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FIXED POINTS FOR REDUCTIVE GROUP ACTIONS ON ACYCLIC VARIETIES

by Martin FANKHAUSER [(1)]

1. Introduction.

The Fixed Point Problem. The base field will be the field of complex numbers $\mathbb{C}$ throughout the paper. Let G be a reductive algebraic group acting algebraically on affine n-space $\mathbb{A}^n$. The *Fixed Point Problem* asks whether every such action has fixed points, see [Kr89b]. In this paper, we consider the following, more general problem : Let G be a reductive group, and X a variety with an algebraic G-action. Then X is called a G-variety. The variety X has the structure of a complex analytic space in a canonical way. The corresponding *strong* topology will be used to consider the singular cohomology ring $H^*(X; A)$, where A will always denote either the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$ or the field $\mathbb{Z}_p$ with p elements. If $H^*(X; A) = A$, i.e., if X has the A-cohomology of a point, then X is called *A-acyclic.* Now the problem can be put this way : If X is a smooth affine and A-acyclic G-variety, what can be said about the set of fixed points X^G ? In particular, is $X^G \neq \varnothing$?

The following results are well known.

Smith Theory, see [Br], Chapter III :

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(1) If G is a torus and X is A-acyclic, then X^G is A-acyclic.

(2) If G is a finite p-group and X is $\mathbb{Z}_p$-acyclic, then X^G is $\mathbb{Z}_p$-acyclic.

Petrie-Randall [PR], p.210, see also Verdier [Ve] : Let G be a finite group having a normal series $P \subset H \subset G$, where P is a p-group, G/H is a q-group (p, q prime) and H/P is a cyclic group. If X is $\mathbb{Z}_p$-acyclic, then the Euler characteristic of the fixed point set is $\chi(X^G) \equiv 1 (\mathrm{mod}\ q)$, and $\chi(X^G) = 1$ if G/H is trivial.

Luna-Kraft-Schwarz [KS], p.4 : If X is A-acyclic and $\dim X/\!\!/G = 1$, then X^G is either a point, or $X^G \cong \mathbb{A}$.

Here $X/\!\!/G$ denotes the algebraic quotient for the action of G on X, i.e., the affine variety corresponding the $\mathbb{C}$-algebra of invariant functions on X (see [Kr84], II.3.2). Note that in all these cases X^G is not empty, and in the situation of Smith Theory as well as in the Situation of Luna-Kraft-Schwarz, X^G is even connected. We will use Smith Theory and Petrie-Randall as the cornerstones for fixed point theorems on semi-simple group actions.

Our first result shows that the dimension of the quotient $X/\!\!/G$ behaves reasonably if G is semisimple. Note that the hypothesis is satisfied if X is A-acyclic (by Smith Theory), and that for $X = \mathbb{A}^n$, the result follows from the factoriality of X.

THEOREM A. — *Let G be a semi-simple group, and X a smooth affine G-variety with non-empty and connected fixed point set X^T, $T \subset G$ a maximal torus. Then the generic fiber of the quotient map $\pi_X : X \to X/\!\!/G$ contains a dense orbit.*

There is an extensive literature on differentiable actions of compact transformation groups on acyclic manifolds. One of the results is, that in order to get fixed points, one has to impose some kind of smallness condition on the action, e.g. by limiting the number of orbit types (see [HS82]) or restricting the dimension of the orbit space (see [HS86]). This was the motivation to study the analogous problem in the algebraic setting. We prove the following two theorems :

THEOREM B. — *Let G be a simple group of rank n, and X a smooth affine and $\mathbb{Z}_2$-acyclic G-variety. If G has no fixed points, then $\dim X/\!\!/G > n - \log_2 n$.*

This result can be strengthened considerably, see the table on page 4.

THEOREM C. — *Let G be a connected reductive group, and X a smooth affine and $\mathbb{Z}$-acyclic G-variety.*

(1) *If* $\dim X/\!/G \leqslant 2$, *then* X^G *is $\mathbb{Z}$-acyclic.*

(2) *If* $\dim X/\!/G = 3$, *then* X^G *is not empty.*

The problem of constructing fixed point free actions for reductive groups on $\mathbb{A}^n$ or even on acyclic varieties is completely open. Note however that if G is not reductive, then there are *affine* actions of G on some $\mathbb{A}^n$ without fixed points, cf. [KP], p.479.

This paper grew out of the author's thesis [Fa], written under the direction of Hanspeter Kraft. I thank Hanspeter Kraft and Gerald Schwarz for their constant support and encouragement, Friedrich Knop, Peter Littelmann and Eldar Straume for their help.

Conventions and notation. For the rest of this paper, a variety is always tacitly assumed to be affine and smooth. If G is an algebraic group, we write G^0 for its identity component and we use the german letter $\mathfrak{g}$ to denote its Lie algebra. If T is a torus, we denote by $\mathcal{X}(T)$ its character group. Let G be connected reductive and $T \subset G$ is a maximal torus. We denote $R(G) \subset \mathcal{X}(T)$ the roots of G, and for $\alpha \in R(G)$ we have the associated reflection s_α on $\mathcal{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$. There is a linear form $\langle \alpha, ? \rangle : \mathcal{X}(T) \to \mathbb{Z}$ such that $s_\alpha(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha$ for every $\lambda \in \mathcal{X}(T)$. We call a subset $\Pi \subset \mathcal{X}(T)$ *α-saturated* if $\lambda - i\alpha \in \Pi$ for every $\lambda \in \Pi$ and any integer i between 0 and $\langle \alpha, \lambda \rangle$. It is well-known that the weight system of a G-module is α-saturated for any root α. We always assume chosen a fundamental Weyl chamber $\mathcal{C}(G) \subset \mathcal{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that it makes sense to talk about Weyl chambers even if G is not semi-simple, e.g. if G is a torus, then $\mathcal{C}(G) = \mathcal{X}(G) \otimes_{\mathbb{Z}} \mathbb{R}$. For the simple groups, their roots and their weights we use the notation of Bourbaki [Bo]. For $\omega \in \mathcal{X}(T)$ a dominant weight, we let V_ω denote the irreducible G-module with highest weight ω. If we want to emphasize the group which is acting, then we write $V_\omega(G)$ instead. θ will always denote the one-dimensional trivial representation. The direct sum of m copies of a representation V will be denoted by mV.

Smoothness of fixed point sets. The following proposition (see [Fa], p.9) is a corollary of the Slice Theorem [Lu]. We will use it to reduce some problems to considering actions on the fixed point set of subgroups.

PROPOSITION. — *Let G be a reductive group, and X a smooth affine G-variety. Then X^H is smooth for any (not necessarily reductive) subgroup $H \subset G$.*

Remark. — Bass [Ba] was the first to construct an action of the unipotent group $(\mathbb{C},+)$ on $\mathbb{C}^3$ which is not triangular : the action has a singular fixed point set. The corollary implies that an action of $(\mathbb{C},+)$ with singular fixed point set cannot be extended to an action of a reductive group containing $(\mathbb{C},+)$.

2. Leitfaden.

We describe the main steps in the proof of Theorems B and C, under the assumption that Theorem A is already proved.

Let G be a connected reductive group with a maximal torus T. Let X be a G-variety such that X^T is non-empty and connected. The following definition, due to Wu-Yi Hsiang (cf. [Hs]), generalizes the well-known definition of the weight system of a G-module. Choose $x \in X^T$, and put

$$\Sigma(X) = \text{the isomorphism type of the } T\text{-module } T_xX.$$

Since by hypothesis X^T is connected, the T-isomorphism type of every tangential representation $T_xX, x \in X^T$ is the same (cf. [Kr89a], p.112/113), and so $\Sigma(X)$ does not depend upon the choice of $x \in X^T$. We call $\Sigma(X)$ the *weight system* of the action, since we can think of it as a set of weights of T with multiplicities. We denote by $\Sigma'(X)$ the set (with multiplicities) of non-zero weights in $\Sigma(X)$.

Remark. — If X is a G-module, then its weight system $\Sigma(X)$ determines the isomorphism type of the representation completely. However, Schwarz' counterexample [Sch] to the Linearization Problem shows that there are families of non-isomorphic actions on $\mathbb{A}^n$ which have the same weight system.

Denote $W(G) = W$ the Weyl group of G. There is a canonical action of W on the character group $\mathcal{X}(T)$. Using the action of W on the connected set X^T induced by the action of the normalizer $\mathrm{Nor}_G(T)$ on X^T, one proves (cf. [Hs], p.37) :

PROPOSITION 2.1. — *The weight system $\Sigma(X)$ is stable under the Weyl group W.*

From now on let X be an A-acyclic G-variety. Then X^T is non-empty and connected by Smith Theory. Thus $\Sigma(X)$ is defined, and if G is semisimple, the hypothesis of Theorem A is satisfied.

The next theorem is a direct translation of a result of Wu-Yi Hsiang (cf. [HH70], p.207) to the algebraic setting.

THEOREM 2.2. — *If $\Sigma(X) \cap R(G) = \varnothing$, then $X^G = X^T$. In particular, X^G is A-acyclic, and* $\dim X^G$ *is the multiplicity of the zero weight in* $\Sigma(X)$.

Proof. — Choose $x \in X^T$. Then $T_x(Gx) \cong \mathfrak{g}/\mathfrak{g}_x \subset T_xX$ as G_x-modules. Restricting to the T-action on T_xX we get that $R(G) - R(G_x^0) \subseteq \Sigma(X)$, hence by hypothesis $R(G) = R(G_x^0)$. This implies that $G_x^0 = G^0 = G$ and $X^T = X^G$. The rest follows from Smith Theory and Luna's Slice Theorem. □

Combining Proposition 2.1 and Theorem 2.2 one sees that $\Sigma(X)$ contains at least one W-orbit of roots if X^G is not A-acyclic, e.g. if the action has no fixed points.

For technical reasons, which will become apparent during the proof of Theorem C, we have to strengthen slightly the statement of Theorem B. Let $x \in X$ be on a closed G-orbit. We will use the notation $\tilde{N}_x$ for the largest G_x^0-submodule in the slice N_x without fixed lines, i.e., we decompose $N_x = N_x^{G_x^0} \oplus \tilde{N}_x$. We denote

$$d(X) := \max\{\dim \tilde{N}_x/\!/G_x^0 \mid x \in X^T\}.$$

PROPOSITION 2.3. — *Let G be a semi-simple group, and X an A-acyclic G-variety. Then* $\dim X/\!/G \geqslant d(X) \geqslant \dim \Sigma'(X) - \dim G$.

Proof. — Fix $x \in X^T$. By the Slice Theorem it follows that

$$\dim X/\!/G = \dim N_x/\!/G_x = \dim N_x/\!/G_x^0 \geqslant \dim \tilde{N}_x/\!/G_x^0,$$

thus $\dim X/\!/G \geqslant d(X)$. The second inequality follows from $\dim \tilde{N}_x - \dim G_x^0 \geqslant \dim \Sigma'(X) - \dim G$, and by Theorem A we have that $\dim \tilde{N}_x/\!/G_x^0 \geqslant \dim \tilde{N}_x - \dim G_x^0$. □

Let now G be a simple group, and denote n its rank. To simplify our discussion, we assume that G is simply laced. Since W acts transitively on $R(G)$, our discussion shows that if X^G is not A-acyclic, then $R(G) \subset \Sigma'(X)$. On the other hand, if $\dim(\Sigma'(X) - R(G)) \geqslant 2n$, then $\dim X/\!/G \geqslant d(X) \geqslant n$.

Thus in order to prove Theorem B we only have to consider actions with : (a) $R(G) \subset \Sigma'(X)$, and (b) $\dim(\Sigma'(X) - R(G)) < 2n$.

Let X be a $\mathbb{Z}_2$-acyclic G-variety, such that $\Sigma(X)$ satisfies conditions (a) and (b). We will determine a reductive subgroup $G' \subset G$ with $T \subset G'$, such that $X^{G'} \neq \varnothing$ and $\dim \tilde{N}_x /\!/ G_x^0 \geqslant n - \log_2 n$ for $x \in X^{G'}$. This yields Theorem B.

Of course, this strategy needs some modifications if G is not simply-laced. More precisely, in §5-9 we show the results in the following table :

type	If X^G is	then $d(X)$ is	$\Sigma(X)$
A_1	not $\mathbb{Z}_2$-acyclic	$\geqslant 2$	$d(X) = 2 \Leftrightarrow \Sigma(X) = \Sigma(V_{2\omega_1})$
A_1	empty	$\geqslant 5$	
A_n, $n = 2, 3$	not $\mathbb{Z}_2$-acyclic	$\geqslant n$	$d(X) = n \Leftrightarrow \Sigma'(X) = R(\mathrm{A}_n)$
A_n, $n = 2, 3$	empty	$\geqslant 16, 33$	
A_n, $n > 3$	not $\mathbb{Z}_2$-acyclic	$> n - \log_2 n$	
C_n, $n = 3, 4$	not $\mathbb{Z}_2$-acyclic	$\geqslant n - 1$	$d(X) = n - 1 \Leftrightarrow \Sigma(X) = \Sigma(V_{\omega_2})$ or $\Sigma(V_{\omega_2}) \oplus \Sigma(V_{\omega_1})$
C_n, $n = 3, 4$	empty	$\geqslant 21, 44$	
C_n, $n > 4$	not $\mathbb{Z}_2$-acyclic	$> n - \log_2 n$	
B_n, $n \leqslant 4$	not $\mathbb{Z}_2$-acyclic	$\geqslant 2n - 1$	$d(X) \leqslant 2n \Leftrightarrow \Sigma'(X) = \Sigma'(V_{2\omega_1})$
D_4	empty	$\geqslant 44$	
$\mathrm{B}_n, \mathrm{D}_n$	not $\mathbb{Z}_2$-acyclic	$\geqslant n$	
E_6	not $\mathbb{Z}_2$-acyclic	$\geqslant 5$	
E_n, $n = 7, 8$	not $\mathbb{Z}_2$-acyclic	$\geqslant n$	
F_4	not $\mathbb{Z}_2$-acyclic	$\geqslant 2$	$d(X) = 2 \Leftrightarrow \Sigma(X) = \Sigma(V_{\omega_4})$
F_4	empty	$\geqslant 44$	
G_2	not $\mathbb{Z}_2$-acyclic	$\geqslant 12$	$\Sigma(X)$ contains at least 3 W-orbits of cardinality 6

Finally, the proof of Theorem C relies on an induction on the number of simple factors of a connected reductive group, using Theorem B. However, there are some small groups which need special care, and an additional acyclicity hypothesis.

