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THE EXTENSION THEOREM OF OHSAWA-TAKEGOSHI AND THE THEOREM OF DONNELLY-FEFFERMAN

by Bo BERNDTSSON (*)

1. Introduction.

In [OT] Ohsawa and Takegoshi proved the following theorem.

THEOREM A. — *Let $\mathcal{D}$ be a bounded and pseudoconvex domain in $\mathbb{C}^n$, and let φ be plurisubharmonic in $\mathcal{D}$. Let H be a hyperplane. Then, for any holomorphic function, f, on $\mathcal{D} \cap H$ there is a holomorphic function F in $\mathcal{D}$ such that $F = f$ on H and*

$$\int_{\mathcal{D}} |F|^2 e^{-\varphi} \leq C_{\mathcal{D}} \int_H |f|^2 e^{-\varphi}.$$

Moreover, $C_{\mathcal{D}}$ depends only on the diameter of $\mathcal{D}$.

This theorem is particularly useful as there is no loss in the estimate and the constant only depends on the size of $\mathcal{D}$.

The original proof of Ohsawa-Takegoshi was based on a $\bar\partial$-theorem involving complete Kähler metrics, inspired by a theorem of Donnelly and Fefferman [DF]. Later, Manivel [M] proved, using related methods, a more general version of Theorem A for sections of vector bundles.

Recently Siu [S] found a simpler proof of Theorem A which avoids the use of general metrics and the somewhat complicated commutator identities

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used in the approach of Ohsawa-Takegoshi. Siu's proof uses only the Hörmander-Kohn-Morrey formalism for the $\bar\partial$-equation in domains in $\mathbb{C}^n$, but the idea from [OT], [DF] of twisting the $\bar\partial$-complex by multiplication with a function is still visible in Siu's proof. Related ideas have also been used recently by McNeal [McN] in his work on estimates for the Bergman kernel.

The aim of this note is to give yet another proof of Theorem A. It is not radically different from the methods of Siu and McNeal in as much as the same crucial integral formula is still used. However, this formula is derived and interpreted in a different, in our opinion more transparent way. We also show how the same methods lead to a simple proof of a more general version of the $\bar\partial$-theorem refereed to above, which also generalizes a recent result of Diederich-Ohsawa [DO].

Consider more generally the problem of extending the function f from a subvariety $V = \{h = 0\}$, where h is holomorphic in $\mathcal{D}$. A well-known scheme to solve such problems is to first construct a local extension, $\tilde f$. Then one multiplies $\tilde f$ with a cut-off function χ, and solves a $\bar\partial$-problem

$$\bar\partial u = \tilde f \frac{\bar\partial \chi}{h}.$$

Then $F = \tilde f \chi - hu$ solves the extension problem. To avoid using any special property of the local extension $\tilde f$, Siu considers cut-off functions with support in an ϵ-neighbourhood of V, and then lets ϵ tend to zero. The difficulty is that one needs an estimate independent of ϵ, and for this the usual Hörmander estimate is not enough.

A variant of this scheme, which we will use here, is to consider the $\bar\partial$-problem

$$\bar\partial u = f \cdot \bar\partial \frac{1}{h}, \tag{1.1}$$

and then put $F = hu$. Clearly F is then holomorphic and

$$F = h\Big(u - \frac{f}{h}\Big) + f,$$

so $F = f$ on V since $u - f/h$ is holomorphic, hence smooth. (This method was introduced in [A].)

An immediate difficulty here is that the right hand side of (1.1) is not a form with L^2-coefficients but a current with measure coefficients. More seriously, it is clearly impossible to even find a solution to (1.1) in L^2_{loc}. It is however, possible to find a solution such that $hu \in L^2$ and we shall show how the L^2-methods can be adapted to prove this directly.

The main tool we use is a differential identity (Lemma 2.1) from [Be2], [Be3] (the one-dimensional case was used in [Be1]). This differential identity implies immediately the integral formula used by Siu and McNeal. Indeed, the two formulas are essentially equivalent, but the differential identity is in our opinion more suggestive and might hopefully have further applications (one is given in [Be3]).(*)

Finally, it's a pleasure to thank Jeff McNeal for stimulating discussions on the topic of this paper.

