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Pieri’s formula for flag manifolds and Schubert polynomials


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PIERI'S FORMULA FOR FLAG MANIFOLDS 
AND SCHUBERT POLYNOMIALS

by Frank SOTTILE

1. Introduction.

Schubert polynomials had their origins in the study of the cohomology of flag manifolds by Bernstein-Gelfand-Gelfand [3] and Demazure [7]. They were later defined by Lascoux and Schützenberger [17], who developed a purely combinatorial theory.

For each permutation $w$ in the symmetric group $S_n$ there is a Schubert polynomial $\mathcal{G}_w$ in the variables $x_1, \ldots, x_{n-1}$. When evaluated at certain Chern classes, a Schubert polynomial gives the cohomology class of a Schubert subvariety of the manifold of complete flags in $\mathbb{C}^n$. In this way, the collection $\{\mathcal{G}_w \mid w \in S_n\}$ of Schubert polynomials determines a basis for the integral cohomology of the flag manifold. Thus there exist integer structure constants $c_{uv}^w$ defined by the identity

$$\mathcal{G}_u \cdot \mathcal{G}_v = \sum_w c_{uv}^w \mathcal{G}_w.$$ 

No combinatorial formula is known, or even conjectured, for these constants. There are, however, a few special cases in which they are known.

One important case is Monk's formula [21], which characterizes the algebra of Schubert polynomials. While this is usually attributed

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to Monk, Chevalley simultaneously established the analogous formula for generalized flag manifolds in a manuscript that was only recently published [6]. Let \( s_k \) be the transposition interchanging \( k \) and \( k + 1 \). Then \( S^k = x_1 + \cdots + x_k = s_1(x_1, \ldots, x_k) \), the first elementary symmetric polynomial. For any permutation \( w \in S_n \), Monk’s formula states

\[
S^w \cdot S^k = S^w \cdot s_1(x_1, \ldots, x_k) = \sum S^{wt_{ab}},
\]

where \( t_{ab} \) is the transposition interchanging \( a \) and \( b \), and the sum is over all \( a \leq k < b \) where \( w(a) < w(b) \) and if \( a < c < b \), then \( w(c) \) is not between \( w(a) \) and \( w(b) \).

The classical Pieri’s formula computes the product of a Schur polynomial by either a complete or an elementary symmetric polynomial. Our main result is a formula for Schubert polynomials and the cohomology of flag manifolds which generalizes both Monk’s formula and the classical Pieri’s formula.

Let \( s_m(x_1, \ldots, x_k) \) and \( s_1^m(x_1, \ldots, x_k) \) be respectively the complete and elementary symmetric polynomials of degree \( m \) in the variables \( x_1, \ldots, x_k \). When evaluated at certain Chern classes, they become the cohomology classes of special Schubert varieties. Let \( \ell(w) \) be the length of a permutation \( w \). We will show:

**THEOREM 1.** — Let \( k, m, n \) be positive integers, and let \( w \in S_n \).

1. \( S^w \cdot s_m(x_1, \ldots, x_k) = \sum_v S^v \), the sum over all \( v = wt_{a_1, b_1} \cdots t_{a_m, b_m} \), where \( a_i \leq k < b_i \) and \( \ell(wt_{a_1, b_1} \cdots t_{a_i, b_i}) = \ell(w) + i \) for \( 1 \leq i \leq m \) with the integers \( b_1, \ldots, b_m \) distinct.

2. \( S^w \cdot s_1^m(x_1, \ldots, x_k) = \sum_v S^v \), the sum over all \( v \) as in 1, except that now the integers \( a_1, \ldots, a_m \) are distinct.

Theorem 1 computes some of the structure constants in the cohomology ring of the flag manifold. If \( n \) is taken large enough, equivalently, if the index of summation is over \( v \in S_{n+m} \), then these cohomological formulas become identities among the Schubert polynomials.

These formulas were stated in a different form by Lascoux and Schützenberger in [17], where an algebraic proof was outlined. They were later independently conjectured in yet another form by Bergeron and Billey [2]. Our formulation facilitates our proofs. Using geometry, we expose a surprising connection to the classical Pieri’s formula (Lemma 11), from which we deduce Theorem 1. In Theorem 5 this connection is used to
determine additional structure constants. Theorem 8 utilizes the formulas of Theorem 1 to give a formula for the multiplication of a Schubert polynomial by a hook Schur polynomial, indicating a relation between multiplication of Schubert polynomials and paths in the Bruhat order on $S_n$. This is exploited in Corollary 9 to deduce an enumerative result about the Bruhat order on $S_n$.

This exposition is organized as follows: Section 2 contains preliminaries about Schubert polynomials while Section 3 is devoted to the flag manifold. In Section 4 we deduce our main results from a geometric lemma proven in Section 5. We remark that while our results are stated in terms of the integral cohomology of the complex manifold of complete flags, our results and proofs are valid for the Chow rings of flag varieties defined over any field.

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2. Schubert polynomials.

In [3], [7] cohomology classes of Schubert subvarieties of the flag manifold were obtained from the class of a point using repeated correspondences in $\mathbb{P}^1$-bundles, which may be described algebraically as “divided differences.” Subsequently, Lascoux and Schützenberger [17] found explicit polynomial representatives for these classes. We outline Lascoux and Schützenberger's construction of Schubert polynomials. For a more complete account, see [20].

For an integer $n > 0$, let $S_n$ be the group of permutations of $[n] = \{1, 2, \ldots, n\}$. Let $t_{a,b}$ be the transposition interchanging $a < b$. Adjacent transpositions $s_i = t_{i+1,i}$ generate $S_n$. The length, $\ell(w)$, of a permutation $w$ is characterized by $\ell(w t_{a,b}) = \ell(w) + 1$ if and only if $w(a) < w(b)$ and whenever $a < c < b$, either $w(c) < w(a)$ or $w(b) < w(c)$.

For each integer $n > 1$, let $R_n = \mathbb{Z}[x_1, \ldots, x_n]$. The group $S_n$ acts on $R_n$ by permuting the variables. For $f \in R_n$, the polynomial $f - s_i f$ is antisymmetric in $x_i$ and $x_{i+1}$, and so is divisible by $x_i - x_{i+1}$. Thus we may define the linear divided difference operator

$$\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i).$$
If \( w = s_{a_1} s_{a_2} \cdots s_{a_p} \) is a factorization of \( w \) into adjacent transpositions with minimal length (\( p = \ell(w) \)), then the composition of divided differences \( \partial_{a_1} \circ \cdots \circ \partial_{a_p} \) depends only upon \( w \), defining an operator \( \partial_w \) for each \( w \in S_n \). Let \( w_0 \) be the longest permutation in \( S_n \), that is \( w_0(j) = n+1-j \). For \( w \in S_n \), define the Schubert polynomial \( \mathfrak{S}_w \) by

\[
\mathfrak{S}_w = \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2} \cdots x_{n-1}).
\]

The degree of \( \partial_i \) is \(-1\), so \( \mathfrak{S}_w \) is homogeneous of degree \( \binom{n}{2} - \ell(w^{-1}w_0) = \ell(w) \).

Let \( S \subset R_n \) be the ideal generated by the non-constant symmetric polynomials. The set \( \{ \mathfrak{S}_w \mid w \in S_n \} \) of Schubert polynomials is a basis for \( \mathbb{Z}[x_1^1 \cdots x_n^{n-1} \mid i_j \leq n-j] \), a transversal to \( S \) in \( R_n \). Thus Schubert polynomials are explicit polynomial representatives of an integral basis for the ring \( H_n = R_n/S \).

Recently, other descriptions have been discovered for Schubert polynomials \([1],[4],[10],[11]\). One may define Schubert polynomials \( \mathfrak{S}_w \) for all \( w \in S_\infty = \bigcup_{n=1}^{\infty} S_n \). Then \( \{ \mathfrak{S}_w \mid w \in S_\infty \} \) is an integral basis for the polynomial ring \( \mathbb{Z}[x_1, x_2, \ldots] \). While our methods involve cohomology calculations and so are \textit{a priori} valid only in the rings \( H_n \), they imply identities among Schubert polynomials in the ring \( \mathbb{Z}[x_1, x_2, \ldots] \).