3. Good quotients for semi-simple groups.

We start with a connected reductive group G acting on a variety X. Moreover, we fix a maximal torus $T \subset G$.

PROPOSITION 3.1. — *Assume that the fixed point set X^T is non-empty and connected.*

(1) *There is a reductive subgroup $H \subset G$ containing T such that $G_x = H$ for all x in an open dense subset of X^T. In particular, $X^T = X^H$.*

(2) *The roots of H^0 are $R(H^0) = R(G) - (\Sigma'(X) \cap R(G))$. In particular, $R(H^0)$ is $W(G)$-invariant and $\Sigma'(N_x) \cap R(H^0) = \varnothing$.*

(3) *The normalizer $L := \mathrm{Nor}_G(H^0)$ acts on X^T, and L contains $\mathrm{Nor}_G(T)$.*

(4) *The representation of H^0 on the tangent space T_xX is independent of $x \in X^T$ and extends to a representation of L.*

Proof. — (1) This is a consequence of the Slice Theorem. All orbits Gx for $x \in X^T$ are closed. Since X^T is irreducible we can assume that they belong all to the same Luna stratum. This implies that in the slice representation N_x ($x \in X^T$) the stabilizers in G_x of all points $y \in N_x^T$ are conjugate and in particular conjugate to the stabilizer of $0 \in N_x$ which is G_x.

(2) On one hand, for any $x \in X^T$, $\mathfrak{g}/\mathfrak{g}_x \cong T_x(Gx) \subset T_xX$, and therefore $(R(G) - R(G_x^0)) \subseteq \Sigma'(X)$. This implies $R(G) - (R(G) \cap \Sigma'(X)) \subseteq R(G_x^0)$. On the other hand, assume that there is an $\alpha \in R(H^0) \cap \Sigma'(X)$. Then the slice N_x contains an irreducible G_x^0-submodule V such that $\alpha \in \Sigma(V)$. Since $\Sigma(V)$ is α-saturated, we have that $\alpha - \alpha = 0 \in \Sigma(V)$, i.e., $V^T \neq \{0\}$. The stabilizer H_v of any vector $v \in V^T - \{0\}$ does not contain H^0. This is a contradiction to (1), hence $R(H^0) \cap \Sigma'(X) = \varnothing$.

(3) It is clear that L and also $\mathrm{Nor}_G(T)$ both act on X^T. For any $x \in X^T$ and $g \in \mathrm{Nor}_G(T)$ we have $G_{gx}^0 = gG_x^0g^{-1}$, and the claim follows from (1).

(4) It is well-known that the representation of H^0 on the tangent space T_xX is independent of $x \in X^T$ (cf. [Kr89a], p.112/113). By (2) and Proposition 2.1, $\Sigma(N_x) = \Sigma(X) - (\Sigma'(X) \cap R(G))$ is $W(G)$-invariant (as a set with multiplicities). Now the claim follows from the next lemma.

LEMMA 3.2. — *Let $H \subset G$ a be a connected reductive subgroup containing T, and such that $R(H)$ is a union of $W(G)$-orbits. A representation*

$\rho: H \to \mathrm{GL}(V)$ extends to a representation of $L := \mathrm{Nor}_G(H)$ if and only if $\Sigma(V)$ is $W(G)$-invariant.

Proof. — Given a reductive group G and a reductive subgroup H containing a maximal torus T of G it is well-known that H has finite index in its normalizer $L := \mathrm{Nor}_G(H)$. More precisely, L/H is canonically isomorphic to $W(L)/W(H)$, the quotient of the corresponding Weyl groups. In our situation, L contains $\mathrm{Nor}_G(T)$ hence $L/H \cong W(G)/W(H)$.

Given the fundamental Weyl chamber $\mathcal{C}(G)$, one has the following partial order on $\mathcal{X}(T)$: $\lambda <_G \mu$ is equivalent to $\mu - \lambda = \sum_{i=1}^{n} n_i \alpha_i$, where $n_i \in \mathbb{N}$ and the α_i are fundamental roots. For $H \subset G$ a maximal rank subgroup, we may assume that $\mathcal{C}(H) \supset \mathcal{C}(G)$. Then $\lambda <_H \mu$ implies $\lambda <_G \mu$.

Choose a maximal weight μ with respect to $<_G$ in $\Sigma(V)$. Write $W(G) \cdot \mu \cap \mathcal{C}(H) = \{\mu_1, \dots, \mu_d\}$, so

$$V_\mu(G)|_H \cong \bigoplus_{i=1}^{d} V_{\mu_i}(H) \oplus \bigoplus_{\lambda \in \Lambda} V_\lambda(H),$$

where for each weight in $\lambda \in \Lambda$, there is an i such that $\lambda <_H \mu_i$, and $\lambda \notin W(G)\mu$. Here we use the fact that there is a unique highest weight for an irreducible G-module, hence every weight in $W(G) \cdot \mu$ occurs with multiplicity one in $\Sigma(V_\mu(G))$. Then $V' := \bigoplus_{i=1}^{d} V_{\mu_i}(H)$ is an irreducible L-module, and $\Sigma(V')$ is $W(G)$-invariant. By $W(G)$-invariance of $\Sigma(V)$, $V|_H$ contains an H-submodule isomorphic to V'. This proves the lemma. □

PROPOSITION 3.3. — *Let L be a reductive group and $H \subset L$ a normal subgroup containing a maximal torus T of L. Assume that there is a subgroup $N \subset L$ normalizing T such that $\mathcal{X}(T)^N = \{0\}$. Then for every representation V of L the quotient $\pi: V \to V/\!\!/H$ is good (i.e., the generic fiber contains a dense orbit).*

Proof. — Since invariant rational functions separate generic orbits we have to show that the field of invariant rational functions $\mathbb{C}(V)^H$ is the field of fractions of the invariant ring $\mathbb{C}[V]^H$. Let $r = \frac{p}{q} \in \mathbb{C}(V)^H$ and assume that p and q have no common divisors. Then both are eigenfunctions with respect to a character χ of H : $p(gv) = \chi(g) \cdot p(v)$ for all $v \in V$, and similarly for q. Now choose representatives $n_1 = 1, n_2, \dots, n_m$ of N/T in

N. Then the function

$$\tilde{p}(v) := \prod_{i=1}^{m} p(n_i v)$$

is an eigenfunction with character $\tilde{\chi} := \sum_i n_i \chi$. Clearly, $\tilde{\chi}$ is invariant under N and so $\tilde{\chi} = 0$ by assumption, i.e., $\tilde{p}$ is an invariant function. Thus

$$r(v) = \frac{p(v)}{q(v)} = \frac{\tilde{p}(v)}{q(v) \cdot p(n_2 v) \cdots p(n_m v)}$$

is a quotient of two invariant regular functions. □

Now we are ready to prove the main result of this chapter.

THEOREM A. — *Let G be a semi-simple group acting on a smooth affine variety X. Assume that the fixed point set X^T of a maximal torus T of G is non-empty and connected. Then the quotient $\pi_X : X \to X/\!\!/G$ is good.*

Proof. — Let $V := N_x$ be the slice representation of $H := G_x$ in a generic point x of X^T. By the Slice Theorem it suffices to prove that the quotient $\pi_V : V \to V/\!\!/H$ is good, or equivalently, that the quotient $\pi : V \to V/\!\!/H^0$ is good. The representation of H^0 on V extends to a representation of $L := \mathrm{Nor}_G(H^0)$ by Proposition 3.1 (4) and L contains the normalizer N of the maximal torus T in G by Proposition 3.1 (3). It is well-known that $\mathcal{X}(T)^N = \{0\}$ for any semi-simple group. Thus, we can apply Proposition 3.3 above and the claim follows. □

Remark. — The assumption that X^T is connected is essential for the theorem as shown by the following example. Assume that G is semi-simple and $\alpha \in \mathcal{X}(T) - \{0\}$. Let T act on $\mathbb{C}^m$ $(m > 1)$ by scalar multiplication via α. Then the associated bundle

$$X := G \times^T \mathbb{C}^m$$

is a smooth G-variety of dimension $\dim G + m - \dim T$ without invariants. The generic orbit has dimension $\dim G + 1 - \dim T$, and X^T consists of $|W|$ points.

4. Rank one groups.

We look at SO_3-actions on a $\mathbb{Z}_2$-acyclic variety X. The condition that X^{SO_3} is empty will force some specific slice representations to occur, for various subgroups (see Oliver [Ol]). This implies that the weight system cannot be to small.

The list of reductive subgroups – up to conjugacy – of SO_3 is well-known : A maximal torus T, its normalizer N, the cyclic subgroup $\mathcal{C}_n \subset T$ of order n, the dihedral group $\mathcal{D}_n \subset N$ of order $2n$, the icosahedral group $\mathcal{I} \cong \mathcal{A}_5$, the octahedral group $\mathcal{O} \cong \mathcal{S}_4$, and the tetrahedral group $\mathcal{T} \cong \mathcal{A}_4$. Note that $N, \mathcal{I}$ and $\mathcal{O}$ are maximal proper subgroups of SO_3. Furthermore, $X^N = (X^T)^W$ is $\mathbb{Z}_2$-acyclic because $W = N/T \cong \mathbb{Z}_2$ and Smith Theory, and $X^{\mathcal{O}} \neq \varnothing$ due to the normal series $\mathcal{O} = \mathcal{S}_4 \supset \mathcal{A}_4 \supset \mathcal{D}_2$ and Petrie-Randall.

Denote by ω a generator of $\mathcal{X}(T)$, i.e., $\omega = 2\omega_1(\mathrm{A}_1) \in R(\mathrm{A}_1) = R(\mathrm{SO}_3)$. Let m_i be the multiplicity of $i\omega$ in $\Sigma(X)$. Then

$$\Sigma(X) = m_0\theta \oplus \bigoplus_{i \geqslant 1} m_i(i\omega \oplus -i\omega), \tag{4.1}$$

due to the W-invariance of the weight system. Denote $M_s := \sum_{i\geqslant 1} m_{si}$ for $s \geqslant 1$.

LEMMA 4.1 (see [HH74], pp.233/34). — *We have* $\operatorname{codim}_{X^T} X^N = M_1 - 2M_2$, *i.e.,* $\dim X^N = m_0 - M_1 + 2M_2$, *and* $\dim X^{\mathcal{D}_{2^s}} = m_0 - M_1 + 2M_2 + M_{2^s}$, $s \in \mathbb{N} - \{0\}$.

Proof. — Choose $x \in X^N$. Then the N-module $T_x X$ is a direct sum

$$T_x \cong m_0'\theta \oplus m_0''\sigma \oplus \bigoplus_{i\geqslant 1} m_i\rho_i$$

of irreducible N-modules. Here σ denotes the one-dimensional non-trivial N-module via the projection $N \to N/T \cong \mathbb{Z}_2$, and ρ_i the two-dimensional irreducible N-module with $\rho_i|_T = i\omega \oplus -i\omega$. Note that $m_0' + m_0'' = m_0$, and $m_0'' = \operatorname{codim}_{X^T} X^N$. We claim that $m_0'' = M_1 - 2M_2$. Then the lemma follows from the fact that $X^{\mathcal{D}_{2^s}}$ is $\mathbb{Z}_2$-acyclic, hence $\dim X^{\mathcal{D}_{2^s}} = \dim (T_x X)^{\mathcal{D}_{2^s}}$.

To prove that $m_0'' = M_1 - 2M_2$ we consider the action of $\mathcal{D}_2$. There are four irreducible representations of this group, each of dimension one. Let ε_0 denote the trivial representation, ε_1 the non-trivial one with kernel $\mathcal{C}_2$, and $\varepsilon_2, \varepsilon_3$ the remaining two. Then of course $\theta|_{\mathcal{D}_2} = \varepsilon_0$, $\sigma|_{\mathcal{D}_2} = \varepsilon_1$, $\rho_i|_{\mathcal{D}_2} = \varepsilon_0 \oplus \varepsilon_1$ for i even, and $\rho_i|_{\mathcal{D}_2} = \varepsilon_2 \oplus \varepsilon_3$ for i odd. Moreover $\mathcal{D}_2 \subset \mathcal{O}$ is normal and the elements of $\mathcal{D}_2 - \{e\}$ are all $\mathcal{O}$-conjugate. Thus the multiplicities in the $\mathcal{D}_2$-representation T_xX satisfy $\mathrm{mult}(\varepsilon_1) = \mathrm{mult}(\varepsilon_2) = \mathrm{mult}(\varepsilon_3)$. Since $\mathrm{mult}(\varepsilon_1) = m_0'' + M_2$ and $\mathrm{mult}(\varepsilon_2) = M_1 - M_2$, the claim follows. □

LEMMA 4.2 (compare also [HH74], pp.233/34).