2. The extension theorem.

We shall use the following two lemmas from [Be2] and [Be3]. For the convenience of the reader proofs are given in an appendix.

LEMMA 2.1. — *Let $\alpha = \sum \alpha_j d\bar{z}_j$ be a smooth $(0,1)$-form in a domain in $\mathbb{C}^n$, and let φ be a smooth function. Then*

$$\sum \frac{\partial^2}{\partial z_j \partial \bar{z}_k}(\alpha_j \bar{\alpha}_k e^{-\varphi}) = -2\,\mathrm{Re}\,\bar\partial \bar\partial^*_\varphi \alpha \cdot \bar{\alpha} e^{-\varphi} + |\bar\partial^*_\varphi \alpha|^2 e^{-\varphi}$$
$$+\sum \left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2 e^{-\varphi} - |\bar\partial \alpha|^2 e^{-\varphi} + \sum \varphi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi}.$$

Here $\bar\partial^*_\varphi$ is the formal adjoint of the $\bar\partial$-operator in $L^2(e^{-\varphi})$, *i.e.*,

$$\bar\partial^*_\varphi \alpha = -e^{\varphi} \sum \frac{\partial}{\partial z_j}(e^{-\varphi}\alpha_j).$$

In the one-dimensional case the left hand side equals

$$\Delta |\alpha|^2 e^{-\varphi},$$

and the formula can be used to obtain pointwise control of α by integrating against a fundamental solution of the Laplace operator. This was used in [Be1] to obtain L^p-estimates for $\bar\partial$ in one variable. In higher dimensions one can instead integrate against $\log |h|$, h holomorphic, and get an estimate for the integral of α over the zero-variety of h. This is what leads to the extension theorem. The integrated version of Lemma 2.1 is

LEMMA 2.2. — *Let $\mathcal{D} = \{\rho < 0\}$ be a smoothly bounded domain in $\mathbb{C}^n$ and let w be a smooth function on $\bar{\mathcal{D}}$. Let α and φ be as in*

(*) *Note added 30/7/96* : Since this paper was written I have realized that Lemma 2.1 is essentially contained in an earlier paper by Siu, [S2].

Lemma 2.1 and assume α satisfies the $\bar\partial$-Neumann boundary conditions on $\partial\mathcal{D}$, $\alpha\cdot\partial\rho = 0$. Then

$$\int w\sum \varphi_{j\bar k}\alpha_j\bar\alpha_k e^{-\varphi} - \int\sum w_{j\bar k}\alpha_j\bar\alpha_k e^{-\varphi} + \int w|\bar\partial^*_\varphi\alpha|^2 e^{-\varphi} + \int w\sum\left|\frac{\partial\alpha_j}{\partial\bar z_k}\right|^2 e^{-\varphi} + \int_{\partial\mathcal{D}} w\sum\rho_{j\bar k}\alpha_j\bar\alpha_k e^{-\varphi}\frac{dS}{|\partial\rho|}$$
$$= 2\,\mathrm{Re}\int w\bar\partial\bar\partial^*_\varphi\alpha\cdot\bar\alpha e^{-\varphi} + \int w|\bar\partial\alpha|^2 e^{-\varphi}.$$

In the proof of Theorem A one may assume that $\mathcal{D}$ is smoothly bounded and strictly pseudoconvex, that φ is smooth on $\bar{\mathcal{D}}$, and that f extends to a holomorphic function in a neighbourhood of $\bar{\mathcal{D}}$, as long as one obtains a constant $C_{\mathcal{D}}$ that only depends on the diameter of $\mathcal{D}$. We will have use for a more or less standard lemma.

LEMMA 2.3. — *Let g be a $\bar\partial$-closed $(0,1)$-current defined in a neighbourhood of $\bar{\mathcal{D}}$. Assume u is a L^1-function in $\mathcal{D}$ and*

$$\int_{\mathcal{D}} g\cdot\bar\alpha e^{-\varphi} = \int_{\mathcal{D}} u\cdot\overline{\bar\partial^*_\varphi\alpha} e^{-\varphi} \tag{2.1}$$

for all smooth, $\bar\partial$-closed $(0,1)$-forms α on $\bar{\mathcal{D}}$, satisfying the $\bar\partial$-Neumann boundary conditions. Then $\bar\partial u = g$ in the sense of distributions.

