\mathbb{R}^{2n}$. Our first goal is to prove the following result which can be thought of as the analogue of Schwartz theorem for integrable mean periodic functions on $\mathbb{C}^n$.

THEOREM 4.2. — *Let $f \in L^p(\mathbb{C}^n), 1 \le p \le 2$ be mean periodic. Then there exists φ in $L^2(\mathbb{R}^n)$ such that $\overline{V}_\varphi(\varphi) \in T(f)$.*

Proof. — As f is mean periodic there is a nontrivial g in $L_0^2(\mathbb{C}^n)$ such that $f \times g = 0$. We let

$$T(f)^\perp = \{g \in L_0^2(\mathbb{C}^n) : f \times g = 0\}. \tag{4.13}$$

Defining $f^*(z) = \overline{f}(-z)$ we easily verify that $W(f^*) = W(f)^*$, the adjoint of $W(f)$. Therefore, if $g \in T(f)^\perp$ one has $W(g)^*W(f)^* = 0$. Since $W(f^*) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a nontrivial bounded operator there is a ψ in $L^2(\mathbb{R}^n)$ such that $\varphi = W(f)^*\psi \neq 0$.

We will show that $\overline{V}_\varphi(\varphi) \in T(f)$. To see this we first observe that

$$W(g^*)\varphi = 0 \quad \text{for all } g \in T(f)^\perp. \tag{4.14}$$

Proceeding as in the proof of Theorem 4.1 we get $\overline{g} \times V_\varphi(\varphi) = 0$ for all $g \in T(f)^\perp$. This proves $\overline{V}_\varphi(\varphi) \in T(f)$.

For the above proof the fact that $W(f)$ is bounded on $L^2(\mathbb{R}^n)$ is crucial. When f is only a tempered distribution with no integrability assumption $W(f)$ need not be a bounded operator. In the case of spherically mean periodic functions we can prove the following result for any function of tempered growth.

THEOREM 4.3. — *Let f be a continuous function of tempered growth. If f is spherically mean periodic then $\varphi_k \in V(f)$ for some k. Consequently $\Phi_{\alpha\beta} \in V(f)$ for all $\alpha \in \mathbb{N}^n$ and $|\beta| = k$.*

Proof. — We first claim that there exists a k such that $R_k(g) = 0$ for all radial g in $V(f)^\perp$. If the claim is not true then for each k we can find a radial $g \in V(f)^\perp$ such that $g \times \varphi_k \neq 0$. But then $f \times g = 0$ gives $f \times \varphi_k = 0$ as $R_k(g) \neq 0$ which will force f to be zero. Hence the claim is true.

Now let φ_k is such that $R_k(g) = 0$ for all g radial in $V(f)^\perp$. We need to show that $\varphi_k \in V(f)$. First observe that if μ is a radial measure such that $f \times \mu = 0$ then for any radial approximate identity g_n we have $\mu \times g_n$ converging to μ weakly. As $\varphi_k \times \mu \times g_n = 0$ for all n we also have $\int \varphi_k d\mu = 0$. If φ_k is not in $V(f)$ then we can find a compactly supported Radon measure μ such that

$$\int \varphi_k(z)d\mu(z) \neq 0, \quad \int h(z)d\mu(z) = 0 \text{ for all } h \in V(f). \tag{4.15}$$

But then the radial measure $\nu = R\mu$ satisfies

$$\int \varphi_k(z)d\nu \neq 0, \quad f \times \nu = 0 \tag{4.16}$$

which is a contradiction. Hence $\varphi_k \in V(f)$.

Finally, $\varphi_k \in V(f)$ implies $\varphi_k \times g = 0$ for all $g \in V(f)^\perp$. Consequently, $\overline{g} \times \varphi_k = 0$ or taking the Weyl transform $W(\overline{g})P_k = 0$ where P_k is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the kth eigen space of the Hermite operator $H = (-\Delta + |x|^2)$ spanned by $\Phi_\beta, |\beta| = k$ (see for example [9]). Therefore, for any β with $|\beta| = k$ and $\alpha \in \mathbb{N}^n$ one has $(W(\overline{g})\Phi_\beta, \Phi_\alpha) = 0$ which is the same as

$$\int \overline{g}(z)\Phi_{\beta\alpha}(z)dz = 0. \tag{4.17}$$

Since $\overline{\Phi}_{\beta\alpha}(-z) = \Phi_{\alpha\beta}(z)$ the above gives

$$\int \Phi_{\alpha\beta}(z)g^v(z)dz = 0 \tag{4.18}$$

for all $g \in V(f)^\perp$ which shows that $\Phi_{\alpha\beta} \in V(f)$.