(1) *If $M_2 = M_4$, then $X^{\mathrm{SO}_3} = X^N$ is $\mathbb{Z}_2$-acyclic, and $M_2 = 0$.*

(2) *If $M_4 = 0$, then $X^{\mathrm{SO}_3} = X^{\mathcal{O}} \neq \varnothing$, and $M_2 = m_2$.*

Proof. — (1) Because $M_2 = M_4$, we have that $\dim X^{\mathcal{D}_2} = \dim X^{\mathcal{D}_4}$, hence $X^{\mathcal{D}_2} = X^{\mathcal{D}_4}$ since $X^{\mathcal{D}_4} \subseteq X^{\mathcal{D}_2}$ and $X^{\mathcal{D}_2}$ is irreducible. This holds for all of the three subgroups of $\mathcal{O}$ which are conjugate to $\mathcal{D}_4$. Since they generate $\mathcal{O}$, we have $X^{\mathcal{O}} = X^{\mathcal{D}_2}$. Since N and $\mathcal{O}$ are maximal closed proper subgroups of SO_3, it follows that $X^{\mathcal{O}} \cap X^N = X^{\mathrm{SO}_3}$. But here $X^{\mathcal{O}} = X^{\mathcal{D}_2} \supseteq X^N$, hence $X^{\mathrm{SO}_3} = X^N$. The tangential representation in a fixed point x of SO_3 is

$$T_xX \cong \bigoplus_{i>0}(m_i - m_{i+1})V_{i\omega} \oplus (m_0 - m_1)\theta, \tag{4.2}$$

hence $m_i \geqslant m_{i+1}$. In particular, if $M_2 = M_4$, then $m_i = 0$ for $i > 1$.

(2) Because $M_4 = 0$ we find that $X^{\mathcal{D}_4} = X^N$. Hence $X^{\mathrm{SO}_3} = X^{\mathcal{O}} \cap X^N = X^{\mathcal{O}}$. By (4.2) it follows that $m_i = 0$ for $i \geqslant 4$, so $M_2 = m_2$. □

The following proposition should be compared to Theorem 2.1 in [HS86].

PROPOSITION 4.3. — *Let X be a $\mathbb{Z}_2$-acyclic SO_3-variety.*

(1) *$\mathrm{codim}_{X^{\mathcal{T}}}(X^N) = M_1 - 2M_2 \geqslant 0$, and $m_1 \geqslant 1$ if the action is not trivial.*

(2) *If $M_2 = 0$ then $X^{\mathrm{SO}_3} = X^N$ is $\mathbb{Z}_2$-acyclic.*

(3) *If $m_{2i} \neq 0$ for only one $i \geqslant 1$, then in fact $m_2 \neq 0$ and $X^{\mathrm{SO}_3} = X^{\mathcal{O}} \neq \varnothing$. If in addition $m_2 = m_3$, then $X^{\mathrm{SO}_3} = X^N$ is $\mathbb{Z}_2$-acyclic.*

(4) *If X is also $\mathbb{Z}_3$-acyclic and $M_3 = 0$, then $X^{\mathrm{SO}_3} = X^{\mathcal{T}} \neq \varnothing$; in fact, its Euler characteristic is $\chi(X^{\mathrm{SO}_3}) = 1$.*

Proof. — (1) By Lemma 4.1, $\mathrm{codim}_{X^{\mathcal{T}}}(X^N) = M_1 - 2M_2$. If $m_1 = 0$, then SO_3 has fixed points on X by Theorem 2.2, and $\Sigma(X) = m_0\theta$ by equation (4.2). Hence the action is trivial.

(2) It is obvious that $M_2 = 0$ implies $M_2 = M_4$, and Lemma 4.2(1) implies the claim.

(3) If i is even, then $M_4 = M_2 = 0$ by Lemma 4.2(1), a contradiction to $m_{2i} \neq 0$. Hence i is odd and $M_4 = 0$, so $X^{\mathrm{SO}_3} = X^{\mathcal{O}}$ and $m_2 = M_2$ by Lemma 4.2(2). For $x \in X^{\mathrm{SO}_3}$ the character of the $\mathcal{O}$-representation on $T_x X$ can be computed from the weight system (see [Ol], p.232). In particular, if $m_2 = m_3$ it follows that $\dim(T_x X)^{\mathcal{O}} = \dim(T_x X)^N$, and the irreducibility of X^N implies that $X^N = X^{\mathcal{O}}$.

(4) Let γ be a 3-cycle in the tetrahedral group $\mathcal{T} \cong \mathcal{A}_4$, and $T' \subset \mathrm{SO}_3$ a torus containing γ. Because X is $\mathbb{Z}_3$-acyclic, so is X^γ, and $\dim X^\gamma = m_0 + 2M_3 = m_0 = \dim X^{T'}$, so $X^{T'} = X^\gamma$. For any $x \in X^{\mathcal{T}}$, $T' \subset G_x$, hence $G_x = \mathrm{SO}_3$ and $X^{\mathrm{SO}_3} = X^{\mathcal{T}}$. On the other hand, $\mathcal{A}_4 \triangleright \mathcal{D}_2 \triangleright (1)$ is a normal series for the tetrahedral group, and by Petrie-Randall it follows that $\chi(X^{\mathcal{T}}) = 1$. □

Corollary 4.4. — *Let G be a simple group of rank 1, and X a $\mathbb{Z}_2$-acyclic G-variety.*

(1) *If X^G is not $\mathbb{Z}_2$-acyclic, then $d(X) \geqslant 2$. Moreover, $d(X) = 2$ implies that $\Sigma(X) = \Sigma(V_{4\omega_1})$.*

(2) *If $X^G = \varnothing$, then $d(X) \geqslant 5$, and in particular $\dim X/\!\!/G \geqslant 5$.*

Proof. — The center C is either trivial or $C \cong \mathbb{Z}_2$, so X^C is $\mathbb{Z}_2$-acyclic. Hence the action of $G/C \cong \mathrm{SO}_3$ on X^C satisfies the hypothesis, and $X^G = (X^C)^{\mathrm{SO}_3}$. Moreover, $d(X) \geqslant d(X^C)$, so we may assume that $G = \mathrm{SO}_3$. If $X^{\mathrm{SO}_3} = \varnothing$, then $M_2 \geqslant 2$ by Proposition 4.3(3), and $M_1 \geqslant 2M_2 \geqslant 4$ by Proposition 4.3(1). Therefore $d(X) \geqslant \dim \Sigma'(X) - \dim \mathrm{SO}_3 = 2M_1 - 3 \geqslant 5$ by Proposition 2.3. To prove (1), assume that X^G is not $\mathbb{Z}_2$-acyclic and $d(X) \leqslant 2$, hence $\dim \Sigma'(X) \leqslant 5$. We have that $M_2 > 0$ by Proposition 4.3(2), hence $m_1 = m_2 = 1$ by Proposition 4.3(1) and (3) and $m_i = 0$ for $i > 2$. It follows that $T_x X \cong V_{4\omega_1}$ for $x \in X^{\mathrm{SO}_3}$, and we are done. □

5. Rank 1 subgroups and saturatedness.

Throughout this chapter, we let G be a connected reductive group with a fixed maximal torus T. For $\alpha \in R(G)$ define $T_\alpha := \ker(\alpha) \subset T$. Its identity component T_α^0 is a corank 1 torus, and the centralizer $G_\alpha = C_G(T_\alpha^0)$ is connected, cf. [Hu], p.140. Of course, G_α is a reductive group of semisimple rank 1, with center T_α, and $\bar{G}_\alpha := G_\alpha/T_\alpha$ is isomorphic to SO_3. The normalizer $N_\alpha := \mathrm{Nor}_{G_\alpha}(T)$ is contained in $N := \mathrm{Nor}_G(T)$ by definition, and it is clear that $\bar{N}_\alpha = N_\alpha/T_\alpha$ is just $\mathrm{Nor}_{\bar{G}_\alpha}(\bar{T})$, where $\bar{T} := T/T_\alpha$ is a maximal torus in $\bar{G}_\alpha$. Therefore $W_\alpha := \bar{N}_\alpha/\bar{T}$ is the Weyl group of $\bar{G}_\alpha$, and since $W_\alpha = N_\alpha/T \subset N/T = W$, it is the subgroup of W generated by the reflection corresponding to the root α.

Let X be a $\mathbb{Z}_2$-acyclic G-variety. Then $X_\alpha := X^{T_\alpha}$ is $\mathbb{Z}_2$-acyclic, since $T_\alpha/T_\alpha^0 \cong \mathbb{Z}_2$ or trivial. X_α has an induced action of $\bar{G}_\alpha \cong SO_3$. Note that the weight system is $\Sigma(X_\alpha) = \Sigma(X) \cap \mathbb{Z}\alpha$, as a subset of $\Sigma(X)$ with multiplicities : For $x \in X_\alpha^{\bar{T}}$, $T_x X_\alpha = T_x(X^{T_\alpha}) = (T_x X)^{T_\alpha}$ by the Slice Theorem.

Proposition 5.1. — *Let X be a $\mathbb{Z}_2$-acyclic G-variety and $\alpha \in R(G)$. If $\Sigma'(X) \cap \mathbb{N}\alpha$ contains no more than three (not necessarily distinct) weights, or there is no $i \geqslant 2$ with $2i\alpha \in \Sigma(X)$, then $\Sigma(X)$ is α-saturated as a set without multiplicities.*

Proof. — Consider the action of $\bar{G}_\alpha$ on X_α. Under both assumptions we are either in case (2) or (3) of Proposition 4.3, hence the action of $\bar{G}_\alpha$ on X_α has fixed points. Choose $x \in X^{G_\alpha} = X_\alpha^{\bar{G}_\alpha}$. Then $\Sigma(T_x X) = \Sigma(X)$ is α-saturated. □

For the rest of this chapter we assume that G is a simple group. Let X be an A-acyclic G-variety, $A = \mathbb{Z}$, $\mathbb{Z}_p$ or $\mathbb{Q}$. We extend the notation introduced in §4. If G is non-simply laced, we have $R(G) = R_l(G) \oplus R_s(G)$ where $R_l(G)$ are the long and $R_s(G)$ are the short roots. If G is simply laced we consider all the roots as long roots. For $\alpha \in R_l(G)$, respectively $\alpha \in R_s(G)$, and $i \in \mathbb{N}$ the multiplicity of $i\alpha$ in $\Sigma(X)$ is independent of the choice of α and will be denoted by m_i, respectively by n_i :

$$\Sigma(X) = m_0\theta \oplus \bigoplus_{i\geqslant 1} m_i\,(\oplus_{\alpha\in R_l} i\alpha) \oplus \bigoplus_{i\geqslant 1} n_i\,(\oplus_{\beta\in R_s} i\beta) \oplus \Gamma,$$

where Γ denotes those weights which are not integral multiples of roots. The multiplicities n_i are – by our convention – always zero for a simply laced group G. We put $M_s := \sum_{i\geqslant 1} m_{si}$, $N_s := \sum_{i\geqslant 1} n_{si}$. Note that $\dim X = m_0 + M_1 \dim(R_l(G)) + N_1 \dim(R_s(G)) + \dim\Gamma$.

The following proposition shows that if $\Sigma(X)$ contains a long root, then it contains all roots.

Proposition 5.2. — *Let X be an A-acyclic variety, G a simple group of type B_n, C_n, F_4 or G_2 acting on X. If $n_1 = 0$, then $m_1 = 0$.*

Proof. — The special case that X is a G-module is an easy exercise, using the fact that the weight system of a G-module is saturated. For the general case, take $y \in X^T$, $T \subset G$ a maximal torus. The hypothesis $n_1 = 0$ implies that $R(G_y)$ contains all the short roots. But $\mathfrak{g}_y$ is a Lie-subalgebra of $\mathfrak{g}$ so for two root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$, it also contains their bracket $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in R(G)$. It is well-known that every long root is the sum of two short roots, hence $G_y = G$ and the proposition now follows from the special case. □

Together with Theorem 2.2 we obtain the following result.

COROLLARY 5.3. — *Let X be an A-acyclic G-variety, G simple.*

(1) *If G is simply laced and $m_1 = 0$, then X^G is A-acyclic and* $\dim X^G = m_0$.

(2) *If G is non-simply laced and $n_1 = 0$, then X^G is A-acyclic and* $\dim X^G = m_0$.

If G is non-simply laced, denote $W' \subset W$ the subgroup generated by the W_α, $\alpha \in R_s(G)$, and let $N' \subset N$ be the subgroup generated by the N_α, $\alpha \in R_s(G)$.