A partition \( \lambda \) is a decreasing sequence \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \) of positive integers, called the \textit{parts} of \( \lambda \). Given a partition \( \lambda \) with at most \( k \) parts, there is a Schur polynomial \( s_{\lambda} = s_{\lambda}(x_1, \ldots, x_k) \), which is symmetric in the variables \( x_1, \ldots, x_k \). For a more complete treatment of Schur polynomials, see \([19]\).

The collection of Schur polynomials forms a basis for the ring of symmetric polynomials, \( \mathbb{Z}[x_1, \ldots, x_k]^{S_k} \). The Littlewood-Richardson rule is a formula for the structure constants \( c_{\mu\nu}^\lambda \) for this basis, called \textit{Littlewood-Richardson coefficients}, which are defined by the identity

\[
s_{\mu} \cdot s_{\nu} = \sum_{\lambda} c_{\mu\nu}^\lambda s_{\lambda}.
\]

If \( \lambda \) and \( \mu \) are partitions satisfying \( \lambda_i \geq \mu_i \) for all \( i \), we write \( \lambda \supset \mu \). This defines a partial order on the collection of partitions, called Young’s lattice. Since \( c_{\mu\nu}^\lambda = 0 \) unless \( \lambda \supset \mu \) and \( \lambda \supset \nu \) (cf. \([19]\)), we see that \( \mathcal{I}_{n,k} = \{ s_{\lambda} \mid \lambda_1 \geq n-k \} \) is an ideal. Let \( A_{n,k} \) be the quotient ring \( \mathbb{Z}[x_1, \ldots, x_k]^{S_k}/\mathcal{I}_{n,k} \). 
To a partition \( \lambda \) we may associate its Young diagram, also denoted \( \lambda \), which is a left-justified array of boxes in the plane with \( \lambda_i \) boxes in the \( i \)th row. If \( \lambda \supset \mu \), then the Young diagram of \( \mu \) is a subset of that of \( \lambda \), and the skew diagram \( \lambda / \mu \) is the set theoretic difference \( \lambda - \mu \). If each column of \( \lambda / \mu \) is either empty or a single box, then \( \lambda / \mu \) is a skew row of length \( m \), where \( m \) is the number of boxes in \( \lambda / \mu \). The transpose \( \mu^t \) of a partition \( \mu \) is the partition whose Young diagram is the transpose of that of \( \mu \). We call the transpose of a skew row a skew column. The map defined by \( s_\lambda \mapsto s_{\lambda^t} \) is a ring isomorphism \( A_{n,k} \rightarrow A_{n,n-k} \).

If \( w \) has only one descent (\( k \) such that \( w(k) > w(k+1) \)), then \( w \) is said to be Grassmannian of descent \( k \) and \( \mathfrak{S}_w \) is the Schur polynomial \( s_\lambda(x_1, \ldots, x_k) \). Here \( \lambda \) is the shape of \( w \), the partition with \( k \) parts where \( \lambda_{k+1-j} = w(j) - j \). For integers \( k, m \) define \( r[k, m] \) and \( c[k, m] \) to be the Grassmannian permutations of descent \( k \) with shapes \((m, 0, \ldots, 0) = m \) and \((1^m, 0, \ldots, 0) = 1^m \), respectively. These are the \( m + 1 \)-cycles

\[
\begin{align*}
  r[k, m] &= (k+m \ k+m-1 \ \ldots \ k+2 \ k+1 \ k) \\
  c[k, m] &= (k-m+1 \ k-m+2 \ \ldots \ k-1 \ k+1).
\end{align*}
\]

3. The flag manifold.

Let \( V \) be an \( n \)-dimensional complex vector space. A flag \( F_\bullet \) in \( V \) is a sequence

\[
\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = V,
\]

of linear subspaces with \( \dim_{\mathbb{C}} F_i = i \). The set of all flags is a \( \frac{1}{2}n(n-1) \) dimensional complex manifold, called the flag manifold and denoted \( F(V) \). Over \( F(V) \), there is a tautological flag \( F_\bullet \) of bundles whose fibre at a point \( F_\bullet \) is the flag \( F_\bullet \). Let \(-x_i\) be the Chern class of the line bundle \( F_i/F_{i-1} \). Then the integral cohomology ring of \( F(V) \) is \( H_n = \mathbb{Z}[x_1, \ldots, x_n]/S \), where \( S \) is the ideal generated by those non-constant polynomials which are symmetric in \( x_1, \ldots, x_n \). This description is due to Borel [5].

Given a subset \( S \subset V \), let \( \langle S \rangle \) be its linear span and for linear subspaces \( W \subset U \) let \( U - W \) be their set theoretic difference. An ordered basis \( f_1, f_2, \ldots, f_n \) for \( V \) determines a flag \( E_\bullet \); set \( E_i = \langle f_1, \ldots, f_i \rangle \) for \( 1 \leq i \leq n \). In this case, write \( E_\bullet = \langle f_1, \ldots, f_n \rangle \). A fixed flag \( F_\bullet \) gives a decomposition due to Ehresmann [9] of \( F(V) \) into affine cells indexed by permutations \( w \) of \( S_n \). The cell determined by \( w \) is

\[
X^w_{\bullet} = \{E_\bullet = \langle f_1, \ldots, f_n \rangle \mid f_i \in F_{n+1-w(i)} - F_{n-w(i)}, \ 1 \leq i \leq n \}.
\]
The complex codimension of $X^o_{w,F}$ is $\ell(w)$ and its closure is the Schubert subvariety $X_{w,F}$. Thus the cohomology ring of $\mathbb{F}(V)$ has an integral basis given by the cohomology classes\(^{(1)}\) $[X_{w,F}]$, called Schubert classes, of the Schubert subvarieties.

Independently, Bernstein-Gelfand-Gelfand \cite{BerGelfGelf79} and Demazure \cite{Dem76} related this description to Borel’s, showing $[X_{w,F}] = \partial_{w^{-1}w_0}[X_{w_0,F}]$. Later, Lascoux and Schützenberger \cite{LasSchu82} defined Schubert polynomials, and since $x_1^{n-1}x_2^{n-2}\ldots x_{n-1}$ equals $[X_{w,F}]$, the class of a point, showed that $[X_{w,F}] = \mathcal{G}_w(x_1,\ldots,x_n)$. We adopt the convention of writing $\mathcal{G}_w$ for the Schubert class $[X_{w,F}]$. Since the composition

$$Z[x_1,\ldots,x_n] \rightarrow Z[x_1,\ldots,x_{n+m}] \rightarrow H_{n+m}$$

is an isomorphism in low degrees, one may deduce identities of Schubert polynomials from product formulas for Schubert classes.

This Schubert basis for cohomology diagonalizes the intersection pairing; if $\ell(w) + \ell(v) = \dim \mathbb{F}(V) = \frac{1}{2}n(n-1)$, then

$$\mathcal{G}_w \cdot \mathcal{G}_v = \begin{cases} 
\mathcal{G}_w & \text{if } v = w_0w \\
0 & \text{otherwise.}
\end{cases}$$

For each $k \leq n = \dim V$, the set of all $k$-dimensional subspaces of $V$ is a $k(n-k)$ dimensional complex manifold, called the Grassmannian of $k$-planes in $V$, written $G_kV$. A fixed flag $F_\bullet$ gives a decomposition of $G_kV$ into cells indexed by partitions $\lambda$ with $k$ parts, none exceeding $n-k$. The closure of such a cell is the Schubert variety

$$\Omega_{\lambda}F_\bullet = \left\{ H \in G_kV \mid \dim H \bigcap F_{n-k+j-\lambda_j} \geq j \text{ for } 1 \leq j \leq k \right\},$$

whose codimension is $\lambda_1+\cdots+\lambda_k = |\lambda|$.