5. Mean periodic functions of arbitrary growth.

In the case of mean periodic functions on the real line the celebrated theorem of Schwartz says that any translation invariant closed subspace of $C(\mathbb{R})$ contains an exponential $e^{i\lambda x}$ with $\lambda \in \mathbb{C}$. This follows from the following fundamental theorem of closed ideals in the space of entire functions of exponential type. Let $\hat{E}(\mathbb{R})$ stand for the space of entire functions of exponential type of one complex variable which is the same as the image under Fourier transform of the set of all compactly supported distributions. Then the fundamental theorem of Schwartz says that if I is a proper closed ideal in $\hat{E}(\mathbb{R})$ then I has a common zero. That is there is $\zeta \in \mathbb{C}$ such that $F(\zeta) = 0$ for all $F \in I$.

Given a mean periodic function f on $\mathbb{R}$ one can consider all compactly supported Radon measures μ verifying $f * \mu = 0$ and form the ideal

$$I = \{\hat{\mu}(\zeta) : f * \mu = 0\}. \tag{5.1}$$

As $f \neq 0$, I is a proper closed ideal in $\hat{E}(\mathbb{R})$. Hence there exists $\zeta \in \mathbb{C}$ such that $\hat{\mu}(\zeta) = 0$ for all μ satisfying $f * \mu = 0$ and this precisely means that $e^{ix\zeta}$ belongs to the translation invariant subspace spanned by f. The exact analogue of the above result turned out to be false in the case of $\mathbb{R}^n, n \geq 2$. It was shown in [6] that there exists six distributions μ_j with compact support such that the ideal generated by $\hat{\mu}_j, j = 1, 2, \cdots 6$ does not have a common zero. On the other hand positive results are known in the case of ideals that are invariant under rotations. It was proved in [4] that any such proper ideal of $\hat{E}(\mathbb{R}^2)$ contains an exponential $e^{i(x\zeta_1 + y\zeta_2)}$ for some $(\zeta_1, \zeta_2) \in \mathbb{C}^2$.

We would like to formulate analogous results for the case of mean periodic functions on $\mathbb{C}^n$. To this end we make use of the Paley-Wiener theorem for the Fourier-Weyl transform formulated in section 3. Let f be a mean periodic function and $T(f)$ be as in the previous section. Define $T(f)^{\perp} = \{g \in L_0^2 : f \times g = 0\}$. To each mean periodic function f we associate

$$I(f) = \{\tilde{g} : g^* \in T(f)^{\perp}\}.$$

Regarding $I(f)$ we have the following result.

PROPOSITION 5.1. — *Let f be mean periodic. Then $I(f)$ is a closed proper left ideal of E_0.*

Proof. — To show that $I(f)$ is an ideal, let $\tilde{g} \in I(f)$ and $\tilde{h} \in E_0$. Then $\tilde{h}(\zeta)\tilde{g}(\zeta) = (h \times g)^{\sim}(\zeta)$ which follows from the definition and

$(h \times g)^* = g^* \times h^*$ shows that $g^* \times h^* \in T(f)^{\perp}$. Hence $\tilde{h}\tilde{g} \in I(f)$. To show tht $I(f)$ is closed is E_0 assume that $\tilde{g}_n \in I(f)$ converges to $\tilde{g}$ in E_0. We need to show that $g \in T(f)^{\perp}$.

As $\tilde{g}_n$ converges to $\tilde{g}$ in E_0 we get $\tilde{g}_n(0)$ converges to $\tilde{g}(0)$ in K. This means $\|g_n - g\|_2 \to 0$. Moreover, all g_n and g are supported in a fixed compact set, say $|z| \leq B$. Thus we have

(5.2)

$$f \times g^*(z) = f \times (g^* - g_n^*)(z) = \int_{|w| \leq B} f(z-w)(g^*(w) - g_n^*(w)) e^{\frac{i}{2} \operatorname{Im} z, \overline{w}} dw$$

and an application of Cauchy-Schwarz gives

$$|f \times g^*(z)|^2 \leq \|g^* - g_n^*\|_2^2 \int_{|w| \leq B} |f(z-w)|^2 dw$$

which goes to zero as $n \to \infty$. This means $f \times g^* = 0$ and hence $g^* \in T(f)^{\perp}$.

Thus we have proved that $I(f)$ is a closed ideal in E_0. This cannot be the whole of E_0; for otherwise $f \times g = 0$ for all $g \in L_0^2$ which will then force $f = 0$ by an approximate identity argument.