PROPOSITION 5.4. — *Let X be a $\mathbb{Z}_2$-acylic G-variety, G simple.*

(1) *If G is simply laced, $m_2 = m_3$ and $m_i = 0$ for $i \geqslant 4$, then $X^G = (X^T)^W$.*

(2) *If G is non-simply laced, $n_2 = n_3$ and $n_i = 0$ for $i \geqslant 4$, then $X^G = (X^T)^{W'}$.*

Proof. — We only carry out the proof in case (2), because the other case is proved similarly. For $\alpha \in R_s(G)$, look at the induced action of $\bar{G}_\alpha$ on X_α. The hypothesis on $\Sigma(X)$ means that we are in the situation of Proposition 4.3(2), or (3) with $m_2 = m_3$. Thus $X^{G_\alpha} = X_\alpha^{\bar{G}_\alpha} = X_\alpha^{\bar{N}_\alpha} = ({X_\alpha}^{\bar{T}})^{W_\alpha} = (X^T)^{W_\alpha}$. The G_α (respectively the W_α), $\alpha \in R_s(G)$, generate G (respectively W'). This implies the claim. □

By Proposition 5.1, the multiples of a root have to occur in strings as long as the weight system is "small". This makes the following lemma a pretty useful complement to Proposition 5.4.

LEMMA 5.5. — *Let X be a $\mathbb{Z}_2$-acylic G-variety.*

(1) *If G is simply laced, $m_1 = m_2$ and $m_i = 0$ for $i \geqslant 3$, then $X^G \neq \varnothing$.*

(2) *If G is of type C_n $(n > 2)$, F_4 or G_2, $n_1 = n_2$ and $n_i = 0$ for $i \geqslant 3$, then $X^G \neq \varnothing$.*

Proof. — Again we only do case (2). Note that W' permutes transitively the short roots. Thus for every short root α_0,

$$G = \langle G_\alpha \mid \alpha \in R_s(G)\rangle = \langle G_{w\alpha_0} \mid w \in W'\rangle = \langle N', G_{\alpha_0}\rangle.$$

The action of $\bar{G}_\alpha$ on X_α satisfies $m_1 = m_2$, and the other m_i's are zero, hence $X^T = X^{N_\alpha}$ by Proposition 4.3(1). It follows that $X^{N'} = X^T$, and that $X^G = X^{N'} \cap X^{G_{\alpha_0}} = X^T \cap X^{G_{\alpha_0}} = X^{G_{\alpha_0}} \neq \varnothing$. □

Remark. — This proof fails if G is of type B_n, because then W' does not act transitively on the short roots.

6. The special linear groups.

We start with a lemma which is a translation of Lemma 3.3 in [HS86], p.27 to the algebraic setting (see also [Fa], pp.27/28).

LEMMA 6.1. — *Let X be an affine SL_n-variety. Let $T \subset \mathrm{SL}_n$ be the diagonal torus, and define the 1-PSG $\lambda : \mathbb{C}^* \xrightarrow{\sim} S \subset T$ by*

$$\lambda(\xi) = \mathrm{diag}(1, \xi, \xi^2, \ldots, \xi^{\frac{n-1}{2}}, \xi^{\frac{1-n}{2}}, \ldots, \xi^{-2}, \xi^{-1}) \quad \text{for } n \text{ odd},$$
$$\lambda(\xi)t = \mathrm{diag}(\xi, \xi^3, \ldots c, \xi^{n-1}, \xi^{1-n}, \ldots, \xi^{-3}, \xi^{-1}) \quad \text{for } n \text{ even}.$$

Let $t_n := \lambda(e^{\frac{2\pi i}{n}})$ for n odd, respectively $t_n := \lambda(e^{\frac{\pi i}{n}})$ for n even, and assume that $X^S = X^{t_n}$. Then $X^{\mathrm{SL}_n} = (X^T)^{c_n}$, where $c_n \in W(\mathrm{SL}_n)$ is a Coxeter element.

LEMMA 6.2. — *Let p be a prime number, and X a $\mathbb{Z}_p$-acyclic variety. Let $q = p^s$ for some $s \in \mathbb{N}$, and assume that SL_q acts on X. If for every $\mu \in \Sigma(X)$ with $\mu(t_q) = 1$ it holds that $\mu \circ \lambda \equiv 1$, then $X^{\mathrm{SL}_q} = (X^T)^{c_q}$, and the fixed point set is $\mathbb{Z}_p$-acyclic.*

Proof. — By Smith Theory, X^S and X^{t_q} are both $\mathbb{Z}_p$-acyclic, hence irreducible. Since $\dim X^S = \dim X^{t_q}$ for the given weight system, and $X^S \subseteq X^{t_q}$, we have the equality $X^S = X^{t_q}$. By Lemma 6.1 we conclude that $X^{\mathrm{SL}_q} = (X^T)^{c_q}$. □

Let X be a $\mathbb{Z}_2$-acyclic SL_n-variety, n a power of 2. If $\Sigma'(X) = R(\mathrm{A}_{n-1})$, then X^{SL_n} is $\mathbb{Z}_2$-acyclic. In a fixed point x of SL_n we have that $\tilde{N}_x = \mathrm{Ad}_{\mathrm{SL}_n}$, so $d(X) = \dim \mathrm{Ad}_{\mathrm{SL}_n} /\!/ \mathrm{SL}_n = n-1$. This observation is the clue for the following proof.

PROPOSITION 6.3. — *Let G be a simple group of type A_n, $n \geqslant 2$, and X a $\mathbb{Z}_2$-acyclic G-variety.*

(1) *If X^G is not $\mathbb{Z}_2$-acyclic, then $d(X) > n - \log_2 n$. Moreover, if $n \geqslant 5$, then $d(X) \geqslant 4$.*

(2) *Assume that $X^G = \varnothing$. If $n = 2$, then $d(X) \geqslant 16$, and if $n = 3$, then $d(X) \geqslant 33$.*

Proof. — (1) We may assume (see §2) that $R(G) \subset \Sigma'(X)$ and $\dim(\Sigma'(X) - R(\mathrm{A}_n)) < 2n$. One easily computes that the only non-trivial $W(\mathrm{A}_n)$-orbits in $\mathcal{X}(T)$ of cardinality $< 2n$ are of the form $\{i\varepsilon_1, \ldots, i\varepsilon_{n+1}\}$ for some $i \in \mathbb{Z}$. There can be at most one such orbit in $\Sigma'(X)$, and by Proposition 5.1, $\Sigma(X)$ is α-saturated for every root α. It follows that $i = \pm 1$ if $\{i\varepsilon_1, \ldots, i\varepsilon_{n+1}\} \subset \Sigma'(X)$.

We construct a maximal rank subgroup G_I of G with semisimple rank $> n - \log_2 n$. Write $n+1$ in the binary system : $n+1 = (b_j b_{j-1} \ldots b_1 b_0)_2$, with $0 \leqslant b_\ell < 2$ for $\ell = 0, \ldots, j$ and $n+1 = \sum_{\ell \geqslant 0} b_\ell 2^\ell$. Define $I := \Big\{k \in \{1, \ldots, n\} \mid \text{there is no } i \geqslant 2 \text{ with } k = \sum_{\ell \geqslant i} b_\ell 2^\ell \Big\}$, and $\tilde{I} := \{\ell \in \mathbb{N}_{>0} \mid b_\ell \neq 0\}$. Number the elements in $\tilde{I}$ such that $\tilde{I} = \{\ell_1, \ldots, \ell_r\}$ with $\ell_1 > \ell_2 > \ldots > \ell_r$. Denoting $\alpha_k = \varepsilon_k - \varepsilon_{k+1}$ the kth simple root, put $T_I = \bigcap_{k \in I} T_{\alpha_k}$. Now the connected group $G_I = C_G(T_I^0)$ has maximal rank, and G_I/T_I^0 is of type $\mathrm{A}_{q_1 - 1} \times \ldots \times \mathrm{A}_{q_r - 1}$, where $q_i = 2^{\ell_i}$ unless $i = r$ and $b_1 = b_0 = 1$, in which case $q_r = 3$. Put $H_I = \mathrm{SL}_{q_1} \times \ldots \times \mathrm{SL}_{q_r}$, and consider the finite group homomorphism $H_I \twoheadrightarrow G_I/T_I^0$. Denote by T_i the maximal torus in SL_{q_i} with image in T/T_I^0. By construction, $\operatorname{rank} H_I > n - \log_2 n$, and in case $n = 5$, $H_I = \mathrm{SL}_4 \times \mathrm{SL}_2$ is of rank 4. Note that $\operatorname{rank} H_I$ equals the semisimple rank of G_I.

Since G_I normalizes T_I^0, it acts on $X_I := X^{T_I^0}$. Look at the induced action of H_I on X_I. This turns X_I into a smooth, affine and $\mathbb{Z}_2$-acyclic H_I-variety, and its non-zero weight system is

$$\Sigma'(X_I) = \Sigma'(X)^{T_I^0}|_{T_1 \times \ldots \times T_r} = \begin{cases} R(H_I), & \text{if } I \neq \varnothing, \\ \Sigma'(X), & \text{if } I = \varnothing. \end{cases}$$

If $q_i = 2^{\ell_i}$, then the hypothesis of Lemma 6.2 is fulfilled for the SL_{q_i}-actions on X_I, hence $({X_I}^{T_i})^{c_{q_i}} = {X_I}^{\mathrm{SL}_{q_i}}$. If $q_r = 3$, then ${X_I}^{\mathrm{SL}_{q_r}} = ({X_I}^{T_r})^{W(\mathrm{A}_2)}$ by Proposition 5.4(1). Therefore we get that

$$X^{G_I} = {X_I}^{\mathrm{SL}_{q_1}\times\ldots\times\mathrm{SL}_{q_r}} = ({X_I}^{T_1\times\ldots\times T_r})^F,$$

where F is a finite group. In fact, F is the 2-group $\langle c_{q_1}\rangle \times \ldots \times \langle c_{q_r}\rangle$ if $q_r = 2^{\ell_r}$, and $F \cong \langle c_{q_1}\rangle \times \ldots \times \langle c_{q_{r-1}}\rangle \times \mathcal{S}_3$ if $q_r = 3$. In any case, there are fixed points by Petrie-Randall.

Choose $x \in X^{G_I}$. Then $\mathrm{Ad}_{G_x^0} \subseteq \tilde{N}_x$, so $\dim \tilde{N}_x /\!/ G_x^0 \geqslant$ the semisimple rank of G_x^0. Since $G_I \subseteq G_x^0$, this is $\geqslant$ the semisimple rank of G_I.

(2) For $n = 2, 3$, $W(\mathrm{A}_n) \cong \mathcal{S}_{n+1}$ has fixed points on X^T by Petrie-Randall. Thus Propositions 5.1, 5.4(1) and Lemma 5.5(1) imply that $M_1 \geqslant 3$. Moreover, if $M_1 = 3$, then $m_1 = 2$ and $m_2 = 1$, thus $\dim \Gamma \geqslant n(n^2-1)$ by saturatedness. This implies that $\dim \Sigma'(X) - \dim \mathrm{A}_n \geqslant 16$ if $n = 2$, respectively $\geqslant 33$ if $n = 3$. □

We complement Proposition 6.3 by a result which takes care of SL_5-actions, to get a general lower bound $\dim X /\!/ \mathrm{A}_n \geqslant 4$ for fixed point free actions on $\mathbb{Z}$-acyclic varieties. The Weyl group $W(\mathrm{A}_4) \cong \mathcal{S}_5$ has no decomposition series that would allow the application of Petrie-Randall. But if we assume that X is also $\mathbb{Z}_5$-acyclic, we get fixed points for SL_5 as long as the conditions of Lemma 6.2 are satisfied.

PROPOSITION 6.4. — *Let G be a simple group of type* A_4, *and X a $\mathbb{Z}_2$- and $\mathbb{Z}_5$-acyclic variety. If G acts on X without fixed points, then $d(X) \geqslant 26$.*

Proof. — Since SL_5 is the simply connected group of type A_4, we can assume that $G = \mathrm{SL}_5$. If $M_1 \geqslant 4$, then $d(X) \geqslant \dim \Sigma'(X) - \dim G \geqslant 56$, so we assume that $M_1 \leqslant 3$. Then the weight system satisfies the following conditions : (a) It is α-saturated for every $\alpha \in R(\mathrm{A}_4)$ (by Proposition 5.1), (b) $R(\mathrm{A}_4) \subset \Sigma'(X)$ (by Theorem 2.2), and (c) there is a weight $\mu \in \Sigma'(X)$ such that $\mu(t_5) = 1$ but $\mu \circ \lambda \neq 1$ (by Lemma 6.2). Now it is easy to see that a weight system which satisfies conditions (a) to (c) has $\dim \Sigma'(X) \geqslant 50$, which is enough to prove the claim. E.g. if $\mu = 5\varepsilon_2$, then also $\mu' = 5\varepsilon_2 - (\varepsilon_2 - \varepsilon_1) - (\varepsilon_2 - \varepsilon_3) \in \Sigma'(X)$, and $|W(\mathrm{A}_4)\mu'| = 30$. The bound comes from the weight $\mu = -\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4$. □

7. The symplectic groups.

Fix a maximal torus T in the symplectic group Sp_n, the simply connected group of type C_n. Sp_n has a maximal rank subgroup isomorphic to GL_n corresponding to the roots $\pm(\varepsilon_i-\varepsilon_j)$, $1 \leqslant i < j \leqslant n$. Thus, using the notation of Lemma 6.1, it makes sense to talk about $t_n \in S \subset \mathrm{SL}_n \subset \mathrm{Sp}_n$ and $c_n \in W(\mathrm{SL}_n) \subset W(\mathrm{Sp}_n)$.