The evaluation of a symmetric polynomial in $k$ variables at the Chern roots $x_1,\ldots,x_n$ of the dual of the tautological $k$-plane bundle on $G_kV$ identifies $H^*G_kV$ with the ring $A_{n,k}$ of §2. The classes $[\Omega_{\lambda}F_\bullet]$ form a basis for the cohomology ring of $G_kV$ and $[\Omega_{\lambda}F_\bullet]$ is $s(\lambda)(x_1,\ldots,x_k)$. We will write $s(\lambda)$ for the Schubert class $[\Omega_{\lambda}F_\bullet]$.

If $Y \subset V$ has codimension $d$, then $G_kY \subset G_kV$ is a Schubert subvariety whose indexing partition is $d^k$, the partition with $k$ parts each equal to $d$. It follows that $\Omega_{(n-k)^k}F_\bullet = \{F_k\}$, so $s_{(n-k)^k}$ is the class of a point.

\(^{(1)}\) Strictly speaking, we mean the classes Poincaré dual to the fundamental cycles in homology.
The Schubert basis diagonalizes the intersection pairing; for a partition \( \lambda \), let \( \lambda^c \) be the partition \( (n-k-\lambda_1, \ldots, n-k-\lambda_l) \). If \(|\mu| + |\lambda| = k(n-k)\), then
\[
s_\lambda \cdot s_\mu = \begin{cases} s_{(n-k)^k} & \text{if } \lambda^c = \mu \\ 0 & \text{otherwise.} \end{cases}
\]

The Schur polynomial \( s_m \) is the complete symmetric polynomial of degree \( m \) in \( x_1, \ldots, x_k \). The Schur polynomial \( s_{1m} \) is the \( m \)th elementary symmetric polynomial in \( x_1, \ldots, x_k \). Pieri’s formula is a formula for multiplying Schur polynomials by either \( s_m \) or \( s_{1m} \). For \( s_m \), suppose \(|\mu| + |\lambda^c| + m = k(n-k)\), then
\[
s_\mu \cdot s_{\lambda^c} \cdot s_m = \begin{cases} s_{(n-k)^k} & \text{if } \lambda/\mu \text{ is a skew row of length } m \\ 0 & \text{otherwise.} \end{cases}
\]

For \( k \leq n \), the association \( E_* \mapsto E_k \) defines a map \( \pi : \mathbb{F}(V) \to G_k V \). The functorial map \( \pi^* \) on cohomology is induced by the inclusion into \( H_n \) of polynomials symmetric in \( x_1, \ldots, x_k \). That is, \( A_{n,k} \hookrightarrow H_n \). If \( \lambda \) is a partition with \( k \) parts and \( w \) the Grassmannian permutation of descent \( k \) and shape \( \lambda \), then \( \pi^* s_\lambda = \mathcal{S}_w \).

Under the Poincaré duality isomorphism between homology and cohomology groups, the functorial map \( \pi_* \) on homology induces a group homomorphism \( \pi_* \) on cohomology. While \( \pi_* \) is not a ring homomorphism, it does satisfy the projection formula (see Example 8.17 of [12]):
\[
\pi_*(\alpha \cdot \pi^* \beta) = (\pi_* \alpha) \cdot \beta,
\]
where \( \alpha \) is a cohomology class on \( \mathbb{F}(V) \) and \( \beta \) is a cohomology class on \( G_k V \).

4. Pieri’s formula for flag manifolds.

An open problem is to find the analog of the Littlewood-Richardson rule for Schubert polynomials. That is, determine the structure constants \( c^w_{uv} \) for the Schubert basis of the cohomology of flag manifolds, which are defined by the identity
\[
\mathcal{S}_w \cdot \mathcal{S}_v = \sum_u c^w_{uv} \mathcal{S}_u.
\]
These constants are positive integers as they count the points in a suitable triple intersection of Schubert subvarieties. They are are known only in some special cases.
For example, if both $w$ and $v$ are Grassmannian permutations of descent $k$ so that $\mathcal{G}_w$ and $\mathcal{G}_v$ are pullbacks of classes from $G_k V$, then the classical Littlewood-Richardson rule gives a formula for the $c_{uv}^w$'s.

Another case is Monk's formula, which states:

$$\mathcal{G}_w \cdot \mathcal{G}_k = \sum \mathcal{G}_{w_{ta,b}},$$

the sum over all $a \leq k < b$ with $\ell(w_{ta,b}) = \ell(w) + 1$. We use geometry to generalize this formula, giving an analog of the classical Pieri's formula.

Let $w, v \in S_n$. Write $w \rightarrow v$ if there exist integers $a_1, b_1, \ldots, a_m, b_m$ with

1. $v = wt_{a_1, b_1} \cdots t_{a_m, b_m}$,
2. $a_i < k < b_i$ and $\ell(w t_{a_1, b_1} \cdots t_{a_i, b_i}) = \ell(w) + i$ for $1 \leq i \leq m$, and
3. the integers $b_1, b_2, \ldots, b_m$ are distinct.

Similarly, $w \rightarrow v$ if we have integers $a_1, \ldots, b_m$ as in (1) and (2) where now

(3)' the integers $a_1, a_2, \ldots, a_m$ are distinct.

Our primary result is the following.

THEOREM 1. — Let $w \in S_n$. Then

1. For all $k$ and $m$ with $k + m < n$, we have $\mathcal{G}_w \cdot \mathcal{G}_{r[k,m]} = \sum_{w \rightarrow v} \mathcal{G}_v$.
2. For all $m < k < n$, we have $\mathcal{G}_w \cdot \mathcal{G}_{c[k,m]} = \sum_{w \rightarrow v} \mathcal{G}_v$.

Theorem 1 may be alternatively stated in terms of the structure constants $c_{uv}^w$.

THEOREM 1'. — Let $w, v \in S_n$. Then

1. For all integers $k, m$ with $k + m \leq n$, $c_{w r[k,m]}^v = \begin{cases} 1 & \text{if } w \rightarrow v \\ 0 & \text{otherwise.} \end{cases}$
2. For all integers $k, m$ with $m \leq k \leq n$, $c_{w c[k,m]}^v = \begin{cases} 1 & \text{if } w \rightarrow v \\ 0 & \text{otherwise.} \end{cases}$

We first show the equivalence of parts 1 and 2 and then establish part 1. An order $<_k$ on $S_n$ is introduced, and we show that $c_{w r[k,m]}^v$ is 0
unless $w <_k v$. A geometric lemma enables us to compute $c_{w r[k, m]}^v$ when $w <_k v$.

**Lemma 2.** Let $w_0$ be the longest permutation in $S_n$, and $k + m \leq n$. Then

1. $w_0 r[k, m] w_0 = c_{[n-k, m]}.$
2. Let $w, v \in S_n$. Then $w \rightarrow v$ if and only if $w_0 w_0 \rightarrow w_0 v w_0$.
3. The map induced by $G_w \mapsto G_{w_0 w w_0}$ is an automorphism of $H_n$.
4. Statements 1 and 2 of Theorem 1' are equivalent.

This automorphism $G_w \mapsto G_{w_0 w w_0}$ is the analog of the map $s_{\lambda}(x_1, \ldots, x_k) \mapsto s_{\lambda}(x_1, \ldots, x_{n-k})$ for Grassmannians.

**Proof.** Statements (1) and (2) are easily verified, as $w_0(j) = n + 1 - j$.

Statement (3) is also immediate, as $G_w \mapsto G_{w_0 w w_0}$ leaves Monk’s formula invariant and Monk’s formula characterizes the algebra of Schubert polynomials.

For (4), suppose $k + m \leq n$ and $w, v \in S_n$ and let $\overline{w}$ denote $w_0 w w_0$.