Our next proposition shows that the ideal associated to a spherically mean periodic function has a nice invariance property. To state this we need to recall the definition of the metaplectic representation. For $\sigma \in U(n) = \operatorname{Sp}(n) \cap O(2n)$ we know that it preserves the symplectic form on $\mathbb{C}^n$ and hence $(z, t) \to (\sigma z, t)$ defines an automorphism of the Heisenberg group H^n. It $\pi_1(z, t)$ is the representation of H_n corresponding to the parameter $\lambda = 1$ then $(z, t) \to \pi_1(\sigma z, t)$ is also a representation agreeing with π_1 on the centre of the group. By a well known theorem of Stone-von Neumann it follows that $\pi_1(\sigma z, t)$ is unitarily equivalent to $\pi_1(z, t)$. Thus there exists a unitary operator $m(\sigma)$ such that

$$\pi(\sigma z) = m(\sigma)\pi(z)m(\sigma)^{-1}. \tag{5.3}$$

The structure of the group $\operatorname{Sp}(n)$ allows us to compute $m(\sigma)$ up to a constant multiple. A further analysis allows us to choose the constants in such a way that

$$m(\sigma)m(\sigma') = \pm m(\sigma\sigma'). \tag{5.4}$$

This permits us to lift $\sigma \to m(\sigma)$ into a single valued representation of the double cover of $\operatorname{Sp}(n)$ giving the metaplectic representation.

Now let f be spherically mean periodic and let $V(f)^{\perp}$ be defined as earlier. Define

$$J(f) = \{\tilde{g} : g^* \in V(f)^{\perp}\} \tag{5.5}$$

be the associated left ideal. For each $\sigma \in U(n)$ define the operator T_σ by

$$T_\sigma \tilde{g}(\zeta) = m(\sigma)^{-1}\tilde{g}(\sigma\zeta)m(\sigma). \tag{5.6}$$

A word of caution about our notation. When $\sigma = a + ib$ is a unitary matrix with a and b real matrices the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is in $\mathrm{Sp}(n)$ and for $\xi \in \mathbb{R}^{2n}, \sigma\xi$ will stand for $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\xi$. With this identification σ preserves the symplectic form: $[z, w] = [\sigma z, \sigma w]$ for $z, w \in \mathbb{C}^n$. If $\zeta \in \mathbb{C}^{2n}$ then $\sigma\zeta$ will stand for the element of $\mathbb{C}^{2n}$ obtained by applying the $2n \times 2n$ matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ to the $2n$-vector ζ. With the above notations the following proposition is an immediate consequence of the definition of $\tilde{g}$.

PROPOSITION 5.2. — *The ideal $J(f)$ is invariant under the action of $T_\sigma, \sigma \in U(n)$.*

Proof. — The invariance of $V(f)$ under the action of $U(n)$ shows that $g_\sigma \in V(f)^\perp$ whenever $g \in V(f)^\perp$ where $g_\sigma(z) = g(\sigma z)$. The proposition will follow once we establish

$$T_\sigma \tilde{g}(\zeta) = \tilde{g}_\sigma(\zeta). \tag{5.7}$$

To see this consider the Fourier-Weyl transform

$$\tilde{g}(\sigma\zeta) = \int e^{-i[z,\sigma\zeta]} g(z)\pi(z)dz. \tag{5.8}$$

By our previous remarks it follows that $[z, \zeta] = [\sigma z, \sigma\zeta]$ even for $\zeta \in \mathbb{C}^{2n}$. Therefore, (5.8) gives

$$\tilde{g}(\sigma\zeta) = \int e^{-i[z,\zeta]} g(\sigma z)\pi(\sigma z)dz. \tag{5.9}$$

From the definition of the metaplectic representation it follows that

$$\tilde{g}(\sigma\zeta) = m(\sigma)\tilde{g}_\sigma(\zeta)m(\sigma)^{-1} \tag{5.10}$$

which proves (5.7). Hence the proposition.

Having associated a closed left ideal with each spherically mean periodic function we may now ask whether the ideal has a common zero. The following proposition answers this question in the negative. The example in the proposition actually shows a much stronger result.

PROPOSITION 5.3. — *There is a spherically mean periodic function f and a compactly supported Radon measure μ such that $f \times \mu = 0$ but $\tilde{\mu}(\zeta)$ does not vanish for any ζ.*

Proof. — We prove the proposition by exhibiting a counter example. Recall that by a proper choice of r we can have $\varphi_k \times \mu_r = 0$. We claim $\tilde{\mu}_r(\zeta) \neq 0$ for any $\zeta \in \mathbb{C}^{2n}$. To prove this claim we assume $n = 1$ (just for the sake of simplicity of notations) and suppose $\tilde{\mu}_r(\zeta) = 0$ for some $\zeta \in \mathbb{C}^2$. Then if h_j are the normalised Hermite functions on $\mathbb{R}$ then we have

$$(\tilde{\mu}_r(\zeta)h_j, h_k) = 0 \qquad \text{for any } j, k. \tag{5.11}$$

This means that

$$\int_{|z|=r} e^{-i[z,\zeta]}\Phi_{jk}(z)d\mu_r = 0 \tag{5.12}$$

where Φ_{jk} are the special Hermite functions.