Proof. — It is enough to prove that (2.1) holds for *any* smooth $(0,1)$-form α with compact support in $\mathcal{D}$. Decompose $\alpha = \alpha^1+\alpha^2$ where $\bar\partial\alpha^1 = 0$ and $\alpha^2 \perp \mathrm{Ker}(\bar\partial)$ in $L^2(e^{-\varphi})$. It follows from the regularity of the $\bar\partial$-Neumann problem that α^1 and α^2 are both smooth. Note that $\alpha^2\perp\mathrm{Im}\,\bar\partial$ implies $\bar\partial^*_\varphi\alpha^2 = 0$ and that α^2 satisfies the $\bar\partial$-Neumann boundary conditions. Hence α^1 satisfies (2.1) by hypothesis. Moreover,

$$\int g\cdot\bar\alpha^2 e^{-\varphi} = 0$$

since g can be approximated by $\bar\partial$-closed $(0,1)$-forms in L^2. Since (2.1) holds for α^1, it also must hold for α. □

LEMMA 2.4. — *Let g, φ, and $\mathcal{D}$ be as in the previous lemma, and assume the inequality*

$$\left|\int g\cdot\alpha e^{-\varphi}\right|^2 \le \int|\bar\partial^*_\varphi\alpha|^2\frac{e^{-\varphi}}{\mu},$$

where $1/\mu$ is an integrable positive function, holds for all $\bar\partial$-closed $(0,1)$-forms α satisfying the $\bar\partial$-Neumann boundary conditions. Then there is a solution, u, to the equation $\bar\partial u = g$ such that

$$\int |u|^2 \mu e^{-\varphi} \leq C.$$

Proof. — By elementary Hilbert space theory there is a function v such that

$$\int g \cdot \alpha e^{-\varphi} = \int v \overline{\bar\partial_\varphi^* \alpha} \frac{e^{-\varphi}}{\mu}$$

for all α of the above type, satisfying

$$\int |v|^2 \frac{e^{-\varphi}}{\mu} \leq C.$$

Let $u = v/\mu$. Clearly u satisfies the stated inequality; in particular u is in L^1 by the hypothesis on μ. By the previous lemma u solves $\bar\partial u = g$, so we are done. □

Choose coordinates so that the hyperplane H in Theorem A is $\{z; z_1 = 0\}$. Suppose $\mathcal{D}$ is included in the set where $|z_1| \leq 1$, and let

$$w = \frac{1}{\pi} \log \frac{1}{|z_1|^2}.$$

Since $w \geq 0$ on $\bar{\mathcal{D}}$, Lemma 2.2 implies, if $\bar\partial\alpha = 0$ and α satisfies the $\bar\partial$-Neumann boundary conditions,

$$\begin{aligned}\int_{z_1=0} |\alpha_1|^2 e^{-\varphi} + \int w |\bar\partial^* \alpha|^2 e^{-\varphi} &\leq 2 \operatorname{Re} \int w \bar\partial \bar\partial_\varphi^* \alpha \cdot \bar\alpha e^{-\varphi} \\ &= 2 \int w |\bar\partial_\varphi^* \alpha|^2 e^{-\varphi} - 2 \operatorname{Re} \int \bar\partial_\varphi^* \alpha \bar\partial w \cdot \bar\alpha e^{-\varphi}.\end{aligned}$$

Hence

$$\int_{z_1=0} |\alpha_1|^2 e^{-\varphi} \leq \frac{1}{\pi} \int \log \frac{1}{|z_1|^2} |\bar\partial_\varphi^* \alpha|^2 e^{-\varphi} + \frac{2}{\pi} \int |\bar\partial_\varphi^* \alpha| \frac{|\alpha_1|}{|z_1|} e^{-\varphi}. \tag{2.2}$$