The isomorphism $G_w \mapsto G_{\overline{w}}$ of (3) shows $c_{w r[k, m]}^v = c_{\overline{w} r[k, m]}^{\overline{w}}$. Part (1) shows $c_{\overline{w} r[k, m]}^v = c_{\overline{w} c[n-k, m]}^v$. Then (2) shows the equality of the two statements of Theorem 1'.

Let $<_k$ be the transitive closure of the relation given by $w <_k w t_{a, b}$ where $a \leq k < b$ and $\ell(w t_{a, b}) = \ell(w) + 1$. We call $<_k$ the $k$-Bruhat order, in [18] it is the $k$-colored Ehresmanœdre.

**Lemma 3.** If $c_{w r[k, m]}^v \neq 0$, then $w <_k v$ and $\ell(v) = \ell(w) + m$.

**Proof.** By Monk’s formula, $w <_k v$ if and only if $G_v$ appears with a non-zero (necessarily positive) coefficient when $G_w (G_{t_{k, k+1}})^{\ell(v) - \ell(w)}$ is written as a sum of Schubert classes.

Since $r[k, m] = t_{k, k+1} \cdot t_{k, k+2} \cdots t_{k, k+m}$, Monk’s formula shows that $G_{r[k, m]}$ is a summand of $(G_{s_k})^m$ with coefficient 1. Thus the coefficient of $G_v$ in the expansion of $G_w \cdot (G_{s_k})^m$ exceeds that of $G_v$ in $G_w \cdot G_{r[k, m]}$. Hence $c_{w r[k, m]}^v = 0$ unless $w <_k v$ and $\ell(v) = \ell(w) + m$.

In Section 5 we use geometry to prove the following lemma.
Lemma 4. — Let $w <_k v$ be permutations in $S_n$. Suppose $v = wt_{a_1}b_1 \cdots t_{a_m}b_m$, where $a_i \leq k < b_i$ and $\ell(wt_{a_1}b_1 \cdots t_{a_i}b_i) = \ell(w) + i$ for $1 \leq i \leq m$. Let $d = n - k - \#\{b_1, \ldots, b_m\}$. Then

1. There is a cohomology class $\delta$ on $G_k V$ such that $\pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0v}) = \delta \cdot s_d$.

2. If $w \xrightarrow{r[k,m]} v$, then there are partitions $\lambda \supset \mu$ where $\lambda/\mu$ is a skew row of length $m$ whose $j$th row has length $\#\{i \mid a_i = j\}$ and $\pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0v}) = s_\mu \cdot s_\lambda = \sum \mu \nu \nu c_{\mu \nu} s_\nu$.

We first use this to compute some structure constants. For $\nu$ a partition with $k$ parts, let $w(\nu)$ be the Grassmannian permutation of descent $k$ and shape $\nu$.

Theorem 5. — Let $w, v \in S_n$ and $k \leq n$ be an integer. Suppose $w <_k v$ and $\ell(v) = \ell(w) + m$. Let $a_1, b_1, \ldots, a_m, b_m$ be such that $v = wt_{a_1}b_1 \cdots t_{a_m}b_m$ where $a_i \leq k < b_i$ and $\ell(wt_{a_1}b_1 \cdots t_{a_i}b_i) = \ell(w) + i$ for $1 \leq i \leq m$. Let $\nu$ be a partition with $k$ parts.

1. If $w \xrightarrow{r[k,m]} v$, the structure constant $c^w_{w(\nu)}$ equals the Littlewood-Richardson coefficient $c^\lambda_{\mu \nu}$, where $\lambda/\mu$ is a skew row of length $m$ whose $j$th row has length $\#\{i \mid a_i = j\}$.

2. If $w \xrightarrow{c[k,m]} v$, the structure constant $c^w_{w(\nu)}$ equals the Littlewood-Richardson coefficient $c^\lambda_{\mu \nu}$, where $\lambda/\mu$ is a skew column of length $m$ whose $j$th column has length $\#\{i \mid b_i = j\}$.

Proof. — Using the involution $\mathcal{G}_w \mapsto \mathcal{G}_{w_0w_0w}$, it suffices to prove part (1). Recall that $\mathcal{G}_{w(\nu)} = \pi^*(s_\nu)$. As $\mathcal{G}_{w_0}$ and $s_{(n-k)^k}$ are the classes of points, $\pi_* \mathcal{G}_{w_0} = s_{(n-k)^k}$. By the projection formula and part (2) of Lemma 4,

\[
c^w_{w(\nu)} s_{(n-k)^k} = \pi_* (c^w_{w(\nu)} \mathcal{G}_{w_0}) = \pi_* (\mathcal{G}_w \cdot \mathcal{G}_{w_0v} \cdot \mathcal{G}_{w(\nu)}) = \pi_* (\sum \mu \nu \nu c_{\mu \nu} s_\nu) = c^\lambda_{\mu \nu} s_{(n-k)^k}.
\]

Proof of Theorem 1'. — By Lemma 3, we need only show that if
$w <_k v$ and $\ell(v) - \ell(w) = m$, then

$$c^v_{w, r[k,m]} = \begin{cases} 1 & \text{if } w \rightarrow v \\ 0 & \text{otherwise.} \end{cases}$$

Begin by multiplying the identity $\mathcal{G}_w \cdot \mathcal{G}_{r[k,m]} = \sum_v c^v_{w,r[k,m]} \mathcal{G}_v$ by $\mathcal{G}_{w_0 v}$ and use the intersection pairing to obtain:

$$\mathcal{G}_w \cdot \mathcal{G}_{w_0 v} \cdot \mathcal{G}_{r[k,m]} = \sum_v c^v_{w,r[k,m]} \mathcal{G}_{w_0}.$$  

Recall that $\mathcal{G}_{r[k,m]} = \pi^* s_m(x_1, \ldots, x_k)$. Apply the map $\pi_*$ and then the projection formula to obtain:

$$\pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0 v} \cdot \pi^* s_m) = c^v_{w,r[k,m]} \pi_*(\mathcal{G}_{w_0})$$

$$\pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0 v}) \cdot s_m = c^v_{w,r[k,m]} s_{(n-k)^k}.$$  

By part (1) of Lemma 4, there is a cohomology class $\delta$ on $G_k V$ with

$$\pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0 v}) \cdot s_m = \delta \cdot s_{d^k} \cdot s_m$$

But $s_{d^k} \cdot s_m = 0$ unless $d + m \leq n - k$. Since $d = n - k - \#\{b_1, \ldots, b_m\} \geq n - k - m$, we see that $c^v_{w,r[k,m]} = 0$ unless $m = \#\{b_1, \ldots, b_m\}$, which implies $w \rightarrow v$.

To complete the proof of Theorem 1', suppose that $w \rightarrow^{r[k,m]} v$. By part (1) of Theorem 5, $c^v_{w,r[k,m]} = c^\lambda_{\mu, m}$, where $\lambda/\mu$ a skew row of length $m$ and $m = (m, 0, \ldots, 0)$. But this equals 1 by the classical Pieri's formula for the Grassmannian.

The formulas of Theorem 1 may be formulated as the sum over certain paths in the $k$-Bruhat order. We explain this formulation here. A (directed) path in the $k$-Bruhat order from $w$ to $v$ is equivalent to a choice of integers $a_1, b_1, \ldots, a_m, b_m$ with $a_i \leq k < b_i$ for $1 \leq i \leq m$ and if $w(0) = w$ and $w(i) = w^{(i-1)} \cdot t_{a_i, b_i}$, then $\ell(w(i)) = \ell(w) + i$ and $w^{(m)} = v$. Here, the path is

$$w = w(0) <_k w(1) <_k w(2) <_k \cdots <_k w^{(m)} = v.$$  

**Lemma 6.** — Let $w, v \in S_n$ and $k, m$ be positive integers. Then

1. $w \rightarrow^{r[k,m]} v$ if and only if there is a path in the $k$-Bruhat order of length $m$ such that

$$w^{(1)}(a_1) < w^{(2)}(a_2) < \cdots < w^{(m)}(a_m).$$
2. \( w \xrightarrow{c^{[k,m]}} v \) if and only if there is a path in the k-Bruhat order of length \( m \) such that
\[
w^{(1)}(a_1) > w^{(2)}(a_2) > \cdots > w^{(m)}(a_m).
\]
Furthermore, these paths are unique.