LEMMA 7.1. — *Let p be a prime number, and X a $\mathbb{Z}_2$- and $\mathbb{Z}_p$-acyclic variety. Let $q = p^s$ for some $s \in \mathbb{N}$, and assume that Sp_q acts on X. If $n_2 = n_3$, $n_i = 0$ for $i \geqslant 4$, and every $\mu \in \Sigma(X) \cap (\bigoplus_{1\leqslant i\leqslant q-1} \mathbb{Z}(\varepsilon_{i+1} - \varepsilon_i))$ which satisfies $\mu(t_q) = 1$ also satisfies $\mu\circ\lambda \equiv 1$, then $X^{\mathrm{Sp}_q} = ((X^T)^{\mathbb{Z}_2^{n-1}})^{c_q}$, and $\chi(X^{\mathrm{Sp}_q}) = 1$.*

Proof. — By Proposition 5.4(2), we know that $X^{\mathrm{Sp}_q} = (X^T)^{W'(\mathrm{Sp}_q)} = ((X^T)^{\mathbb{Z}_2^{q-1}})^{S_q}$. Denoting by C the center of $\mathrm{GL}_q \subset \mathrm{Sp}_q$, the weight system $\Sigma(X^C)$ for the SL_q-action on X^C is

$$\Sigma(X^C) = \Sigma(X)^C = \Sigma(X) \cap \left(\bigoplus_{1\leqslant i\leqslant q-1} \mathbb{Z}(\varepsilon_{i+1} - \varepsilon_i)\right).$$

By Lemma 6.2, the hypothesis implies that $X^{\mathrm{GL}_q} = (X^C)^{\mathrm{SL}_q} = (X^T)^{c_q}$, and in particular $(X^T)^{c_q} = (X^T)^{S_q}$. Therefore $X^{\mathrm{Sp}_q} = ((X^T)^{\mathbb{Z}_2^{q-1}})^{c_q}$, and $\chi(X^{\mathrm{Sp}_q}) = 1$ by Petrie-Randall. □

PROPOSITION 7.2. — *Let G be a simple group of type C_n, and X a $\mathbb{Z}_2$-acyclic G-variety.*

(1) *If X^G is not $\mathbb{Z}_2$-acyclic, then $d(X) > n - \log_2 n$.*

(2) *Assume that $X^G = \varnothing$. If $n = 3$, then $d(X) \geqslant 21$, and if $n = 4$, then $d(X) \geqslant 44$.*

Proof. — (1) By Corollary 5.3(2), $R_s(\mathrm{C}_n) \subset \Sigma'(X)$. We may assume that $\dim \Sigma'(X) - \dim R_s(\mathrm{C}_n) < 4n$, since otherwise $d(X) \geqslant n$ by Proposition 2.3. The only $W(\mathrm{C}_n)$-orbits of cardinality $< 4n$ are of the form $\{i\varepsilon_1, -i\varepsilon_1, \dots, i\varepsilon_n, -i\varepsilon_n\}$ for some $i \in \mathbb{N}$ if $n \geqslant 4$. Moreover, such a weight system is α-saturated for every $\alpha \in R(\mathrm{C}_n)$, hence $i = 1$ or 2 (if such an orbit occurs at all).

For $n = 3$, the orbit $W(\mathrm{C}_3)(i\omega_3)$ has also cardinality $< 4n = 12$. If such an orbit occurs in $\Sigma'(X)$, then we get that $\Sigma'(X) = R_s(\mathrm{C}_3) \oplus$

$W(C_3)(i\omega_3)$. By Proposition 5.4(2) and Petrie-Randall, the action has fixed points, hence $\Sigma(X)$ is the weight system of a C_3-representation. This is absurd

So for any n, we have to consider two cases.

CASE 1 : $\Sigma'(X) = R_s(C_n) \oplus \{2\varepsilon_1, -2\varepsilon_1, \dots, 2\varepsilon_n, -2\varepsilon_n\} = R(C_n)$.

The subgroup $A_1^n \subset C_n$ corresponding to the long roots has fixed points by Proposition 4.3(2), since $X^{A_1^n} = (X^T)^{W(A_1)^n} = (X^T)^{\mathbb{Z}_2^n}$ is $\mathbb{Z}_2$-acyclic. Take $x \in X^{A_1^n}$. Then of course $\tilde{N}_x = \mathrm{Ad}_{G_x^0}$, G_x^0 is semi-simple of rank n and $d(X) = \dim \mathrm{Ad}_{G_x^0} /\!/ G_x^0 = n$.

CASE 2 : $\Sigma'(X) = R_s(C_n) \oplus \{\varepsilon_1, -\varepsilon_1, \dots, \varepsilon_n, -\varepsilon_n\}$ or $\Sigma'(X) = R_s(C_n)$.

We use the inclusion of groups $A_{n-1} \subset C_n$: By the construction in Proposition 6.3, there is a maximal rank subgroup $G_I \subset A_{n-1}$ with semi-simple part of type $A_{q_1-1} \times \dots \times A_{q_r-1}$, with $q_i = 2^{s_i}$ or 3 (if $i = r$). It is easy to see that G_I together with the previously described A_1^n generates a maximal rank subgroup G_I' of type $C_{q_1} \times \dots \times C_{q_r}$ in C_n. G_I' has fixed points by Lemma 7.1. Take an $x \in X^{G_I'}$. The classification in [BdS], p.219 shows that G_x^0 is of type $C_{k_1} \times \dots \times C_{k_s}$ with $\sum_{i=1}^{s} k_i = n$, since every maximal rank semisimple subgroup of C_n is of this form. Moreover, because $G_I \subset G_x^0$, we have that $s \leqslant r < \log_2 n$. The slice representation $N_x|_{G_x^0}$ contains $V_{\omega_2}(C_{k_1}) \oplus \dots \oplus V_{\omega_2}(C_{k_s})$, and thus

$$d(X) \geqslant \sum_{i=1}^{s} \dim(V_{\omega_2}(C_{k_i}) /\!/ C_{k_i}) = \sum_{i=1}^{s} (k_i - 1) = n - s > n - \log_2 n.$$

(2) Note that $W(C_n)$ has fixed points on X^T for $n = 3, 4$ by Petrie-Randall. Hence if $N_1 \leqslant 3$, then $n_1 = 2$, $n_2 = 1$, and $n_i = 0$ for $i \geqslant 3$. In case we had $m_1 = 0$, we would get A_1^n-fixed points, and with the weight $2(\varepsilon_1+\varepsilon_2)$ we also have the weight $2(\varepsilon_1 + \varepsilon_2) - 2\varepsilon_2 = 2\varepsilon_1$ in $\Sigma'(X)$, a contradiction to $m_1 = 0$. Therefore $m_1 \geqslant 1$, and $d(X) \geqslant \dim \Sigma'(X) - \dim G \geqslant n(4n - 5)$. If $N_1 \geqslant 4$, then $d(X) \geqslant 3n(2n - 3)$. □

Similarly to Proposition 6.4 one shows :

PROPOSITION 7.3. — *Let G be a simple group of type C_5, and X a $\mathbb{Z}_2$- and $\mathbb{Z}_5$-acyclic variety. If G acts on X without fixed points, then* $d(X) \geqslant 65$.

8. Spinor and special orthogonal groups.

LEMMA 8.1. — *Let X be a $\mathbb{Z}_2$-acyclic G-variety.*

(1) *If G is of type B_n, $n_2 = n_3$ and $n_i = 0$ for $i \geqslant 4$, then X^G is $\mathbb{Z}_2$-acyclic.*

(2) *If G is of type D_n, $M_1 \leqslant 3$ and $\Gamma = \varnothing$, then X^G is $\mathbb{Z}_2$-acyclic.*

Proof. — (1) By Proposition 5.4(2), $X^G = (X^T)^{W'} = (X^T)^{\mathbb{Z}_2^n}$, which is $\mathbb{Z}_2$-acyclic by Smith Theory.

(2) Since $M_1 \leqslant 3$, $\Sigma(X)$ is α-saturated for every $\alpha \in R(\mathrm{D}_n)$ by Proposition 5.1. Since $\Gamma = \varnothing$, this implies that $m_i = 0$ for $i > 1$. Let $T_{n-1} \subset \mathrm{B}_{n-1}$ and $T_n \subset \mathrm{D}_n$ be maximal tori such that $T_{n-1} \subset T_n$ under the inclusion $\mathrm{B}_{n-1} \subset \mathrm{D}_n$. We consider the action of B_{n-1} on X. It has the weight system

$$\Sigma(X|_{\mathrm{B}_{n-1}}) = \Sigma(X)|_{T_{n-1}} = m_0\theta \oplus m_1\Big(\bigoplus_{\alpha\in R_l(\mathrm{B}_{n-1})} \alpha\Big) \oplus 2m_1\Big(\bigoplus_{\beta\in R_s(\mathrm{B}_{n-1})} \beta\Big).$$

The proof of part (1) shows that $X^{\mathrm{B}_{n-1}} = (X^{T_{n-1}})^{\mathbb{Z}_2^{n-1}}$. Since $W(\mathrm{B}_{n-1})$ embeds into $W(\mathrm{D}_n)$ canonically, $\mathbb{Z}_2^{n-1}$ acts on X^{T_n}. The stabilizer G_x in a fixed point $x \in (X^{T_n})^{\mathbb{Z}_2^{n-1}}$ has maximal rank and contains a semi-simple subgroup of type B_{n-1}. Using the classification of maximal rank subgroups of D_n (see [BdS], p.219), we get that $G_x^0 \simeq \mathrm{D}_n$, i.e., $X^{\mathrm{D}_n} = (X^{T_n})^{\mathbb{Z}_2^{n-1}}$.

□

PROPOSITION 8.2. — *Let G be a simple group of type B_n or D_n, and X a $\mathbb{Z}_2$-acyclic G-variety.*

(1) *If X^G is not $\mathbb{Z}_2$-acyclic, then $d(X) \geqslant n$.*

(2) *If $n = 2, 3, 4$, $G \simeq \mathrm{B}_n$ and X^G is not $\mathbb{Z}_2$-acyclic, then $d(X) \geqslant 2n - 1$.*

(3) *If G is of type D_4 and X^G is not empty, then $d(X) \geqslant 44$.*

Proof. — (1) It is enough to show that $\dim \Sigma'(X) - \dim G \geqslant n$. Assume to the contrary that $\dim \Sigma'(X) - \dim G < n$. In particular, we have that $M_1 \leqslant 3$. By Proposition 5.1, the weight system $\Sigma(X)$ is α-saturated for every long root α.

Assume that $G \simeq \mathrm{B}_n$. By Corollary 5.3(2), $n_1 \geqslant 1$, and by Lemma 8.1(1), $n_i \geqslant 1$ for some $i \geqslant 2$. Take such an $i \geqslant 2$ with $i\varepsilon_1 \in \Sigma'(X)$. Thus also $i\varepsilon_1 - (\varepsilon_1 - \varepsilon_2) = (i-1)\varepsilon_1 + \varepsilon_2 \in \Sigma(X)$. The length of the orbit

$W(\mathrm{B}_n)[(i-1)\varepsilon_1+\varepsilon_2]$ is $\geqslant 2n(n-1)$, and $\dim\Sigma'(X) \geqslant N_1\cdot 2n+2n(n-1) \geqslant \dim \mathrm{B}_n + n$, a contradiction.

Now consider a group G of type D_n. If $M_1 \geqslant 2$, then we had that $\dim\Sigma'(X) - \dim \mathrm{D}_n \geqslant n$. Hence $m_1 = M_1 = 1$, and by Lemma 8.1(2), Γ is not empty. Since the smallest $W(\mathrm{D}_n)$-orbits have cardinality $\geqslant 2n$, we get that $\dim\Sigma'(X) \geqslant \dim R(\mathrm{D}_n) + 2n = \dim \mathrm{D}_n + n$, which is again a contradiction.

(2) The proof of (1) shows that the smallest possible cardinality for $\Sigma'(X)$ is realized by the weight system

$$\Sigma'(X) = \{\pm\varepsilon_1,\ldots,\pm\varepsilon_n\} \oplus \{\pm 2\varepsilon_1,\ldots,\pm 2\varepsilon_n\} \oplus R_l(\mathrm{B}_n). \tag{8.1}$$

The D_n-action on X induced by $\mathrm{D}_n \hookrightarrow \mathrm{B}_n$ (the identification of $R(\mathrm{D}_n)$ with $R_l(\mathrm{B}_n)$) has fixed points by Proposition 5.4(1) and Petrie-Randall, and in a point $x \in X^{\mathrm{D}_n}$, either $G_x^0 \simeq \mathrm{B}_n$ or $G_x^0 \simeq \mathrm{D}_n$ by [BdS], p.219. In both cases, $\tilde{N}_x \cong V_{2\omega_1}$, hence $\dim \tilde{N}_x/\!\!/G_x^0 \geqslant 2n-1$. If $\Sigma'(X)$ is of a different form, then it is very easy to see that $\dim\Sigma'(X) \geqslant \dim \mathrm{B}_n + 2n$.

(3) Because $W(\mathrm{D}_4)$ has fixed points on X^T, we get that $M_1 \geqslant 3$ by Propositions 5.1, 5.4(1) and Lemma 5.5(1), so $\dim\Sigma'(X) - \dim \mathrm{D}_4 \geqslant 44$.