Using the explicit formulas for $\Phi_{jk}(re^{i\theta})$ we see that (5.12) becomes (in view of (2.16) and (2.17))

$$\int_0^{2\pi} e^{-i[re^{i\theta},\zeta]}e^{im\theta}d\theta = 0 \tag{5.13}$$

for all $m = 0, \pm 1, \pm 2, \cdots$. But this is not possible as the function $\theta \to e^{-i[re^{i\theta},\zeta]}$ is nontrivial. This contradiction proves the claim.

The above example shows that it is not reasonable to expect a common zero for $J(f)$. On the other hand we have proved (in Theorem 4.3) that if f is tempered and spherically mean periodic then $\exists \varphi_k$ in $V(f)$. This means that $\varphi_k \times g^* = 0$ if $g^* \in V(f)^\perp$ and consequently $g \times \varphi_k = 0$. This shows that $W(g)\Phi_\alpha = 0$ for all $|\alpha| = k$ and $g^* \in V(f)^\perp$. Since $W(g) = \tilde{g}(0)$ we can state Theorem 4.3 as follows: if f is spherically mean periodic and is of tempered growth then

$$\bigcap\{\operatorname{Ker}\tilde{g}(0) : g^* \in V(f)^\perp\} \tag{5.14}$$

is non empty. Therefore, it is reasonable to ask the following question even in the general case: Does there exist a $\zeta \in \mathbb{C}^{2n}$ such that

$$\bigcap\{\operatorname{Ker}\tilde{g}(\zeta) : g^* \in V(f)^\perp\} \tag{5.15}$$

is non empty? By considering the Hermite basis we rephrase the above question as follows. Let

$$J_\alpha(f) = \{\tilde{g}(\zeta)\Phi_\alpha : \tilde{g} \in J(f)\}. \tag{5.16}$$

We are interested in knowing if $J_\alpha(f)$ has a common zero for some α.

THEOREM 5.1. — *Let f be a spherically mean periodic function on $\mathbb{C}^n$. Then the following conditions are equivalent:*

(i) *ζ is a common zero of $J_\alpha(f)$*

(ii) $e^{i[z,\overline{\zeta}]}\Phi_{\alpha\alpha}(z) \in V(f)$.

Proof. — That (i) implies (ii) is clear since

$$\int e^{-i[z,\zeta]}\Phi_{\alpha\alpha}(z)g(z)dz = (\tilde{g}(\zeta)\Phi_\alpha, \Phi_\alpha) = 0 \tag{5.17}$$

for all $g^* \in V(f)^\perp$. This means that

$$\int e^{i[z,\overline{\zeta}]}\Phi_{\alpha\alpha}(z)\overline{g}(z)dz = 0 \tag{5.18}$$

and consequently $e^{i[z,\overline{\zeta}]}\Phi_{\alpha\alpha}(z)$ belongs to $V(f)$. Conversely, suppose we are given (ii) which means

$$\int e^{i[z-w,\overline{\zeta}]}\Phi_{\alpha\alpha}(z-w)e^{\frac{i}{2}\operatorname{Im} z.\overline{w}}g^*(w)dw = 0 \tag{5.19}$$

for all $g^* \in V(f)^\perp$. Recalling the definition of g^* the above means

$$\int e^{-i[w,\zeta]}\Phi_{\alpha\alpha}(z+w)e^{\frac{i}{2}\operatorname{Im} z\cdot\overline{w}}g(w)dw = 0$$

for all $g^* \in V(f)^\perp$.

Now we can expand $\Phi_{\alpha\alpha}(z+w)e^{\frac{i}{2}\operatorname{Im} z.\overline{w}}$ in terms of the orthonormal system $\{\Phi_{\beta\gamma}(z)\}$. Since

$$\int \Phi_{\alpha\alpha}(z+w)e^{\frac{i}{2}\operatorname{Im} z.\overline{w}}\overline{\Phi}_{\beta\gamma}(z)dz \tag{5.20}$$

$$= \int \Phi_{\alpha\alpha}(w-z)e^{\frac{i}{2}\operatorname{Im} w.\overline{z}}\Phi_{\gamma\beta}(z)dz$$

$$= \Phi_{\alpha\alpha} \times \Phi_{\gamma\beta}(w)$$

$$= \delta_{\alpha\gamma}(2\pi)^{\frac{n}{2}}\Phi_{\alpha\beta}(w), \tag{5.21}$$

in view of orthogonality properties of $\Phi_{\beta\gamma}$ we obtain

$$\Phi_{\alpha\alpha}(z+w)e^{\frac{i}{2}\operatorname{Im} z.\overline{w}} = (2\pi)^{\frac{n}{2}}\sum_\beta \Phi_{\alpha\beta}(w)\Phi_{\beta\alpha}(z). \tag{5.22}$$