Proof. — If \( w \xrightarrow{r^{[k,m]}} v \), one may show that the set of values \( \{w^{(i)}(a_i)\} \) and the set of transpositions \( \{t_{a_i b_i}\} \) depend only upon \( w \) and \( v \), and not on the particular path chosen from \( w \) to \( v \) in the k-Bruhat order.

It is also the case that rearranging the set \( \{w^{(i)}(a_i)\} \) in order, as in (1), may be accomplished by interchanging transpositions \( t_{a_i b_i} \) and \( t_{a_j b_j} \) where \( a_i \neq a_j \) (necessarily \( b_i \neq b_j \)). Both (1) and the uniqueness of this representation follow from these observations. Statement (2) follows for similar reasons. \( \square \)

For a path \( \gamma \) in the k-Bruhat order, let \( \text{end}(\gamma) \) be the endpoint of \( \gamma \). We state a reformulation of Theorem 1.

**Corollary 7 (Path formulation of Theorem 1).** — Let \( w \in S_n \).

1. \( \mathfrak{S}_w \cdot \mathfrak{S}_{r^{[k,m]}} = \sum_{\gamma} \mathfrak{S}_{\text{end}(\gamma)} \), the sum over all paths \( \gamma \) in the k-Bruhat order which start at \( w \) such that
\[
w^{(1)}(a_1) < w^{(2)}(a_2) < \cdots < w^{(m)}(a_m),
\]
where \( \gamma \) is the path \( w <_k w^{(1)} <_k w^{(2)} <_k \cdots <_k w^{(m)} \).

Equivalently, \( c_{w r^{[k,m]}}^v \) counts the number of paths \( \gamma \) in the k-Bruhat order from \( w \) to \( v \) such that
\[
w^{(1)}(a_1) < w^{(2)}(a_2) < \cdots < w^{(m)}(a_m).
\]

2. \( \mathfrak{S}_w \cdot \mathfrak{S}_{c^{[k,m]}} = \sum_{\gamma} \mathfrak{S}_{\text{end}(\gamma)} \), the sum over all paths \( \gamma \) in the k-Bruhat order which start at \( w \) such that
\[
w^{(1)}(a_1) > w^{(2)}(a_2) > \cdots > w^{(m)}(a_m),
\]
where \( \gamma \) is the path \( w <_k w^{(1)} <_k w^{(2)} <_k \cdots <_k w^{(m)} \).

Equivalently, \( c_{w r^{[k,m]}}^v \) counts the number of paths \( \gamma \) in the k-Bruhat order from \( w \) to \( v \) such that
\[
w^{(1)}(a_1) > w^{(2)}(a_2) > \cdots > w^{(m)}(a_m).
\]
This is the form of the conjectures of Bergeron and Billey [2], and it exposes a link between multiplying Schubert polynomials and paths in the Bruhat order. Such a link is not unexpected. The Littlewood-Richardson rule for multiplying Schur functions may be expressed as a sum over certain paths in Young’s lattice of partitions. A connection between paths in the Bruhat order and the intersection theory of Schubert varieties is described in [14]. We believe the eventual description of the structure constants $c_{uv}^w$ will be in terms of counting paths of certain types in the Bruhat order on $S_n$, and that there will be appropriate generalizations for the other classical groups. This should yield new enumerative results about the Bruhat orders on their respective Weyl groups, in the spirit of Corollary 9 below.

Using multiset notation for partitions, $(p,1^{q-1})$ is the hook shape partition whose Young diagram is the union of a row of length $p$ and a column of length $q$. Define $h[k; p, q]$ to be the Grassmannian permutation of descent $k$ and shape $(p,1^{q-1})$. Then $G_{h[k; p, q]} = \pi^* s_{(p,1^{q-1})}$. This permutation, $h[k; p, q]$, is the $p+q$-cycle

$$(k-q+1 \ k-q+2 \ldots \ k-1 \ k \ k+p \ k+p-1 \ldots k+1).$$

THEOREM 8.— Let $q \leq k$ and $k+p \leq n$ be integers. Set $m = p+q-1$. For $w \in S_n$,

$$G_w \cdot G_{h[k; p, q]} = \sum \mathcal{S}_{\text{end}(\gamma)},$$

the sum over all paths $\gamma : w <_k w^{(1)} <_k w^{(2)} <_k \ldots <_k w^{(m)}$ in the $k$-Bruhat order with

$$w^{(1)}(a_1) < \ldots < w^{(p)}(a_p) \quad \text{and} \quad w^{(p)}(a_p) > w^{(p+1)}(a_{p+1}) > \ldots > w^{(m)}(a_m).$$

Alternatively, the sum over those paths $\gamma$ with

$$w^{(1)}(a_1) > \ldots > w^{(q)}(a_q) \quad \text{and} \quad w^{(q)}(a_q) < \ldots < w^{(m)}(a_m).$$

Setting either $p = 1$ or $q = 1$, we recover Theorem 1. If we consider the coefficient $c_{w, h[k; p, q]}^v$ of $G_v$ in the product $G_w \cdot G_{h[k; p, q]}$, we obtain:

COROLLARY 9.— Let $w, v \in S_n$, and $p, q$ be positive integers where $\ell(v) - \ell(w) = p + q - 1 = m$. Then the number of paths $w <_k w^{(1)} <_k w^{(2)} <_k \ldots <_k w^{(m)} = v$ in the $k$-Bruhat order from $w$ to $v$ with

$$w^{(1)}(a_1) < \ldots < w^{(p)}(a_p) \quad \text{and} \quad w^{(p)}(a_p) > w^{(p+1)}(a_{p+1}) > \ldots > w^{(m)}(a_m)$$

equals the number of paths with

$$w^{(1)}(a_1) > \ldots > w^{(q)}(a_q) \quad \text{and} \quad w^{(q)}(a_q) < \ldots < w^{(m)}(a_m).$$
Proof of Theorem 8. — By the classical Pieri’s formula,
\[ s_p \cdot s_{1(q-1)} = s_{(p+1,1^{q-2})} + s_{(p,1^{q-1})}. \]
Expressing these as Schubert classes on the flag manifold (applying \( \pi^* \)), we have:
\[ \mathcal{G}_r[k,p] \cdot \mathcal{G}_{c[k,q-1]} = \mathcal{G}_h[k;p+1,q-1] + \mathcal{G}_h[k;p,q]. \]
Induction on either \( p \) or \( q \) (with \( m \) fixed) and Corollary 7 completes the proof.

5. Geometry of intersections.

We deduce Lemma 4 by studying certain intersections of Schubert varieties. A key fact we use is that if \( X_w F_w \) and \( X_v G_v \) intersect generically transversally, then
\[ [X_w F_w \cap X_v G_v] = [X_w F_w] \cdot [X_v G_v] = \mathcal{G}_w \cdot \mathcal{G}_v \]
in the cohomology ring. Flags \( F_w \) and \( G_v \) are opposite if for \( 1 \leq i \leq n \), \( F_i + G_{n-i} = V \). The set of pairs of opposite flags form the dense orbit of the general linear group \( GL(V) \) acting on the space of all pairs of flags. Using this observation and Kleiman’s Theorem concerning the transversality of a general translate [16], we conclude that for any \( w, v \in S_n \) and opposite flags \( F_w \) and \( G_v \), \( X_w F_w \) and \( X_v G_v \) intersect generically transversally.