□

We have proved that $d(X) \geqslant 4$ for fixed point free actions of a simple group G of type B_n and D_n, unless $G \simeq \mathrm{B}_2$. In the latter case, we will need a fixed point lemma. Consider the group SO_5. It contains maximal 2-tori of rank 4, i.e., a subgroup $Q \cong \mathbb{Z}_2^4$ which is not contained in any subgroup $Q' \subset \mathrm{SO}_5$ with $Q' \cong \mathbb{Z}_2^n$, $n \geqslant 5$. If $N(Q) = \mathrm{Nor}_{\mathrm{SO}_5}(Q)$, then $N(Q)/Q \cong \mathcal{S}_5$. Choose a representative $t_5 \in \mathrm{SO}_5$ for a Coxeter element in $N(Q)/Q$. Then t_5 is a regular element, i.e., contained in a unique maximal torus T. Choose an element t_5' of order 5 and a maximal torus T' in Spin_5 which map to t_5 respectively T under the canonical covering homomorphism $\mathrm{Spin}_5 \to \mathrm{SO}_5$. In the following lemma, take the weight system $\Sigma(X)$ with respect to T'.

LEMMA 8.3. — *Let X be a $\mathbb{Z}_2$- and $\mathbb{Z}_5$-acyclic* Spin_5*-variety. If $\mu(t_5') \neq 1$ for every $\mu \in \Sigma'(X)$, then $\chi(X^{\mathrm{Spin}_5}) = 1$.*

Proof. — Since $\mu(t_5') \neq 1$ for every $\mu \in \Sigma'(X)$, it follows that $X^{T'} = X^{t_5'}$: Both sets are $\mathbb{Z}_5$-acyclic, have the same dimension and $X^{T'} \subseteq X^{t_5'}$. Denote C the center of Spin_5, so $C \cong \mathbb{Z}_2$. Then $Y := X^C$ is a $\mathbb{Z}_2$-acyclic $\mathrm{Spin}_5/C \cong \mathrm{SO}_5$-variety. We already showed that $Y^T = X^{T'} = Y^{t_5}$. Choose $y \in (Y^Q)^{t_5}$. Then $G_y \subset \mathrm{SO}_5$ contains the maximal torus T as well as Q.

In particular, the orbit $Gy \subset Y$ is closed and the group G_y is reductive. It follows show that $G_y = \mathrm{SO}_5$. Hence $X^{\mathrm{Spin}_5} = Y^{\mathrm{SO}_5} = (Y^Q)^{t_5}$, and $\chi((Y^Q)^{t_5}) = 1$ by Petrie-Randall. □

PROPOSITION 8.4. — *Let G be a simple group of type* B_2, *and X a $\mathbb{Z}_2$- and $\mathbb{Z}_5$-acyclic G-variety. If $X^G = \varnothing$, then $d(X) \geqslant 15$.*

Proof. — We may assume that $G \cong \mathrm{Spin}_5$, since this is the simply connected group of type B_2. The point is that the weight system in (8.1) satisfies the hypothesis of Lemma 8.3 : The element t_5' is regular, therefore if $i\alpha(t_5') = 1$ for some $i \in \mathbb{Z}$, $\alpha \in R(\mathrm{B}_2)$, then 5 divides i. Hence the action has fixed points.

The proof is now completely analogous to the proof of Proposition 6.4. The bound comes from the weight system

$$\Sigma'(X) = \Sigma'(V_{2\omega_1}) \oplus \left\{ \frac{1}{2}(\pm 3\varepsilon_1 \pm \varepsilon_2), \frac{1}{2}(\pm\varepsilon_1 \pm 3\varepsilon_2), \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2) \right\},$$

where $\dim \tilde{N}_x /\!/ G_x^0 = 15$ in a fixed point of A_1^2. □

9. The exceptional groups.

PROPOSITION 9.1. — *Let G be a simple group of type E_n, and X a $\mathbb{Z}_2$-acyclic G-variety. Assume that X^G is not $\mathbb{Z}_2$-acyclic. If $G \simeq E_6$, then $d(X) \geqslant 5$. If $G \simeq E_n$, $n = 7$ or 8, then $d(X) \geqslant n$.*

Proof. — There are no non-trivial $W(\mathrm{E}_n)$-orbits in $\mathcal{X}(T)$ of cardinality $\leqslant 2n$. Hence the conditions $R(\mathrm{E}_n) \subset \Sigma'(X)$ and $\dim(\Sigma'(X) - R(\mathrm{E}_n)) \leqslant 2n$ imply that $\Sigma'(X) = R(\mathrm{E}_n)$. Thus for any $x \in X^T$, it follows that $\tilde{N}_x = \mathrm{Ad}_{G_x^0}$, so $d(X) \geqslant \dim \mathrm{Ad}_{G_x^0} /\!/ G_x^0$, which equals the semi simple rank of G_x^0. Let G' denote the following maximal rank subgroups : $\mathrm{D}_5 \times \mathbb{C}^* \subset \mathrm{E}_6$, $\mathrm{A}_1 \times \mathrm{D}_6 \subset \mathrm{E}_7$ or $\mathrm{D}_8 \subset \mathrm{E}_8$, cf. [BdS], p.219. By Proposition 4.3 and Lemma 8.1(2), $X^{G'}$ is $\mathbb{Z}_2$-acyclic. Evaluating the semi-simple rank of G_x^0 for $x \in X^{G'}$ yields the claim. □

Part (1) of the following Lemma will be used in §10.

LEMMA 9.2. — *Let X be a $\mathbb{Z}_2$-acylic G-variety.*

(1) *If $G = \mathrm{F}_4$, $M_1 = 0$, $n_i = 0$ for $i \geqslant 2$ and $\Gamma = \varnothing$, and X is also $\mathbb{Z}_3$-acyclic, then $X^{\mathrm{F}_4} = (X^T)^{c_3}$ is $\mathbb{Z}_3$-acyclic, where $c_3 \in S_3 \subset W(\mathrm{F}_4)$ is an element of order 3.*

(2) *If* $G = \mathrm{G}_2$ *and* $n_i = 0$ *for* $i \geqslant 2$, *then* X^{G_2} *is* $\mathbb{Z}_2$*-acyclic.*

Proof. — In both cases, X^G is not empty by Proposition 5.4(2) and Petrie-Randall.

(1) In a point $x \in X^{\mathrm{F}_4}$, the tangential representation is $n_1 V_{\omega_4} \oplus (m_0 - 2n_1)\theta$. The fixed points satisfy $V_{\omega_4}^{\mathrm{F}_4} = (V_{\omega_4}^T)^{c_3}$, $c_3 \in \mathcal{S}_3$ any element of order 3 (cf. [Vi], p.492, No. 22). Since $(X^T)^{c_3}$ is $\mathbb{Z}_3$-acyclic, hence irreducible, and $(X^T)^{c_3} \supseteq X^{\mathrm{F}_4}$ with both sets of the same dimension, it follows that they coincide.

(2) $\Sigma(X)$ is the weight system of a G_2-module, so $m_i = 0$ for $i \geqslant 2$ and $\Gamma = \varnothing$ follow from saturatedness. For any $x \in X^{\mathrm{G}_2}$ we have a G_2-isomorphism $T_x X \cong (n_1 - m_1)V_{\omega_1} \oplus m_1 V_{\omega_2} \oplus (m_0 - m_1 - n_1)\theta$. $V_{\omega_1}^T$ is a one-dimensional, non-trivial $W(\mathrm{G}_2)$-module, where $W(\mathrm{G}_2)$ acts through its abelianization $\tilde{W} \cong \mathcal{D}_2$, and consequently $V_{\omega_1}^{\mathrm{G}_2} = (V_{\omega_1}^T)^{\mathcal{D}_2}$. Analogously, $V_{\omega_2} \cong \mathrm{Ad}_{\mathrm{G}_2}$, and $V_{\omega_2}^{\mathrm{G}_2} = (V_{\omega_2}^T)^{w_0}$, where $w_0 \in \mathcal{D}_2$ is the longest element in the Weyl group. Since $(X^T)^{\mathcal{D}_2}$ is $\mathbb{Z}_2$-acyclic, thus irreducible, and $X^{\mathrm{G}_2} \subseteq (X^T)^{\mathcal{D}_2}$, we have that $X^{\mathrm{G}_2} = (X^T)^{\mathcal{D}_2}$, because both sets have the same dimension. □

Proposition 9.3. — *Let* X *be a* $\mathbb{Z}_2$*-acyclic* G*-variety.*

(1) *Assume that* $G = \mathrm{F}_4$. *If* $X^G = \varnothing$, *then* $d(X) \geqslant 44$. *If* X^G *is not* $\mathbb{Z}_2$*-acyclic, then* $d(X) \geqslant 2$, *and* $d(X) = 2 \Leftrightarrow \Sigma(X) = \Sigma(V_{\omega_4})$.

(2) *If* $G = \mathrm{G}_2$, *and* X^G *is not* $\mathbb{Z}_2$*-acyclic, then* $d(X) \geqslant 12$.

Proof. — (1) If $X^G = \varnothing$, then $N_1 \geqslant 3$ by Propositions 5.1, 5.4(2), Lemma 5.5(2) and Petrie-Randall. The only possibility for $N_1 = 3$ is $n_1 = 2$, $n_2 = 1$. In this case, we get that $m_1 \geqslant 1$ since $\Sigma(X)$ is α-saturated for every $\alpha \in R_s(\mathrm{F}_4)$. It follows that $d(X) \geqslant N_1|R_s(\mathrm{F}_4)| + M_1|R_l(\mathrm{F}_4)| - \dim \mathrm{F}_4 \geqslant 44$. Assume that $d(X) \leqslant 2$ and that X^G is not $\mathbb{Z}_2$-acyclic. Then the action is non-trivial, and it has fixed points. The Slice Theorem implies that $T_x X \cong V_{\omega_4}$ for $x \in X^{\mathrm{F}_4}$, and (1) follows.

(2) If $N_1 \leqslant 2$, then $n_1 = n_2 = 1$ by Lemma 9.2(2). Such an action has fixed points, and choosing $x \in X^G$, it follows that $\dim \tilde{N}_x \geqslant 27$, since $2\omega_1 \in \Sigma(\tilde{N}_x)$. If $N_1 = 3$, then either X^G is $\mathbb{Z}_2$-acyclic, or $n_1 = 2$, $n_2 = 1$ and $m_1 \geqslant 1$. If now $M_1 = m_1 = 1$ and $\Gamma = \varnothing$, then the subgroup of the long roots, which is of type A_2, has fixed points, and $\dim \tilde{N}_x /\!/ G_x^0 \geqslant 12$ in a point $x \in X^{\mathrm{A}_2}$. □

10. Splitting subgroups and splitting weight systems.

In this chapter, we let G_i ($i = 1, \dots, s$) be connected reductive groups, and we put $G := G_1 \times \dots \times G_s$.

DEFINITION 10.1. — A closed subgroup $H \subset G$ is said to be *splitting* (with respect to the given decomposition), if $H = (H \cap G_1) \times \dots \times (H \cap G_s)$ (cf. [HS86], p.5).

Remark 10.2. — Let H be a reductive subgroup of G. If H has a maximal torus which is splitting with respect to the decomposition $G = G_1 \times \dots \times G_s$, then by conjugacy of maximal tori in H, every maximal torus in H is splitting. If moreover H is connected, then the union of all H-conjugates of a maximal torus is dense in H. This implies that a reductive connected subgroup is splitting if and only if it has a maximal torus which is splitting.

In particular, the identity component of a maximal rank reductive subgroup is always splitting, since every maximal torus in G is splitting. This implies that for $x \in X^T$ the identity component G_x^0 of the stabilizer is splitting.

Let X be a $\mathbb{Z}_2$-acyclic G-variety.

DEFINITION 10.3. — *Let $T_i \subset G_i$ be a maximal torus, $i = 1, \dots, s$.*

(1) *A weight $\mu \in \Sigma(X)$ is called a mixed weight, if there are two distinct i and k such that $\mu|_{T_i}$ and $\mu|_{T_k}$ are both non-trivial.*

(2) *The weight system $\Sigma(X)$ is called splitting (with respect to the given decomposition) if there are no mixed weights.*

(3) *Consider non-trivial decompositions $\{1, \dots, s\} = I_1 \dot{\cup} I_2$, and put $H_j := \times_{i \in I_j} G_i$, $j = 1, 2$. We call the weight system $\Sigma(X)$ totally non-splitting, if $\Sigma(X)$ contains mixed weights with respect to every such decomposition $G = H_1 \times H_2$.*

Remark 10.4. — If V is a G-module, then $\Sigma(V)$ is splitting with respect to a decomposition $G = G_1 \times \dots \times G_s$ if and only if the G-module V is a direct sum of G_i-modules.

The goal of this chapter is to prove Theorem C under the additional hypothesis that the weight system is splitting with respect to a direct product decomposition $G = G_1 \times \dots \times G_s$, with each G_i simple.