Using (5.22) in (5.20) we get
(5.23)

$$\sum_\beta \Phi_{\beta\alpha}(z)\int e^{-i[w,\zeta]}\Phi_{\alpha\beta}(w)g(w)dw = \sum_\beta \Phi_{\beta\alpha}(z)(\tilde{g}(\zeta)\Phi_\alpha\Phi_\beta) = 0.$$

As $\{\Phi_{\beta\gamma}\}$ is orthonormal the above is possible only if

$$(\tilde{g}(\zeta)\Phi_\alpha, \Phi_\beta) = 0 \quad \text{for all } \beta \tag{5.24}$$

which means $\tilde{g}(\zeta)\Phi_\alpha = 0$. Since this is true for all $g^* \in V(f)^\perp$ this proves that (ii) implies (i).

In the above proof we haven't used the invariance of $V(f)$ under the action of $U(n)$ and as such the theorem remains true in the case of mean periodic functions also. In the case of spherically mean periodic function we have the following strengthening of the theorem. Let $J_k(f)$ be the span of $\{ J_\alpha(f) \ : |\ \alpha \ |= k\}$.

COROLLARY 5.1. — *Let f be a spherically mean periodic function. If ζ is a common zero of $J_k(f)$ and $\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{2n}^2 = a^2, a \in \mathbb{C}$ then any $w \in \mathbb{C}^{2n}$ with $w_1^2 + \cdots + w_{2n}^2 = a^2$ is also a common zero of $J_k(f)$.*

Proof. — The proof depends on yet another property of the metaplectic representation $m(\sigma)$. For each $\sigma \in U(n) = \mathrm{Sp}(n) \cap O(2n), m(\sigma)$ leaves invariant the eigen space spanned by $\{ \Phi_\alpha \ : |\ \alpha \ |= k\}$. For a proof of this see Folland [5].

If ζ and w are as in the hypothesis then $w = \sigma\zeta$ for some $\sigma \in U(n)$. Now

$$\tilde{g}(\sigma\zeta) = m(\sigma)\tilde{g}_\sigma(\zeta)m(\sigma)^{-1} \tag{5.25}$$

shows that

$$(\tilde{g}(\sigma\zeta)\Phi_\alpha, \Phi_\beta) = (\tilde{g}_\sigma(\zeta)m(\sigma)^{-1}\Phi_\alpha, m(\sigma)^{-1}\Phi_\beta) \tag{5.26}$$

and consequently

$$(\tilde{g}(\sigma\zeta)\Phi_\alpha, \Phi_\beta) = \sum_{|\nu|=k} c_{\alpha\nu}(\tilde{g}_\sigma(\zeta)\Phi_\nu, m(\sigma)^*\Phi_\beta). \tag{5.27}$$

If ζ is a common zero of $J_k(f)$ the right hand side is zero and hence $\tilde{g}(\sigma\zeta)\Phi_\alpha = 0$ for all $g^* \in V(f)^\perp$. This proves the corollary.

The above theorem strongly suggests that functions of the form $e^{i[z,\zeta]}\Phi_{\alpha\alpha}$ or more generally $e^{i[z,\zeta]}V_\varphi(\varphi)$ are the natural counterparts of the exponentials $e^{ix\zeta}$. The analogue of the Schwartz theorem will be the following :

If f is a spherically mean periodic function on $\mathbb{C}^n$ then for some $\zeta \in \mathbb{C}^{2n}$ and some $\varphi \in L^2(\mathbb{R}^n)$ the function $e^{i[z,\zeta]}V_\varphi(\varphi)$ belongs to $V(f)$.

We are unable to prove this conjecture except in the case of tempered mean periodic functions. General results concerning proper ideals in the space of entire functions of exponential type don't apply here as $J_\alpha(f)$ is not an ideal. Nevertheless, from $J_\alpha(f)$ we can form an ideal to which we can apply known results. But then the problem becomes that of checking whether the associated ideal is proper or not.