Deodhar [8] studies the intersection of two Schubert cells
\[ X_w^O F_w \cap X_{w_0 v} G_v. \]
He shows the intersection is non-empty precisely when \( w \leq v \) in the (ordinary) Bruhat order. In this case, that intersection is decomposed into locally closed subvarieties \( D_{x} \), each isomorphic to \( (\mathbb{C}^x)^a \times \mathbb{C}^b \), where \( x \) runs over certain subexpressions of reduced words of \( v \), with \( a \) and \( b \) satisfying \( \ell(v) - \ell(w) = a + 2b \), and with a unique index \( \sigma' \) with \( b = 0 \). It follows that \( X_w F_w \cap X_{w_0 v} G_v \) is irreducible with a dense subset \( D_{x} \simeq (\mathbb{C}^x)^{\ell(v)-\ell(w)}. \)

These facts hold for the Schubert subvarieties of \( G_k V \) as well. Namely, if \( \lambda \) and \( \mu \) are any partitions with \( \mu \subset \lambda \) and \( F_\lambda \) and \( G_\mu \) are opposite flags, then \( \Omega_\mu F_\lambda \cap \Omega_\lambda G_\mu \) is an irreducible, generically transverse intersection containing a dense subset isomorphic to \( (\mathbb{C}^x)^{\lambda|-\mu|}. \)

Let \( F_w \) and \( F_w' \) be opposite flags in \( V \). Let \( e_1, \ldots, e_n \) be a basis for \( V \) such that \( e_i \) generates the one dimensional subspace \( F_{n+1-i} \cap F'_i \). We deduce Lemma 4 from the following two results.
LEMMA 10. — Let \( w, v \in S_n \) with \( w <_k v \) and \( \ell(v) - \ell(w) = m \). Suppose that \( v = wt_{a_1}b_1 \cdots t_{a_m}b_m \) with \( a_i \leq k < b_i \) and \( \ell(wt_{a_1}b_1 \cdots t_{a_i}b_i) = \ell(w) + i \) for \( 1 \leq i \leq m \). Let \( \pi : \mathbb{F}(V) \to G_k V \) be the canonical projection. Define \( Y = \{ e_{w(j)} \mid j \leq k \text{ or } w(j) \neq v(j) \} \). Then \( Y \) has codimension \( d = n - k - \# \{ b_1, \ldots, b_m \} \) and
\[
\pi \left( X_w F_* \cap X_{w_0 v} F'_* \right) \subset G_k Y.
\]
Also, if \( E_* \in X_w F_* \cap X_{w_0 v} F'_* \), then there exist a basis \( f_1, \ldots, f_n \) for \( V \) with \( E_* = \langle f_1, \ldots, f_n \rangle \), where, if \( j > k \) with \( w(j) = v(j) \), then \( f_j = e_{w(j)} \).

LEMMA 11. — Let \( w, v \in S_n \) with \( w \rightarrow v \) and let \( a_1, \ldots, b_m \) be as in the statement of Lemma 10. Then there exist opposite flags \( G_* \) and \( G'_* \) and partitions \( \lambda \supset \mu \), with \( \lambda/\mu \) a skew row of length \( m \) whose \( j \)th row has length \( \# \{ i \mid a_i = j \} \) such that
\[
\pi(X_w F_* \cap X_{w_0 v} F'_*) = \Omega_\mu G_* \cap \Omega_\lambda G'_*,
\]
and the map \( \pi|_{X_w F_* \cap X_{w_0 v} F'_*} : X_w F_* \cap X_{w_0 v} F'_* \to \Omega_\mu G_* \cap \Omega_\lambda G'_* \) has degree 1.

Lemma 11 vividly exhibits the connection to the classical Pieri’s formula that was mentioned in the Introduction. A typical geometric proof of Pieri’s formula for Grassmannians (see [13], [15]) involves showing a triple intersection of Schubert varieties
\[
(1) \quad \Omega_\mu G_* \cap \Omega_\lambda G'_* \cap \Omega_m G''_*
\]
is transverse and consists of a single point, when \( G_*, G'_* \), and \( G''_* \) are in suitably general position.

One could construct a proof of Theorem 1 along those lines, studying a triple intersection of Schubert subvarieties
\[
(2) \quad X_w G_* \cap X_{w_0 v} G'_* \cap X_{r[k,m]} G''_*,
\]
where \( G_*, G'_* \), and \( G''_* \) are in suitably general position. Doing so, one observes that the geometry of the intersection of (2) is governed entirely by the geometry of an intersection similar to that in (1). In part, that is because \( X_{r[k,m]} G''_* = \pi^{-1}\Omega_m G''_* \). This is the spirit of our method, which may be seen most vividly in Lemmas 14 and 15.

Proof of Lemma 4. — Since \( F_* \) and \( F'_* \) are opposite flags,
\[
X_w F_* \cap X_{w_0 v} F'_*
\]
is a generically transverse intersection, so in the cohomology ring
\[ X_w F_\bullet \cap X_{w_0 V} F_\bullet' = [X_w F_\bullet] \cdot [X_{w_0 V} F_\bullet'] = \mathcal{G}_w \cdot \mathcal{G}_{w_0 V}. \]

Let \( Y \) be the subspace of Lemma 10. Since \( \pi(X_w F_\bullet \cap X_{w_0 V} F_\bullet') \subset G_k Y \), the class \( \pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0 V}) \) is a cohomology class on \( G_k Y \). However, all such classes are of the form \( \delta \cdot [G_k Y] \), for some cohomology class \( \delta \) on \( G_k V \). Since \( d \) is the codimension of \( Y \), we have \( [G_k Y] = s_{dk} \), establishing part (1) of Lemma 4.

For part (2), suppose further that \( w \longrightarrow v \). If \( \rho \) is the restriction of \( \pi \) to \( X_w F_\bullet \cap X_{w_0 V} F_\bullet' \), then
\[ \pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0 V}) = \pi_* \left( [X_w F_\bullet \cap X_{w_0 V} F_\bullet'] \right) = \deg \rho \left[ \pi \left( X_w F_\bullet \cap X_{w_0 V} F_\bullet' \right) \right]. \]

By Lemma 11, \( \deg \rho = 1 \) and \( \pi(X_w F_\bullet \cap X_{w_0 V} F_\bullet') = \Omega_\mu G_\bullet \cap \Omega_\chi G_\bullet' \). Since \( G_\bullet \) and \( G_\bullet' \) are opposite flags, we have
\[ \pi_*(\mathcal{G}_w \cdot \mathcal{G}_{w_0 V}) = 1 \cdot \left[ \Omega_\mu G_\bullet \cap \Omega_\chi G_\bullet' \right] = [\Omega_\mu G_\bullet] \cdot [\Omega_\chi G_\bullet'] = s_\mu \cdot s_\chi \]
\[ = \sum \lambda c_\mu^\lambda s_\nu \cdot s_\nu'. \]

The last equality follows by the Littlewood-Richardson rule and the identity \( c_\mu^\lambda = c_\mu^\nu \).

We deduce Lemma 10 from two additional lemmas. We first make a definition. Let \( W \subsetneq V \) be a codimension 1 subspace and let \( e \in V - W \) so that \( V = \langle W, e \rangle \). For \( 1 \leq p \leq n \), define an expanding map \( \psi_p : \mathcal{F}(W) \longrightarrow \mathcal{F}(V) \) as follows:
\[ (\psi_p E_\bullet)_i = \begin{cases} E_i & \text{if } i < p \\ (E_{i-1}, e) & \text{if } i \geq p. \end{cases} \]

Note that if \( E_\bullet = (f_1, \ldots, f_n) \), then \( \psi_p E_\bullet = (f_1, \ldots, f_{p-1}, e, f_p, \ldots, f_n) \).

For \( w \in S_n \) and \( 1 \leq p \leq n \), define \( w|_p \in S_{n-1} \) by
\[ w|_p(j) = \begin{cases} w(j) & \text{if } j < p \text{ and } w(j) < w(p) \\ w(j+1) & \text{if } j \geq p \text{ and } w(j) < w(p) \\ w(j) - 1 & \text{if } j < p \text{ and } w(j) > w(p) \\ w(j+1) - 1 & \text{if } j \geq p \text{ and } w(j) > w(p). \end{cases} \]

If we represent permutations as matrices, \( w|_p \) is obtained by crossing out the \( p \)th row and \( w(p) \)th column of the matrix for \( w \).