LEMMA 10.5. — *Let G be a simple group. Suppose that X^G is not $\mathbb{Z}_2$-acyclic and $d(X) \leqslant 3$. Then we have one of the following cases :*

(1) *$d(X) = 2$, and $G \simeq \mathrm{A}_1, \mathrm{A}_2, \mathrm{C}_3$ or F_4. Furthermore $X^G \neq \varnothing$, and there is a 2-group $W_2 \subset W(G)$ such that $\dim \tilde{N}_y /\!\!/ G_y^0 \geqslant 1$ for $y \in (X^T)^{W_2}$. The weight system has the following form :*

$$\begin{aligned}
&(a)\ G \simeq \mathrm{A}_1 : \Sigma'(X) = \Sigma'(V_{4\omega_1}),\\
&(b)\ G \simeq \mathrm{A}_2 : \Sigma'(X) = R(\mathrm{A}_2),\\
&(c)\ G \simeq \mathrm{C}_3 : \Sigma'(X) = R_s(\mathrm{C}_3) \text{ or } \Sigma'(X) = R_s(\mathrm{C}_3) \oplus \Sigma'(V_{\omega_1}),\\
&(d)\ G \simeq \mathrm{F}_4 : \Sigma'(X) = R_s(\mathrm{F}_4).
\end{aligned}$$

(2) *$d(X) = 3$, and there is a 2-group $W_2 \subset W(G)$ such that $\dim \tilde{N}_y /\!\!/ G_y^0 \geqslant 2$ for $y \in (X^T)^{W_2}$.*

Proof. — We already showed that $G \simeq \mathrm{A}_n$ ($1 \leqslant n \leqslant 4$), B_2, C_n ($3 \leqslant n \leqslant 5$) or F_4. Moreover, it was shown that actions with $d(X) \leqslant 2$ have fixed points, and the list in (1) is exhaustive. We only have to determine the respective $W_2 \subset W(G)$.

$G \simeq \mathrm{A}_1$: Put $W_2 := \{e\}$.

$G \simeq \mathrm{A}_2$: Pick any $\alpha \in R(\mathrm{A}_2)$ and put $W_2 := W_\alpha \subset W(\mathrm{A}_2)$.

$G \simeq \mathrm{A}_n, n = 3, 4$: The proof of Proposition 6.3 shows that $\dim \tilde{N}_y /\!\!/ G_y^0 \geqslant 3$ for $y \in (X^T)^{c_4}$, where c_4 is a Coxeter element in $W(H_I) \cong \mathcal{S}_4 \subseteq W(G)$.

$G \simeq \mathrm{B}_2$: Put $W_2 := W(\mathrm{A}_1) \times W(\mathrm{A}_1) = W(\mathrm{D}_2) \subset W(\mathrm{B}_2)$.

$G \simeq \mathrm{C}_3$: G contains a subgroup of type $\mathrm{C}_2 \times \mathbb{C}^*$, whose fixed point set is just $((X^T)^{\mathbb{Z}_2})^{c_2}$ by Lemma 7.1. Put $W_2 := \langle c_2 \rangle \rtimes \mathbb{Z}_2$.

$G \simeq \mathrm{C}_n$, $n = 4, 5$: Put $W_2 := \langle c_4 \rangle$, where c_4 is a Coxeter element in $W(\mathrm{A}_3) \subset W(\mathrm{C}_n)$.

$G \simeq \mathrm{F}_4$: If $d(X) \leqslant 3$, then $\Sigma'(X) = R_s(\mathrm{F}_4)$. In this case, the subgroup $\mathrm{B}_4 \subset \mathrm{F}_4$ has fixed points $X^{\mathrm{B}_4} = (X^T)^{\mathbb{Z}_2^4}$, and $\dim \tilde{N}_y /\!\!/ G_y^0 \geqslant 1$ for $y \in X^{\mathrm{B}_4}$. Putting $W_2 := \mathbb{Z}_2^4 \subset W(\mathrm{B}_4) \subset W(\mathrm{F}_4)$ yields the claim. □

Recall that a G-variety X is called *fix-pointed*, if every closed G-orbit in X is a fixed point.

LEMMA 10.6. — *Let G be a reductive group and V a G-module such that $\Sigma(V) \cap R(G) \neq \varnothing$. Then V is not fix-pointed.*

Proof. — Assume that $V^G = \{0\}$, and choose $\alpha \in \Sigma(V) \cap R(G)$. Since $\Sigma(V)$ is α-saturated for every $\alpha \in R(G)$, we get that $\alpha - \alpha = 0 \in \Sigma(V)$, so $V^T \neq \{0\}$. Since the G-orbit through any $v \in V^T$ is closed, we have closed orbits $\neq \{0\}$. □

The proof of the next proposition needs Lemmata 10.5 and 10.6, together with the following remarks : Let X be a smooth affine variety with Euler characteristic $\chi(X) = 1$, e.g. X is $\mathbb{Z}_2$- or $\mathbb{Z}_3$-acyclic. Of course, if $\dim X = 0$, then X is a point, and it is well-known that if $\dim X = 1$, then $X \cong \mathbb{A}$, the complex line. In particular, these varieties are automatically $\mathbb{Z}$- and $\mathbb{Z}_p$-acyclic, for any prime p. If $\dim X^G = \dim X /\!\!/ G$, then the action is fix-pointed. Consequently $X^G = X^T$, so X^G inherits the acyclicity properties of X.

PROPOSITION 10.7. — *Let G_i be a simple group ($i = 1, \dots, s$), and $G = G_1 \times \dots \times G_s$. Let X be a $\mathbb{Z}_2$-acyclic G-variety. Assume that the weight system $\Sigma(X)$ is splitting with respect to the decomposition $G = G_1 \times \dots \times G_s$.*

(1) *If $\dim X /\!\!/ G \leqslant 1$, then X^G is $\mathbb{Z}_2$-acyclic, hence either a point or $\mathbb{A}$.*

(2) *If $\dim X /\!\!/ G \leqslant 2$, then $X^G \neq \varnothing$.*

(3) *If X is also $\mathbb{Z}_3$-acyclic and $\dim X /\!\!/ G \leqslant 2$, then X^G is $\mathbb{Z}_2$-acyclic. Hence either X^G is a point, $X^G \cong \mathbb{A}$ or the action is fix-pointed with $X^G = X^T$.*

(4) *If X is moreover $\mathbb{Z}_3$- and $\mathbb{Z}_5$-acyclic and $\dim X /\!\!/ G = 3$, then $X^G \neq \varnothing$.*

Proof. — By induction on s.

$s = 1$: If $\dim X /\!\!/ G \leqslant 1$, then X^G is $\mathbb{Z}_2$-acyclic by Lemma 10.5. If $\dim X /\!\!/ G = 2$ and X^G is not $\mathbb{Z}_2$-acyclic, then we are in one of the cases of Lemma 10.5(1). If moreover X is $\mathbb{Z}_3$-acyclic, then $\chi(X^G) = 1$ by Proposition 4.3(4), Lemmata 6.2, 7.1 and 9.2(1). Since $\dim X^G = 0$ in all these cases, X^G is a point. In case (4) eventually, $X^G \neq \varnothing$, because $\mathbb{Z}_5$-acyclicity enables us to use Propositions 6.4, 7.3 and 8.4.

$s > 1$: For $x \in X^T$, write $H_i := G_i \cap G_x^0$, and let $\tilde{N}_{x,i}$ be the largest H_i-submodule of N_x without fixed lines. Since $\Sigma(X)$ is splitting, we have that $\tilde{N}_x = \tilde{N}_{x,1} \oplus \dots \oplus \tilde{N}_{x,s}$.

First assume that there is some $i \in \{1, \dots, s\}$ such that X^{G_i} is $\mathbb{Z}_2$-acyclic. Then we use the induction hypothesis for the action of $G_1 \times \dots \times \hat{G}_i \times \dots \times G_s$ on X^{G_i}. This is possible for the following reason :

If $X^{G_i} = X^{T_i}$, then X^{G_i} satisfies the same acyclicity assumptions as X, and induction applies. If $X^{G_i} \neq X^{T_i}$, then choose $x \in X^{T_1\times\ldots\times G_i\times\ldots\times T_s}$. Because $\Sigma(X)$ is splitting, we have that $N_x = \tilde{N}_{x,i} \oplus N_x^{G_i}$. Since $N_x^{G_i} \subset N_x$ is the slice representation for the action of $G_1 \times \ldots \times \hat{G}_i \times \ldots \times G_s$ on X^{G_i}, we get that

$$\dim X/\!/G = \dim N_x/\!/G_x = \dim \tilde{N}_{x,i}/\!/G_i + \dim(X^{G_i}/\!/G_1\times\ldots\times\hat{G}_i\times\ldots\times G_s).$$

By Theorem 2.2, $\Sigma'(X)$ as well as $\Sigma'(\tilde{N}_{x,i})$ contain roots of G_i, so $\dim \tilde{N}_{x,i}/\!/G_i > 0$ by Lemma 10.6. Therefore $\dim(X^{G_i}/\!/G_1 \times \ldots \times \hat{G}_i \times \ldots \times G_s) < \dim X/\!/G$. If we started in case (4), we are now in case (1) or (2), and if we started in case (2) or (3), we are in case (1), and the induction hypothesis applies.

Second, if no X^{G_i} is $\mathbb{Z}_2$-acyclic, we get a contradiction. For $x \in X^T$, we have that $\dim X/\!/G \geqslant \sum_{i=1}^{r} \dim(\tilde{N}_{x,i}/\!/H_i)$. In particular, considering X as a G_i-variety, we see that $(G_i, \Sigma'(X|_{G_i}))$ satisfies the hypothesis of Lemma 10.5. Let $W_2^{(1)} \subset W(G_1)$ be the 2-subgroup of Lemma 10.5. In case (2), the G_2-action on $(X^{T_1\times\hat{T}_2\times T_3\times\ldots\times T_s})^{W_2^{(1)}}$ has a fixed point x, and $\dim N_x/\!/G_x^0 \geqslant \dim \tilde{N}_{x,1}/\!/H_1 + \dim \tilde{N}_{x,2}/\!/G_2 \geqslant 1 + 2$, a contradiction.

In case (4), the same argument reduces the problem to the case where $s = 2$, and the pairs $(G_1, \Sigma'(X|_{G_1}))$ and $(G_2, \Sigma'(X|_{G_2}))$ occur in the list of Lemma 10.5(1). If then $x \in X^{G_1\times G_2}$, we get that $\dim \tilde{N}_x/\!/G_1 \times G_2 = 4$. Hence to get a contradiction, it is enough to prove that $G_1 \times G_2$ has fixed points. But this is a consequence of $\mathbb{Z}_3$-acyclicity.

E.g. $G_1 \cong G_2 \simeq \mathrm{A}_1$ and $\Sigma'(X) = \Sigma'(V_{4\omega_1^{(1)}} \oplus V_{4\omega_1^{(2)}})$: The center acts trivially, hence $G_1\times G_2 \cong \mathrm{SO}_3\times\mathrm{SO}_3$. Let $\mathcal{T}_i \subset G_i$ be tetrahedral groups. By Proposition 4.3(4), $X^{\mathrm{SO}_3\times\mathrm{SO}_3} = X^{\mathcal{T}_1\times\mathcal{T}_2}$. Since we have the decomposition series $\mathcal{T}_1 \times \mathcal{T}_2 \rhd \mathcal{D}_2^{(1)} \times \mathcal{D}_2^{(2)} \rhd \mathcal{D}_2^{(1)} \times \mathcal{D}_2^{(2)}$ meeting the conditions of Petrie-Randall, $X^{\mathcal{T}_1\times\mathcal{T}_2} \neq \varnothing$. For the other types of groups, we use Lemmata 6.2, 7.1 and 9.2(1) : If $G_i \simeq \mathrm{A}_2$ or F_4, there is a $c_3 \in W(G_i)$ of order 3, such that $X^{G_i} = (X^{T_i})^{c_3}$; if $G_i \simeq \mathrm{C}_3$, there is a subgroup $\tilde{W}_i \cong \mathbb{Z}_3 \ltimes \mathbb{Z}_2^2 \subset W(G_i)$ such that $X^{G_i} = (X^{T_i})^{\tilde{W}_i}$. Using Petrie-Randall, this finally implies the claim. □

11. Fixed points for connected reductive groups.

We first consider some totally non-splitting weight systems, starting with an example.

Example 11.1. — Let $G \simeq \mathrm{F}_4 \times \mathrm{A}_1$, and X a $\mathbb{Z}_2$-acyclic G-variety. Fix maximal tori $T_4 \subset \mathrm{F}_4$ and $T_1 \subset \mathrm{A}_1$. Assume that neither X^{F_4} nor X^{A_1} is $\mathbb{Z}_2$-acyclic, and that $\Sigma(X)$ is totally non-splitting. We are going to show that $\dim \tilde{N}_x /\!\!/ G_x^0 \geqslant 5$ for any $x \in X^{T_4 \times T_1}$.