Let $\hat{E}(\mathbb{R}^{2n})$ stand for the space of entire functions of $\zeta \in \mathbb{C}^{2n}$ of exponential type. To $J_k(f)$ we associate an ideal $I_k(f) \subset \hat{E}(\mathbb{R}^{2n})$ as follows. $I_k(f)$ is the closed ideal generated by

$$\{(F(\zeta), \Phi_\beta) \ : \ F(\zeta) \in J_k(f), \ \beta \in \mathbb{N}^n\}. \tag{5.28}$$

In other words $I_k(f)$ is the closed ideal generated by

$$\{(\tilde{g}(\zeta)\Phi_\alpha, \Phi_\beta) \ : \mid \alpha \mid = k, \beta \in \mathbb{N}^n, g^* \in V(f)^\perp\}.$$

We have the following result:

THEOREM 5.2. — *Let f be a spherically mean periodic function. Then $V(f)$ contains an exponential $e^{i[z,\bar{\zeta}]}\Phi_{\alpha\alpha}(z)$, $\mid \alpha \mid = k$ if and only if $I_k(f)$ is a proper ideal in $\hat{E}(\mathbb{R}^{2n})$.*

Proof. — It is clear from the definition that $V(f)$ contains the exponential $e^{i[z,\bar{\zeta}]}\Phi_{\alpha\alpha}$, $\mid \alpha \mid = k$ if and only if ζ is a common zero of $I_k(f)$. So, it is enough to show that when $I_k(f)$ is proper then it has a common zero. To prove this we will apply the therorem of Brown et al which says that any proper closed ideal in $\hat{E}(\mathbb{R}^{2n})$ which is invariant under rotations contains a common zero.

Thus it is enough to verify that $I_k(f)$ is rotation invariant. But this is again a consequence of the invariance of $J_k(f)$ under the action of the metaplectic representation. If $F \in I_k(f)$ is of the form

$$F(\zeta) = \sum_{j=1}^{m} G_j(\zeta)(F_j(\zeta), \Phi_\beta)$$

then

$$F(\sigma\zeta) = \sum_{j=1}^{m} G_j(\sigma\zeta)(F_j(\sigma\zeta)\Phi_\alpha, \Phi_\beta).$$

As in the proof of corollary we can show that $(F_j(\sigma\zeta), \Phi_\beta)$ is again a linear combination of $(H_j(\zeta), \Phi_\mu)$ with $H_j(\zeta) \in J_k(f)$ and hence $F(\sigma\zeta) \in I_k(f)$ as well.

Thus an affirmative answer to our conjecture on mean periodic functions rests on a positive answer to the following question: if f is spherically mean periodic then is it true that for some k the ideal $I_k(f)$ is proper? As $I_k(f)$ is the ideal generated by $J_k(f)$ we are unable to come up with an answer to the above question. In future, we may have better luck.

6. Mean periodic functions on the reduced Heisenberg group.

In this section we study mean periodic functions on the reduced Heisenberg group G. Let H^n be the Heisenberg group defined in section 2 and let Γ be the subgroup $\{(0,k); k \in Z\}$. Then the group $G = H^n/\Gamma$ is called the reduced Heisenberg group or Heisenberg group with compact centre. Elements of G are still denoted by (z,t) with the understanding that $0 \leq t < 2\pi$. The relevant representations of G are $\pi_j(z,t)$ where j is a nonzero integer. For more about this group we refer to Folland [5]. We apply the results of the previous section to study mean periodic functions on G.

A continuous function f defined on G is said to be mean periodic if the closed subspace generated by f and all its translates is a proper subspace of $C(G)$ the space of continuous functions on G. Here the translation means the Heisenberg translation:

$$\tau(w,s)f(z,t) = f((z,t)(w,s)^{-1}) = f(z-w, t-s-\frac{1}{2}\operatorname{Im}(z.\overline{w})). \tag{6.1}$$

Equivalently, f is mean periodic if and only if there exists a compactly supported Radon measure μ such that $f * \mu = 0$. Again $f * \mu$ stands for the convolution of f and μ on the group G. In a similar fashion we say that f is spherically mean periodic on G if the smallest closed subspace of $C(G)$ generated by $f((z,t)(w,s)), (w,s) \in G$ and $f((\sigma z,t)), \sigma \in U(n)$ is proper. As before we denote the subspaces by $T(f)$ and $V(f)$.

From the results of the previous two sections we can deduce the following results concerning mean periodic functions on the reduced Heisenberg group.

THEOREM 6.1. — *Let f be an integrable mean periodic function on G. Then $T(f)$ contains either a function of the form $e^{ikt}(\varphi, \pi_k(z)\varphi)$ for some $k \in \mathbb{Z} \setminus \{0\}$ and $\varphi \in L^2(\mathbb{R}^n)$ or a function $g(z)$ independent of t.*

Proof. — Since f is nontrivial, for some integer k the function

$$f_k(z) = \int_0^{2\pi} f(z,t)e^{-ikt}dt \tag{6.2}$$

is a nontrivial function on $\mathbb{C}^n$. Let $\tau_k(w)$ be the k-twisted translations on $\mathbb{C}^n$ defined by

$$\tau_k(w)g(z) = g(z+w)e^{i\frac{k}{2}\operatorname{Im} z.\overline{w}}. \tag{6.3}$$