Next we apply Lemma 2.2 once more, this time with $w = w_1 = 1 - |z_1|^2$. We then get

$$\begin{aligned}\int |\alpha_1|^2 e^{-\varphi} &\leq \int (1 - |z_1|^2) |\bar\partial_\varphi^* \alpha|^2 e^{-\varphi} + 2 \int |\bar\partial_\varphi^* \alpha| |\alpha_1| |z_1| e^{-\varphi} \\ &\leq \int (1 - |z_1|^2) |\bar\partial_\varphi^* \alpha|^2 e^{-\varphi} + 2 \int |\bar\partial_\varphi^* \alpha|^2 |z_1|^2 e^{-\varphi} + \frac{1}{2} \int |\alpha_1|^2 e^{-\varphi}\end{aligned}$$

so

$$\int |\alpha_1|^2 e^{-\varphi} \leq 4 \int |\bar\partial_\varphi^* \alpha|^2 e^{-\varphi}.$$

Using this in (2.2), together with the elementary inequality $x(\log(1/x)+2) \leq 2$ for $0 < x \leq 1$, we find

$$\int_{z_1=0} |\alpha_1|^2 e^{-\varphi} \leq \frac{1}{\pi} \int \log \frac{1}{|z_1|^2} |\bar\partial_\varphi^* \alpha|^2 e^{-\varphi} + \frac{2}{\pi} \int |\bar\partial_\varphi^* \alpha|^2 \frac{e^{-\varphi}}{|z_1|^2}$$
$$+ \frac{2}{\pi} \int |\bar\partial_\varphi^* \alpha|^2 e^{-\varphi} \leq \frac{4}{\pi} \int |\bar\partial_\varphi^* \alpha|^2 \frac{e^{-\varphi}}{|z_1|^2}.$$

Now, let $g = f \cdot \bar\partial \frac{1}{z_1}$, and assume

$$\int_{z_1=0} |f|^2 e^{-\varphi} = 1.$$

Then if α is a smooth $\bar\partial$-closed $(0,1)$-form satisfying the $\bar\partial$-Neumann boundary conditions, we get

$$\left|\int g \cdot \bar\alpha e^{-\varphi}\right|^2 = \pi^2 \left|\int_{z_1=0} f \bar\alpha_1 e^{-\varphi}\right|^2$$
$$\leq \pi^2 \int_{z_1=0} |\alpha_1|^2 e^{-\varphi} \leq 4\pi \int |\bar\partial_\varphi^* \alpha|^2 \frac{e^{-\varphi}}{|z_1|^2}.$$

This is in principle the inequality we need, but there is a minor problem arising from the fact that $1/|z_1|^2$ is not integrable. To remedy this, we note that if we instead choose w_1 in the last part of the argument as $w_1 = 1 - |z_1|^{2\delta}$, where $\delta \leq 1$, we get instead the inequality

$$\left|\int g \cdot \bar\alpha e^{-\varphi}\right|^2 = \pi^2 \left|\int_{z_1=0} f \bar\alpha_1 e^{-\varphi}\right|^2 \leq C_\delta \int |\bar\partial_\varphi^* \alpha|^2 \frac{e^{-\varphi}}{|z_1|^{2\delta}},$$

where C_δ tends to 4π as δ tends to 1. By Lemma 2.4 there is a solution u_δ to $\bar\partial u_\delta = g$ satisfying

$$\int |u_\delta|^2 |z_1|^{2\delta} e^{-\varphi} \leq C_\delta.$$

By the argument from the introduction $F_\delta = z_1 u_\delta$ solves the extension problem and clearly

$$\int |F_\delta|^2 e^{-\varphi} \leq C_\delta.$$

An immediate passage to the limit gives us a solution F such that

$$\int |F|^2 e^{-\varphi} \leq 4\pi,$$

so we have proved Theorem A, with $C_D = 4\pi$ if D is included in the set where $|z_1| \leq 1$ (Siu obtained $C_D = 64\pi/9(1+1/4e)^{1/2}$). Note also that a similar argument shows that there is a solution to the extension problem satisfying

$$\int |F_\delta|^2 |z_1|^{2\delta-2} e^{-\varphi} \leq C_\delta$$

for any $\delta > 0$.