**Lemma 12.** Let \( W \subsetneq V \) and \( e \in V - W \) with \( V = \langle W, e \rangle \). Let \( G_\bullet \) be a complete flag in \( W \). For \( 1 \leq p \leq n \) and \( w \in S_n \),
\[ \psi_p \left( X_{w|_p} G_\bullet \right) \subset X_w \left( \psi_{w_0 w(p)}(G_\bullet) \right). \]
Proof. — Let $E_\bullet \in X_{w|_p} G_\bullet$. Then $W$ has a basis $f_1, \ldots, f_{n-1}$ with $E_\bullet = \langle f_1, \ldots, f_{n-1} \rangle$ and for each $1 \leq i \leq n-1$, $f_i \in G_{n-w|_p(i)}$. Then we necessarily have $\psi_p(E_\bullet) = \langle \phi_1, \ldots, \phi_n \rangle = \langle f_1, \ldots, f_{p-1}, e, f_p, \ldots, f_{n-1} \rangle$. Noting

$$\left(\psi_{w_0 w(p)}(G_\bullet)\right)_{n+1-j} = \begin{cases} G_{n+1-j} & \text{if } j > w(p) \\ \langle e, G_{n-j} \rangle & \text{if } j \leq w(p), \end{cases}$$

we see that $\phi_i \in \left(\psi_{w_0 w(p)}(G_\bullet)\right)_{n+1-w(i)}$. Thus

$$\psi_p \left( X_{w|_p} G_\bullet \right) \subset X_w \left( \psi_{w_0 w(p)}(G_\bullet) \right). \tag*{□}$$

**Lemma 13.** — Let $W \subseteq V$ and $e \in V - W$ with $V = \langle W, e \rangle$ and let $G_\bullet$ and $G_\bullet'$ be opposite flags in $W$. Suppose that $w <_k v$ are permutations in $S_n$ and $p > k$ an integer such that $w(p) = v(p)$. Let $w_0^{(i)}$ be the longest permutation in $S_i$. Then

1. $\ell(v|_p) - \ell(w|_p) = \ell(v) - \ell(w)$ and $w|_p <_k v|_p$.

2. $\psi_p \left( X_{w|_p} G_\bullet \cap X_{w_0^{(n-1)}(v|_p)} G_\bullet' \right) = X_w \left( \psi_{w_0^{(n)} w(p)}(G_\bullet) \right) \cap X_{w_0^{(n)} v} \left( \psi_{v(p)}(G_\bullet') \right)$.

3. If $E_\bullet \in X_w \left( \psi_{w_0^{(n)} w(p)}(G_\bullet) \right) \cap X_{w_0^{(n)} v} \left( \psi_{v(p)}(G_\bullet') \right)$, then $E_p = \langle E_{p-1}, e \rangle$.

4. If $F_\bullet$ and $F_\bullet'$ are opposite flags in $V$ and $E_\bullet \in X_w F_\bullet \cap X_{w_0^{(n)} v} F_\bullet'$, then $E_k \subseteq F_{n-w(p)} + F_{w(p)-1}$.

**Proof.** — First recall that $\ell(vt_{a,b}) = \ell(v) + 1$ if and only if $v(a) < v(b)$ and if $a < j < b$, then $v(j)$ is not between $v(a)$ and $v(b)$. Thus if $\ell(vt_{a,b}) = \ell(v) + 1$ and $p \notin \{a, b\}$, we have $\ell(vt_{a,b}|_p) = \ell(v|_p) + 1$. Statement (1) follows by induction on $\ell(v) - \ell(w)$.

For (2), since $(w_0^{(n)} v)|_p = w_0^{(n-1)}(v|_p)$ and $w_0^{(n)} w_0^{(n)} v = v$, Lemma 12 shows

$$\psi_p \left( X_{w|_p} G_\bullet \cap X_{w_0^{(n-1)}(v|_p)} G_\bullet' \right) \subset X_w \left( \psi_{w_0^{(n)} w(p)}(G_\bullet) \right) \cap X_{w_0^{(n)} v} \left( \psi_{v(p)}(G_\bullet') \right).$$

The flags $\psi_{w_0^{(n)} w(p)}(G_\bullet)$ and $\psi_{v(p)}(G_\bullet')$ are opposite flags in $V$, since $G_\bullet$ and $G_\bullet'$ are opposite flags in $W$. Then part (1) shows both sides have the same dimension. Since $\psi_p$ is injective, they are equal.
To show (3), let $E_\bullet \in X_w \left( \psi_{w_0^{(n)}(w(p))}(G_\bullet) \right) \cap X_{w_0^{(n)}(v)}(\psi_{w(p)}(G_\bullet'))$. By (2), there is a flag $E_\bullet' \in X_{w_0^{(n)}(v)} G_\bullet \cap X_{w_0^{(n-1)}(v'_j)} G_\bullet'$ with $\psi_p(E_\bullet') = E_\bullet$, so $E_\bullet = \langle E_{p-1}, e \rangle = \langle E'_{p-1}, e \rangle$.

For (4), let $W = F_{n-w(p)} + F_{w(p)-1}$ and $\mathfrak{v}$ any nonzero vector in the one dimensional space $F_{n+1-w(p)} \cap F_{w(p)}$. The distinct subspaces in $F_\bullet \cap W$ define a flag $G_\bullet$, and those in $F_\bullet' \cap W$ define a flag $G_\bullet'$. In fact, $\psi_{w_0^{(n)}(w(p))}(G_\bullet) = F_\bullet$ and $\psi_{w(p)}(G_\bullet') = F_\bullet'$, and $G_\bullet$ and $G_\bullet'$ are opposite flags in $W$. By (2),

$$\psi_p \left( X_{w_0^{(n)}(v)} G_\bullet \cap X_{w_0^{(n-1)}(v'_j)} G_\bullet' \right) = X_{w} F_\bullet \cap X_{w_0^{(n)}(v)} F_\bullet'.$$

Thus flags in $X_{w} F_\bullet \cap X_{w_0^{(n)}(v)} F_\bullet'$ are in the image of $\psi_p$. As $k < p$, $(\psi_p E_\bullet)_k = E_k \subset W$, establishing part (4).

**Proof of Lemma 10.** — Let $F_\bullet$ and $F_\bullet'$ be opposite flags in $V$, let $w < k v$ and let $E_\bullet \in X_w F_\bullet \cap X_{w_0} F_\bullet'$. Define a basis $e_1, \ldots, e_n$ for $V$ by $F_{n+1-j} \cap F_\bullet' = \langle e_j \rangle$ for $1 \leq j \leq n$. Suppose $v = w t_{a_1} b_1 \cdots t_{a_m} b_m$ with $a_i \leq k < b_i$. Let $\{p_1, \ldots, p_d\}$ be the complement of $\{b_1, \ldots, b_m\}$ in $\{k+1, \ldots, n\}$. For $1 \leq i \leq d$, let $Y_i = \langle e_1, \ldots, e_{w(p_i)+1}, e_{w(p_i)+1}, \ldots, e_n \rangle$. Since $w(p_i) = v(p_i)$ and $k < p_i$, we see that $Y_i = F_{n-w(p_i)} + F_{w(p_i)-1}$, so part (4) of Lemma 13 shows $E_k \subset Y_i$. Thus

$$E_k \subset \bigcap_{i=1}^d Y_i = \langle e_{w(j)} \mid j < k \text{ or } j = b_i \rangle = Y.$$
Proof. — Let $\mu_k = 0$ and $\mu_j = r_{j+1} + \cdots + r_k$ for $1 \leq j < k$ and $\lambda_j = r_j + \mu_j$ for $1 \leq j \leq k$. Choose a basis $e_1, \ldots, e_n$ for $V$ such that

\[
L_j = \langle e_{k+1-j+\mu_j}, e_{k+2-j+\mu_j}, \ldots, e_{k+1+r_j-j+\mu_j} = e_{k+1-j+\lambda_j} \rangle
\]

\[
M = \langle e_{m+k+1}, \ldots, e_n \rangle.
\]