Then whatever we have proved in the case of twisted mean periodic functions remains true in the case of the (obviously defined) k-twisted mean periodic functions. In particular, Theorem 4.2 has an analogue for such functions. Our strategy is to apply Theorem 4.2 to f_k. From equation (6.2) it follows that

$$e^{iks} f_k(z+w)e^{i\frac{k}{2}\operatorname{Im} z.\overline{w}} = \int_0^{2\pi} f((z,t)(w,s))e^{-itk}dt \tag{6.4}$$

which shows that functions of the form

$$e^{ikt} f_k(z+w)e^{i\frac{k}{2}\operatorname{Im} z.\overline{w}} \tag{6.5}$$

belong to $T(f)$. First assume that $k \neq 0$.

Let $T_k(f_k)$ stand for the smallest closed subspace of $C(\mathbb{C}^n)$ generated by $\tau_k(w)f_k$. Then by the remarks we made regarding Theorem 4.2 we can conclude that for some $\varphi \in L^2(\mathbb{R}^n)$ the function $(\varphi, \pi_k(z)\varphi)$ belongs to $T_k(f_k)$. In view of (6.5) it is now immediate that $e^{ikt}(\varphi, \pi_k(\varphi)) \in T(f)$.

Now if $k = 0$ then (6.5) shows that $T(f)$ contains $g(z)$ whenever g belongs to the ordinary translation invariant subspace generated by f_0. But as f_0 is integrable and nontrivial this space is the whole of $C(\mathbb{C}^n)$. This completes the proof of the Theorem.

In a similar way we can prove the following analogue of Theorem 4.3 for spherically mean periodic functions of tempered growth on G.

THEOREM 6.2. — *Let f be a spherically mean periodic function of tempered growth. Then $V(f)$ contains either an exponential $\exp(i(x \cdot \zeta' + y \cdot \zeta''))$ with $(\zeta', \zeta'') = \zeta \in \mathbb{C}^{2n}$ or a function of the form $e^{-ijt}\varphi_k^j(z)$ where $\varphi_k^j(z) = \varphi_k(|j|z)$ for some $j \in \mathbb{Z} \setminus \{0\}$ and a nonnegative integer k.*

Proof. — The proof is similar to that of the previous theorem. If for some $j \neq 0$ the function

$$f_j(z) = \int_0^{2\pi} f(z,t)e^{-ijt}dt$$

is non trivial then we can appeal to an analogue of Theorem 4.3 for the subspace $V_j(f_j)$ to conclude that $V(f)$ contains $e^{-ijt}\varphi_k^j(z)$. If $j = 0$ then the ordinary translation and rotation invariant subspace of $\mathbb{C}^n$ generated by f_0 will contain and exponential by the theorem of Brown et al [4]. The same exponential will then belong to $V(f)$. The details are left to the reader.

We conclude this section by stating the following conjecture: if f is a spherically mean periodic function (of arbitrary growth) then $V(f)$ contains either an exponential or a function of the form $e^{i[z,\zeta]}e^{-ijt}\varphi_k^j(z)$ for some $\zeta \in \mathbb{C}^{2n}, j \in \mathbb{Z}\setminus\{0\}$ and k a nonnegative integer. An affirmative answer to this conjecture depends on an answer to the corresponding conjecture on $\mathbb{C}^n$ for twisted mean periodic functions.

7. The Pompeiu problem with a twist.

The Pompeiu problem extensively studied by several authors is very closely related to the theory of mean periodic functions. Before introducing the twisted version of the Pompeiu problem let us recall the formulation of the problem on $\mathbb{R}^n$. A compactly supported Radon measure μ is said to have the Pompeiu property if there is no nontrivial continuous function f satisfying

$$f_\sigma * \mu = 0, \qquad \text{for all } \sigma \in SO(n). \tag{7.1}$$

A bounded measurable subset $D \subset \mathbb{R}^n$ is said to have Pompeiu property if the characteristic function χ_D has the same, *i.e.*,

$$f_\sigma * \chi_D = 0 \qquad \text{for all } \sigma \in SO(n). \tag{7.2}$$

It is known (and not difficult to prove) that no ball $B_r(x_0)$ of radius $r > 0$ in $\mathbb{R}^n$ has the Pompeiu property. An open problem that goes under the name of Pompeiu problem is to show that any simply connected bounded set D with real analytic boundary without the Pompeiu property is a ball. For an excellent survey of the Pompeiu problem see Bagchi-Sitaram [3] and for an extensive bibliography we refer to [14].