It is also worth mentioning that we can apply exactly the same argument with the coordinate function z_1 replaced by a general holomorphic function h bounded by 1 in $\mathcal{D}$ (just replace the choice of w and w_1 by $1/\pi \log |h|^2$ and $1 - |h|^2$ respectively). We then obtain a version of the more general results obtained in [M] and [OhT]:

THEOREM 2.1. — *Let $\mathcal{D}$ be a bounded and pseudoconvex domain in $\mathbb{C}^n$, and let φ be plurisubharmonic in $\mathcal{D}$. Let $V = \{z \in \mathcal{D}; h(z) = 0\}$ be a hypersurface defined by a holomorphic function bounded by 1 in $\mathcal{D}$. Then, for any holomorphic function, f, on V there is a holomorphic function F in $\mathcal{D}$ such that $F = f$ on V and*

$$\int_{\mathcal{D}} |F|^2 e^{-\varphi} \leq 4\pi \int_V |f|^2 \frac{e^{-\varphi}}{|\partial h|^2}.$$

3. The $\bar\partial$-theorem.

It is clear that by choosing w in Lemma 2.2 to be any positive plurisuperharmonic function, one gets estimates for the integrals of $\bar\partial$-closed forms just like in the previous section. We shall now use this to give a simple proof of the theorem of Donnelly and Fefferman.

Let M be a complex manifold of dimension n, equipped with a complete Kähler metric Ω.

Assume Ω has a global potential ψ, so that $\Omega = i\partial\bar\partial\psi$, and assume ψ satisfies the crucial condition that $\partial\psi$ is uniformly bounded when measured in the Ω-metric. The theorem of Donnelly and Fefferman says that under these assumptions we can solve the equation

$$\bar\partial u = g$$

with an estimate

$$\|u\|_\Omega \leq C\|g\|_\Omega$$

for any $\bar\partial$-closed (p,q)-form g, provided $p + q \neq n$. The instance of this theorem that is relevant to the previous discussion is when $(p,q) = (n,1)$, and we shall concentrate on this case in the sequel. We shall also assume that $M = \mathcal{D}$ is a bounded pseudoconvex domain in $\mathbb{C}^n$, and we can then identify $(0, q)$-forms with (n, q)-forms in the natural way. The theorem then says that for any $\bar\partial$-closed $(0,1)$-form $g = \Sigma g_j d\bar z_j$ we can solve $\bar\partial u = g$ with the estimate

$$\int_{\mathcal{D}} |u|^2 \leq C \int_{\mathcal{D}} \sum \psi^{j\bar k} g_j \bar g_k, \tag{3.1}$$

where $(\psi^{jk}) = (\psi_{j\bar k})^{-1}$, and both integrals are taken with respect to Lebesgue measure. We shall now show how the arguments of Section 2 lead to a simple proof of a more general statement.

The assumption $\|\partial\psi\|_\Omega \leq C$ means that

$$|\partial\psi(z) \cdot a|^2 \leq C^2 \sum \psi_{j\bar k}(z) a_j \bar a_k \tag{3.2}$$

for all $z \in \mathcal{D}$ and $a \in \mathbb{C}^n$. Rescaling we may, of course, assume $C = 1$, and then (3.2) just says that $e^{-\psi}$ is plurisuperharmonic. We shall now apply Lemma 2.2 with

$$w = e^{-\delta\psi}$$

where $0 < \delta < 1$.

Let α be a $\bar\partial$-closed $(0,1)$-form satisfying the $\bar\partial$-Neumann boundary conditions, and assume φ is plurisubharmonic and smooth on $\bar{\mathcal{D}}$. We then get from Lemma 2.2

$$\delta(1-\delta) \int \sum \psi_{j\bar k} \alpha_j \bar\alpha_k e^{-\phi-\delta\psi} \leq \int |\bar\partial^*_\phi \alpha|^2 e^{-\phi-\delta\psi} + 2 \int |\bar\partial^*_\phi \alpha||\alpha \cdot \bar\partial\psi|^2 e^{-\phi-\delta\psi}.$$