Let $F_\bullet = \langle e_n, \ldots, e_1 \rangle$ and $F_\bullet' = \langle e_1, \ldots, e_n \rangle$. Then

\[
F_{n-k+j-\mu_j} = M \bigoplus L_1 \bigoplus \cdots \bigoplus L_j
\]

\[
F_{n-k+(k+1-j)-\lambda_j} = F_{k+1-j+\lambda_j} = L_j \bigoplus \cdots \bigoplus L_k.
\]

If $H \in \Omega_\mu F_\bullet \cap \Omega_{\lambda^c} F_\bullet'$, then $\dim H \cap F_{n-k+j-\mu_j} \geq j$ for $1 \leq j \leq k$ and

\[
\dim H \cap F_{n-k+(k+1-j)-\lambda_j} \geq k + 1 - j,
\]

for $1 \leq j \leq k$. Thus for $1 \leq j \leq k$,

\[
\dim H \cap F_{n-k+j-\mu_j} \cap F_{n-k+(k+1-j)-\lambda_j} \geq 1.
\]

But $F_{n-k+j-\mu_j} \cap F_{n-k+(k+1-j)-\lambda_j} = L_j$, so $\dim H \cap L_j \geq 1$ for $1 \leq j \leq k$. Since $L_j \cap L_i = \{0\}$ if $j \neq i$, we see that $\dim H \cap L_j = 1$. Thus

\[
\Omega_\mu F_\bullet \cap \Omega_{\lambda^c} F_\bullet' \subset \{ H \in G_k V \mid \dim H \cap L_j = 1 \text{ for } 1 \leq j \leq k \}.
\]

We show these varieties have the same dimension, establishing their equality: since $|\lambda| = |\mu| + m$, and $F_\bullet$ and $F_\bullet'$ are opposite flags, $\Omega_\mu F_\bullet \cap \Omega_{\lambda^c} F_\bullet'$ has dimension $m$. But the map $H \mapsto (H \cap L_1, \ldots, H \cap L_k)$ defines an isomorphism between $\{ H \in G_k V \mid \dim H \cap L_j = 1 \text{ for } 1 \leq j \leq k \}$ and $\mathbb{P}L_1 \times \cdots \times \mathbb{P}L_k$, which has dimension $\sum (\dim L_j - 1) = m$. Here, $\mathbb{P}L_j$ is the projective space of one dimensional subspaces of $L_j$.

We relate this to intersections of Schubert varieties in the flag manifold.

**Lemma 15.** — Suppose that $w \xrightarrow{r[k,m]} v$ and $v = wt_{a_1,b_1} \cdots t_{a_m,b_m}$ with $a_i \leq k < b_i$ and $\ell(wt_{a_1,b_1} \cdots t_{a_i,b_i}) = \ell(w) + i$ for $1 \leq i \leq m$. Let $F_\bullet$ and $F_\bullet'$ be opposite flags in $V$ and let $\langle e_i \rangle = F_{n+1-i} \cap F_i'$. Define

\[
L_j = \langle e_j, e_{w(b_i)} \mid a_i = j \rangle
\]

\[
M = \langle e_{w(p)} \mid k < p \text{ and } w(p) = v(p) \rangle.
\]

Then

1. $\dim L_j = 1 + \# \{ i \mid a_i = j \}$ and $V = M \bigoplus L_1 \bigoplus \cdots \bigoplus L_k$. 


2. If $E_\bullet \in X_wF_\bullet \cap X_{w_0}F_\bullet'$, then $\dim E_k \cap L_j = 1$ for $1 \leq j \leq k$.

3. Let $\pi$ be the map induced by $E_\bullet \mapsto E_k$. Then

$$\pi : X_wF_\bullet \cap X_{w_0}F_\bullet' \to \left\{ H \in G_kV \mid \dim H \cap L_j = 1 \text{ for } 1 \leq j \leq k \right\}$$

is surjective and of degree 1.

\textbf{Proof.} — Part (1) is immediate.

For (2) and (3), note that both \{ $H \in G_kV \mid \dim H \cap L_j = 1$ for $1 \leq j \leq k$ \} and $X_wF_\bullet \cap X_{w_0}F_\bullet'$ are irreducible and have dimension $m$. We exhibit an $m$ dimensional subset of each over which $\pi$ is an isomorphism.

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{C}^\times)^m$ be an $m$-tuple of nonzero complex numbers. We define a basis $f_1, \ldots, f_n$ of $V$ depending upon $\alpha$ as follows.

$$f_j = \begin{cases} 
    e_{w(j)} + \sum_{i : a_i = j} \alpha_i e_{w(b_i)} & \text{if } j \leq k \\
    e_{w(j)} & \text{if } j > k \text{ and } j \not\in \{b_1, \ldots, b_m\} \\
    \sum_{i : a_i = \overline{a}_q \wedge w(b_i) \geq w(j)} \alpha_i e_{w(b_i)} & \text{if } j = b_q > k.
\end{cases}$$

Let $i_1 < \cdots < i_s$ be those integers $i_l$ with $a_{i_l} = j$. Since $t_{a_i} b_i$ lengthens the permutation $w_{t_{a_1} b_1 \cdots t_{a_{i_l}} b_{i_l-1}}$, we see that

$$w(j) < w(b_{i_1}) < \cdots < w(b_{i_s}) \quad \| \quad v(b_{i_1}) < v(b_{i_2}) < \cdots < v(j).$$

Thus the first term in $f_j$ is proportional to $e_{w(j)}$. Hence $f_j \in F_{n+1-w(j)} - F_{n-w(j)}$, and so $f_1, \ldots, f_n$ is a basis of $V$ and the flag $E_\bullet(\alpha) = \langle f_1, \ldots, f_n \rangle$ is in $X_wF_\bullet$.

Note that $f'_1, \ldots, f'_n$ is also a basis for $E_\bullet(\alpha)$, where $f'_j$ is given by

$$f'_j = \begin{cases} 
    f_j & \text{if } j \leq k \\
    f_{a_q} - f_j & \text{if } j = b_q > k.
\end{cases}$$

Here, the last term in each $f'_j$ is proportional to $e_{v(j)}$, so $f'_j \in F'_{v(j)} = F'_{n+1-w_0v(j)}$, showing that $E_\bullet(\alpha) \in X_{w_0}F_\bullet'$.

Since $f_j \in L_j$ for $1 \leq j \leq k$, we have $\dim E_\bullet(\alpha) \cap L_j = 1$ for $1 \leq j \leq k$. As \{ $E_\bullet(\alpha) \mid \alpha \in (\mathbb{C}^\times)^m$ \} is a subset of $X_wF_\bullet \cap X_{w_0}F_\bullet'$ of dimension $m$, it is dense. Thus if $E_\bullet \in X_wF_\bullet \cap X_{w_0}F_\bullet'$, then $\dim E_k \cap L_j = 1$ for $1 \leq j \leq k$. 

The set \( \{(E_\alpha(\alpha))_k | \alpha \in (\mathbb{C}^\infty)^m \} \) is a dense subset of
\( \{H \in G_kV | \dim H \bigcap L_j = 1 \text{ for } 1 \leq j \leq k \} \cong \mathbb{P}L_1 \times \cdots \times \mathbb{P}L_k. \)
Since \( \pi \) is an isomorphism of this set with \( \{E_\alpha(\alpha) | \alpha \in (\mathbb{C}^\infty)^m \} \), the map
\( \pi : X_wF_0 \cap X_{w_0}F_0' \to \{H \in G_kV | \dim H \bigcap L_j = 1 \text{ for } 1 \leq j \leq k \} \)
is surjective of degree 1, proving the lemma. \( \square \)

We note finally that Lemma 11 is an immediate consequence of
Lemmas 14 and 15(3).

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