First of all, we may assume that $\dim \Sigma'(X) < \dim G + 5 = 60$. For $\omega \in \mathcal{X}(T_4) - \{0\}$, the cardinality of the $W(\mathrm{F}_4)$-orbit through ω in $\mathcal{X}(T_4)$ is either $|W(\mathrm{F}_4)\omega| = 24$ or $|W(\mathrm{F}_4)\omega| \geqslant 48$. For $\omega \in \mathcal{X}(T_1) - \{0\}$ we have always $|W(\mathrm{A}_1)\omega| = 2$. Since $R_s(\mathrm{F}_4) \subset \Sigma'(X|_{\mathrm{F}_4})$ by Theorem 2.2, the presence of a mixed weight implies that

$$\Sigma'(X) = R_s(\mathrm{F}_4) \otimes (i\omega_1(\mathrm{A}_1) \oplus -i\omega_1(\mathrm{A}_1)) \oplus \text{ some more weights of } \mathrm{A}_1.$$

Choose $x \in X^{T_4 \times T_1}$ generic. Since $R(\mathrm{F}_4) \cap \Sigma'(X) = \varnothing$, we have that $\mathrm{F}_4 \subset G_x^0$. On the other hand, $\mathrm{A}_1 \not\subset G_x^0$: If $\mathrm{A}_1 \subset G_x^0$, then it follows that $R(\mathrm{A}_1) \cap \Sigma'(X) = \varnothing$, and because $R(\mathrm{A}_1) \subset \Sigma'(X|_{\mathrm{A}_1})$, we had $i = 2$, which is absurd. Hence $G_x^0 \cong \mathrm{F}_4 \times T_1$. Since the generic F_4-orbit in $2V_{\omega_4}$ has codimension 6 (see e.g. [El], p.51), Theorem A implies that $\dim \tilde{N}_x /\!\!/ G_x^0 \geqslant 5$. As an easy application of the Slice Theorem one shows that $\dim \tilde{N}_y /\!\!/ G_y^0 \geqslant \dim \tilde{N}_x /\!\!/ G_x^0$ for any $y \in X^{T_4 \times T_1}$, proving the claim.

LEMMA 11.2. — *Let G_i $(i = 1 \ldots, s,\ s > 1)$ be simple groups, and $G = G_1 \times \ldots \times G_s$. Let X be a $\mathbb{Z}_2$-acyclic G-variety. Assume that the weight system $\Sigma(X)$ is totally non-splitting, and that no X^{G_i} is $\mathbb{Z}_2$-acyclic. Then $\dim \tilde{N}_x /\!\!/ G_x^0 \geqslant 4$ for every $x \in X^T$.*

Proof. — Example 11.1 shows how to treat actions where a simple factor $G_i \simeq \mathrm{F}_4$ has mixed weights only with a simple factor $G_j \simeq \mathrm{A}_1$. A similar argument as in Example 11.1 also settles the cases $G \simeq \mathrm{A}_1^n$ $(n < 4)$, $\mathrm{A}_1 \times \mathrm{A}_2$, $\mathrm{A}_1^n \times \mathrm{C}_3$ $(n \leqslant 2)$ and $\mathrm{A}_1 \times \mathrm{C}_4$. In all the other cases, use the data in the table on page 1277 :

In the column "W-orbits" we list the least cardinalities for *different* W-orbits which have to occur in $\Sigma'(X|_{G_i})$. Essentially, this uses Corollary 5.3, but also Proposition 4.3 and the proofs of Proposition 8.2(1) and 9.3(2). We denote $m := \min_{\omega \in \mathcal{X}(T_i) - \{0\}} \{|W(G_i)\omega|\}$, the least cardinality of a non-trivial $W(G_i)$-orbit in $\mathcal{X}(T_i)$. An elementary computation gives $\dim \Sigma'(X) \geqslant \dim G + 4$. □

We need another lemma before we can prove Theorem C.

type of G_i	$\dim G_i$	W-orbits	m
A_1	3	$2, 2$	2
A_n, $n > 1$	$n^2 + 2n$	$n^2 + n$	$n + 1$
B_n	$2n^2 + n$	$2(n^2 - n), 2n, 2n$	$2n$
C_n	$2n^2 + n$	$2(n^2 - n)$	$2n$
D_n	$2n^2 - +n$	$2(n^2 - n)$	$2n$
E_6	78	72	26
E_7	133	126	56
E_8	248	240	240
F_4	52	24	24
G_2	14	$6, 6, 6$	6

LEMMA 11.3. — *For $i = 1, \ldots, s$, let G_i be a simple group with maximal torus T_i. Let $s > r \geqslant 1$, and put $K := G_1 \times \ldots \times G_r$, $L := G_{r+1} \times \ldots \times G_s$. Let X be a $\mathbb{Z}_2$-acyclic $K \times L$-variety.*

Assume that $Y = X^K$ is $\mathbb{Z}_2$-acyclic, and that $\Sigma(Y)$, the weight system of the induced action of L on Y, is splitting. If $X^{G_i} \neq X^{T_i}$ for every $i = 1, \ldots, r$, and $Y^{G_j} \neq Y^{T_j}$ for every $j = r + 1, \ldots, s$, then $\dim(X /\!/ K \times L) > \dim(Y /\!/ L)$.

Proof. — Look at the L-action on Y and choose $y \in Y^{T_{r+1} \times \ldots \times T_s}$ generic. By Theorem 2.2 and splitness of $\Sigma(Y)$, it follows from $Y^{G_j} \neq Y^{T_j}$ that $\Sigma(Y) \cap R(G_j) \neq \varnothing$ for every $j = r + 1, \ldots, s$. By Proposition 3.1(2) this implies that $H_j := G_j \cap L_y^0 \subsetneq G_j$, and in particular, there is no factor of type A_n with $n > 2$ in $H_{r+1} \times \ldots \times H_s$. Renumbering, we may assume that there is a t with $r + 1 \leqslant t \leqslant s$ such that H_j is semisimple if $j \leqslant t$, and $H_j = T_j$ if $j > t$.

Now look at y as a point on the $K \times L$-variety X. The slice N_y decomposes as

$$N_y = N_1 \oplus \ldots \oplus N_l \oplus N_y',$$

where $N_y' := N_y^K$ is the slice in y for the L-action on Y and each N_i is an irreducible $K \times \mathrm{Nor}_L(L_y^0)$-module (by §3).

Without restriction we may assume that $\operatorname{rank} G_1 \geqslant \operatorname{rank} G_j$ for $2 \leqslant j \leqslant r$, and that $\Sigma(N_1) \cap R(G_1) \neq \varnothing$ (since $X^{G_1} \neq X^{T_1}$ by hypothesis). Then

$$N_1 \cong V_1 \otimes \ldots \otimes V_r \otimes V_{r+1} \otimes \ldots \otimes V_t \otimes V_{t+1} \otimes \ldots \otimes V_s,$$

where for $1 \leqslant i \leqslant r$, V_i is an irreducible G_i-module, for $r < i \leqslant t$, V_i is an irreducible $\mathrm{Nor}_{G_i}(H_i)$-module, and for $t < i \leqslant s$, V_i is an irreducible $\mathrm{Nor}_{G_i}(T_i)$-module. For $r < i \leqslant t$ let $V_i' \subset V_i$ be an irreducible H_i-submodule. There is no representation equivalent to $(\mathbb{C}^m, \mathrm{SL}_m)$ with $m > \dim V_1$ in the list (V_j, G_j) $(1 \leqslant j \leqslant r)$, (V_j', H_j) $(r < j \leqslant t)$. This implies that $V_1 \otimes \ldots \otimes V_r \otimes V_{r+1}' \otimes \ldots \otimes V_t'$ is not a prehomogeneous $K \times H_{r+1} \times \ldots \times H_t$-vector space, using the classification in [SK], pp.143/44. Hence $\dim(V_1 \otimes \ldots \otimes V_t /\!/ K \times H_{r+1} \times \ldots \times H_t) \geqslant 1$.

It follows from the next lemma that $\dim(N_1 /\!/ K \times L_y^0) > 0$, implying that

$$\dim(X /\!/ K \times L) \geqslant \dim(N_1 /\!/ K \times L_y^0) + \dim N_y' /\!/ L_y^0 > \dim N_y' /\!/ L_y^0 = \dim Y /\!/ L,$$

which was the claim. □

LEMMA 11.4. — *Let G, N be reductive groups. Let $T \subset N$ be a normal torus such that $T = N^0$ and $\mathcal{X}(T)^N = \{0\}$. Given a G-module V_1 with $\dim V_1 /\!/ G \geqslant 1$ and an N-module $V_2 \neq \{0\}$, it follows that $\dim(V_1 \otimes V_2 /\!/ G \times T) \geqslant 1$.*

Proof. — Choose $p \in \mathbb{C}[V_1]^G$ homogeneous of positive degree. Let $(e_1, \ldots, e_m)$ be a T-eigenbasis of V_2. Define $p_i \in \mathbb{C}[V_1 \otimes V_2]^G$ by $p_i(\sum v_j \otimes e_j) := p(v_i)$, $i = 1, \ldots, m$. Then p_i is a T-eigenfunction to some character χ_i, and $\tilde{p} := p_1 \ldots p_m$ is a T-eigenfunction to the character $\tilde{\chi} = \sum \chi_i$. Since V_2 is an N-module, $\tilde{\chi}$ is N-invariant, hence $\tilde{\chi} = 0$ by hypothesis. It follows that $\tilde{p} \in \mathbb{C}[V_1 \otimes V_2]^{G \times T}$, and this proves Lemmata 11.3 and 11.4. □

The following is a stronger version of Theorem C.

THEOREM 11.5. — *Let G be a connected reductive group, and X a $\mathbb{Z}_2$-acyclic G-variety.*

(1) *If $\dim X /\!/ G \leqslant 2$, then the action has fixed points.*

(2) *If X is also $\mathbb{Z}_3$-acyclic and $\dim X /\!/ G \leqslant 2$, then X^G either is a point, $X^G \cong \mathbb{A}$ or the action is fix-pointed with $X^G = X^T$ (hence $\mathbb{Z}_2$- and $\mathbb{Z}_3$-acyclic).*

(3) *If X is also $\mathbb{Z}_3$- and $\mathbb{Z}_5$-acyclic, and $\dim X /\!/ G = 3$, then the action has fixed points.*

Proof. — First some easy reductions : Let C be the identity component of the center of G. Since C is a torus, X^C satisfies the same acyclicity assumptions as X, and the semi-simple group G/C acts on X^C. Since

$\dim(X^C/\!/G/C) \leqslant \dim X/\!/G$ and $(X^C)^{G/C} = X^G$, without loss of generality we may assume that G is semi-simple. In this case, we take a finite homomorphism $\tilde{G} = G_1 \times \ldots \times G_s \longrightarrow G$, where the G_i are simple groups. Of course we can look at the induced action of $\tilde{G}$ on X instead of the action of G. If $T_i \subset G_i$ are maximal tori, and for some i it holds that $X^{G_i} = X^{T_i}$, then we can as well look at the action of $G_1 \times \ldots \times \hat{G}_i \times \ldots \times G_s$ on X^{G_i}. To summarize, we assume without loss of generality that $G = G_1 \times \ldots \times G_s$ with G_i simple, and $X^{T_i} \neq X^{G_i}$ for $i = 1, \ldots, s$.

Now choose $r \geqslant 0$ maximal such that there are distinct $i_1, \ldots, i_r$ with a $\mathbb{Z}_2$-acyclic fixed point set $X^{G_{i_1} \times \ldots \times G_{i_r}}$. Renumbering, we can assume that $i_j = j$ for $j = 1, \ldots, r$. Denote $K = G_1 \times \ldots \times G_r$, $L = G_{r+1} \times \ldots \times G_s$ and $Y = X^K$. Then Y is a $\mathbb{Z}_2$-acyclic L-variety, and $\dim Y/\!/L \leqslant \dim X/\!/G$.

Case 1 : Assume that $\Sigma(Y)$ contains mixed weights. Then there exists a $t > 1$ and $\{i_1, \ldots, i_t\} \subset \{r{+}1, \ldots, s\}$ such that, putting $H = G_{i_1} \times \ldots \times G_{i_t}$: (a) The weight system $\Sigma(Y|_H)$ is totally non-splitting, (b) for any $j \in \{r+1, \ldots, s\} - \{i_1, \ldots, i_t\}$, there is no mixed weight of $H \times G_j$ in $\Sigma(Y)$. Choose $y \in Y^{T_{r+1} \times \ldots \times T_s}$. There is a direct sum decomposition into L_y^0-modules $N_y = V_1 \oplus V_2$, where $V_2 = N_y^{H \cap L_y^0}$ and V_1 is a trivial $G_j \cap L_y^0$-module for $j \in \{r+1, \ldots, s\} - \{i_1, \ldots, i_t\}$. Because no Y^{G_i} is $\mathbb{Z}_2$-acyclic $(r+1 \leqslant i \leqslant s)$, Lemma 11.2 applied to the action of H on Y yields that $\dim(V_1/\!/H \cap L_y^0) \geqslant 4$. This implies that $\dim N_y/\!/L_y \geqslant 4$, a contradiction. Therefore this case cannot occur.

Case 2 : $\Sigma(Y)$ is splitting with respect to the decomposition $L = G_{r+1} \times \ldots \times G_s$. If $r = 0$, Proposition 10.7 yields the properties of $Y^L = X^G$ we claimed. If $r \geqslant 1$, then $\dim Y/\!/L < \dim X/\!/G$ by Lemma 11.3, and applying Proposition 10.7(1) and (2) proves the theorem. □

Corollary (Theorem C). — Let G be a connected reductive group, and X a $\mathbb{Z}$-acyclic G-variety.

(1) If $\dim X/\!/G \leqslant 2$, then X^G is $\mathbb{Z}$-acyclic.

(2) If $\dim X/\!/G = 3$, then X^G is not empty.

Proof. — If X is $\mathbb{Z}$-acyclic, it is $\mathbb{Z}_p$-acyclic for every prime p, so the hypothesis (2) respectively (3) is satisfied. Since X^T is $\mathbb{Z}$-acyclic by Smith Theory, we are done. □

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