By the condition on ψ this is dominated by

$$(1 + 2\delta/(1-\delta)) \int |\bar\partial^* \alpha_\phi|^2 e^{-\phi-\delta\psi} + \delta(1-\delta)/2 \int \sum \psi_{j\bar k} \alpha_j \bar\alpha_k e^{-\phi-\delta\psi},$$

so we obtain

$$\int \sum \psi_{j\bar k} \alpha_j \bar\alpha_k e^{-\varphi-\delta\psi} \leq \frac{4}{\delta(\delta-1)^2} \int |\bar\partial^*_\varphi \alpha|^2 e^{-\varphi-\delta\psi} \tag{3.3}$$

for any smooth ∂-closed $(0,1)$-form α that satisfies the $\bar\partial$-Neumann boundary conditions. Now let g be a $\bar\partial$-closed $(0,1)$-closed $(0,1)$ form in $\mathcal{D}$ and assume

$$\int \sum \psi^{j\bar k} g_j \bar g_k e^{-\varphi+\delta\psi} \leq 1.$$

Then, by (3.3)

$$\left| \int g \cdot \bar\alpha e^{-\varphi} \right|^2 \leq \frac{4}{\delta(\delta-1)^2} \int |\bar\partial^*_\varphi \alpha|^2 e^{-\varphi-\delta\psi}$$

for all forms α of the type we are dealing with. By Lemma 2.4 this implies:

THEOREM 3.1. — *Let $\mathcal{D}$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let φ be plurisubharmonic in $\mathcal{D}$. Let ψ be plurisubharmonic and assume ψ satisfies*

$$|\partial\psi(z)\cdot a|^2 \le \sum \psi_{j\bar{k}}(z) a_j \bar{a}_k$$

for all $z \in \mathcal{D}$ and $a \in \mathbb{C}^n$. Let $0 < \delta < 1$. Then for any $\bar\partial$-closed $(0,1)$-form g in $\mathcal{D}$ there is a solution u to the equation $\bar\partial u = g$ such that

$$\int |u|^2 e^{-\varphi+\delta\psi} \le \frac{4}{\delta(\delta-1)^2} \int \sum \psi^{j\bar{k}} g_j \bar{g}_k e^{-\varphi+\delta\psi}.$$

In particular, choosing $\varphi = \delta\psi$, we get the theorem of Donnelly and Feffermann, even without assuming the metric to be complete. Somewhat weaker results were previously obtained by Diederich-Ohsawa [DO] and Diederich-Herbort [DH]. Their theorems have the same feature of allowing plurisubharmonic weights with "the wrong sign" in the exponent, but do not specify the full range of permitted values of δ, and still assumes completeness of the metric. Note that already the example $\mathcal{D} = \{z \in \mathbb{C}; |z| < 1\}$ and

$$\psi = \log \frac{1}{1-|z|^2}$$

shows that Theorem 3.1 would be false with $\delta = 1$, and even with the constant replaced by $c/(\delta-1)$.

4. Appendix.

Here we shall give the proofs of Lemmas 2.1 and 2.2.

Proof of Lemma 2.1. — This is of course nothing but a direct computation. In the proof we will use the notation $\delta_j = e^{\varphi}\partial/\partial z_j e^{-\varphi}$, and also write $\bar\partial_k$ for $\partial/\partial\bar{z}_k$. Note that

$$\partial/\partial z_k(u\bar{v}e^{-\varphi}) = (\delta_k u)\bar{v}e^{-\varphi} + u\overline{\bar\partial_k v}e^{-\varphi},$$

$$\partial/\partial\bar{z}_k(u\bar{v}e^{-\varphi}) = (\bar\partial_k u)\bar{v}e^{-\varphi} + u\overline{\delta_k v}e^{-\varphi},$$

and that δ_j and $\bar\partial_k$ satisfy the commutator relations

$$[\delta_j, \bar\partial_k] = \varphi_{j\bar{k}}.$$

Using these formulas we find

$$\frac{\partial^2}{\partial z_j \partial \bar z_k}(\alpha_j \bar\alpha_k e^{-\varphi}) = (\delta_j \bar\partial_k \alpha_j)\bar\alpha_k e^{-\varphi} + (\bar\partial_k \alpha_j)(\overline{\bar\partial_j \alpha_k})e^{-\varphi} + \alpha_j \overline{\bar\partial_j \delta_k \alpha_k} e^{-\varphi}$$
$$+(\delta_j \alpha_j)(\overline{\delta_k \alpha_k})e^{-\varphi}.$$