Motivated by the equation (7.1) we make the following definition. We say that a compactly supported Radon measure μ on $\mathbb{C}^n$ has the twisted Pompeiu properly if there is no nontrivial continuous function f satisfying

$$f_\sigma \times \mu = 0 \qquad \text{for all } \sigma \in U(n). \tag{7.3}$$

Similar definition holds for bounded measurable sets. We also say that μ has the weak twisted Pompeiu property if there is no nontrivial continuous function f of tempered growth satisfying (7.3). From the results of sections 4 and 5 we will deduce some results concerning the twisted Pompeiu property.

In the case of $\mathbb{R}^n$ it is well known that balls and spheres fail to have the Pompeiu property. But it is not true in the case of weak twisted Pompeiu property.

THEOREM 7.1. — *The measure μ_r has the weak twisted Pompeiu property if and only if r is not a zero of the function $\varphi_k(t)$ for any k.*

Proof. — If r is a zero of φ_k then it follows from (4.12) that $\varphi_k \times \mu_r = 0$ and hence μ_r fails to have the weak twisted Pompeiu property. Conversely suppose μ_r has the weak twisted Pompeiu property and let f be such that (7.3) is true. Then f is a spherically mean periodic function of tempered growth and hence by Theorem 4.3 $\varphi_k \times \mu_r = 0$ for some k. But this means $\varphi_k(r) = 0$.

Likewise in the case of balls also the weak twisted Pompeiu property depends on the radius.

THEOREM 7.2. — *Let B_r be the ball $|z| \le r$ in $\mathbb{C}^n$. Then there is a countable set Q such that for any $r \notin Q$ the ball B_r has the weak twisted Pompeiu property and for $r \in Q, B_r$ fails to have the property.*

Proof. — If B_r fails to have the weak twisted Pompeiu property then as before for some k we should have $\varphi_k \times \chi_{B_r} = 0$ which means

$$\int_0^r \varphi_k(t)t^{2n-1}dt = 0. \tag{7.4}$$

The above function can be explictly calculated to be

$$Q_k(r) = \int_0^r \varphi_k(t)t^{2n-1}dt = c_k + e^{-\frac{r^2}{2}}P_k(r) \tag{7.5}$$

where C_k is a constant and P_k a polynomial of degree $(k+n-1)$ and it has been shown in [1] that $Q_k(r)$ has zeros on $(0,\infty)$. If we let Q stand for the set of all zeros of Q_k for all $k = 0, 1, 2, \cdots$ then it is clear that $B_r(0)$ fails to have the weak twisted Pompeiu property iff $r \in Q$.

Another example of a set with the weak twisted Pompeiu property is given by the sector

$$S_{\alpha\beta}(r) = \{z = \rho e^{i\theta} : 0 \le \rho \le r, \alpha \le \theta \le \beta\} \tag{7.6}$$

in $\mathbb{C}$. We have:

THEOREM 7.3. — *$S_{\alpha\beta}(r)$ has the weak twisted Pompeiu property if $(\beta-\alpha)$ is not a rational multiple of 2π.*

Proof. — Again the proof uses Theorem 4.3. If $S_{\alpha\beta}(r)$ fails to have the Pompeiu property then for some k and all $j=0,1,2,\cdots$ we should have

$$\int_{S_{\alpha\beta}(r)} \Phi_{jk}(-z)dz = 0 \tag{7.7}$$

which means

$$\int_0^r \int_\alpha^\beta \Phi_{jk}(-\rho e^{i\theta})\rho d\theta d\rho = 0. \tag{7.8}$$

Using the explicit formulas (2.16) and (2.17) we see that the inner integral

$$\int_\alpha^\beta \Phi_{jk}(-\rho e^{i\theta})d\theta \neq 0 \tag{7.9}$$

as $(\beta-\alpha)$ is not a rational multiple of 2π. But then (7.8) cannot hold for all j as (7.9) involves various Laguerre functions. Hence $S_{\alpha\beta}(r)$ has the weak twisted Pompeiu property.

From Theorem 4.3 we know that $D\subset\mathbb{C}^n$ fails to have the weak twisted Pompeiu property iff $\varphi_k\times\chi_D=0$ for some k. This is the analogue of the condition $\widehat{\chi}_D(\zeta)=0$ for all $\zeta_1^2+\zeta_2^2=\alpha^2$ for some $\alpha\in\mathbb{C}$ in the case of the Pompeiu problem on $\mathbb{R}^2$. Since calculating the Weyl transform of χ_D is more difficult than calculating the Fourier transform deciding when $\varphi_k\times\chi_D=0$ for some k seems to be very difficult unless D has some symmetry properties. As in the case of the Pompeiu problem we can now ask the following questions: if D a bounded simply connected set in $\mathbb{C}^n$ with smooth boundary fails to have the weak twisted Pompeiu property, then is it true that $D=B_r(z_0)$ for some $r>0$ and $z_0\in\mathbb{C}^n$?

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