But

$$\sum \alpha_j \overline{\bar\partial_j \delta_k \alpha_k} e^{-\varphi} = -\sum \alpha_j \overline{\bar\partial_j \bar\partial^* \alpha} e^{-\varphi} = -\alpha \cdot \overline{\bar\partial \bar\partial^* \alpha} e^{-\varphi},$$

since $\bar\partial^* \alpha = -\sum \delta_j \alpha_j$. Using the commutation relations we also see that

$$\sum (\delta_j \bar\partial_k \alpha_j)\bar\alpha_k e^{-\varphi} = -\bar\partial \bar\partial^* \alpha \cdot \bar\alpha e^{-\varphi} + \sum \varphi_{jk} \alpha_j \bar\alpha_k e^{-\varphi}.$$

Hence we obtain after summing

$$\sum =: \sum \frac{\partial^2}{\partial z_j \partial \bar z_k}(\alpha_j \bar\alpha_k e^{-\varphi}) = -2\,\mathrm{Re}\,\bar\partial \bar\partial^* \alpha \cdot \bar\alpha e^{-\varphi} + |\bar\partial^* \alpha|^2 e^{-\varphi}$$
$$+\sum \bar\partial_k \alpha_j \overline{\bar\partial_j \alpha_k} e^{-\varphi} + \sum \varphi_{jk} \alpha_j \bar\alpha_k e^{-\varphi}.$$

But

$$\sum \bar\partial_k \alpha_j \overline{\bar\partial_j \alpha_k} = \sum |\bar\partial_j \alpha_k|^2 - |\bar\partial \alpha|^2,$$

so we are done.

Proof of Lemma 2.2. — If we integrate Lemma 2.1 over D, we obtain the fundamental identity used in the proof of Hörmander's theorem. Here we shall multiply by $w > 0$ before integrating. We must then evaluate

$$I =: \int w \sum \frac{\partial^2}{\partial z_j \partial \bar z_k}(\alpha_j \bar\alpha_k e^{-\varphi}).$$

Stokes' theorem gives

$$I = -\int \sum w_j \bar\partial_k(\alpha_j \bar\alpha_k e^{-\varphi}) + \int_\partial \sum w \bar\partial_k(\alpha_j \bar\alpha_k e^{-\varphi})\rho_j dS/|\partial\rho|$$
$$= \int \sum w_{jk} \alpha_j \bar\alpha_k e^{-\varphi} + \int_\partial \sum w \bar\partial_k(\alpha_j \bar\alpha_k e^{-\varphi})\rho_j dS/|\partial\rho|$$
$$+ \int_\partial \sum w_j \alpha_j \overline{\rho_k \alpha_k} e^{-\varphi} dS/|\partial\rho|.$$

Now introduce the notation $\alpha \cdot \partial\rho = \alpha_n$, and assume that $\alpha_n = 0$ on the boundary. This clearly makes the last boundary integral disappear. Moreover, and just like when $w = 1$, the first boundary term also simplifies in this case. To see this we argue as in [H2] or [KF]. First $\alpha_n = 0$ implies

$$\sum \rho_j \bar\partial_k(\alpha_j \bar\alpha_k e^{-\varphi}) = \sum \bar\alpha_k(\bar\partial_k \alpha_j)\rho_j e^{-\varphi}.$$

Then one makes the important observation that since $\alpha_n = 0$ on the boundary we can write $\sum \alpha_j \rho_j = \rho g$ near the boundary for some smooth g. Therefore

$$\sum \bar{\alpha}_k \bar{\partial}_k \sum \alpha_j \rho_j = 0$$

on the boundary, or in other words,

$$\sum \bar{\alpha}_k (\bar{\partial}_k \alpha_j) \rho_j = -\sum \alpha_j \bar{\alpha}_k \rho_{j\bar{k}}.$$

All in all we have then shown that

$$I = \int \sum w_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi} - \int_\partial w \sum \rho_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi} dS/|\partial\rho|.$$

Combining this with Lemma 2.1 we obtain Lemma 2.2.

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