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ON BOUNDARY SLOPES OF IMMERSED INCOMPRESSIBLE SURFACES

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1. Introduction.

Let $M$ be a compact, orientable, irreducible 3-manifold with $\partial M$ a torus. Hatcher [H] showed that there are only finitely many slopes on $\partial M$ realized by boundary curves of embedded incompressible, $\partial$-incompressible surfaces in $M$.

In this paper we show that there can be infinitely many slopes on $\partial M$ realized by the boundary curves of immersed, incompressible, $\partial$-incompressible surfaces in $M$ which are embedded in a neighborhood of $\partial M$.

2. Notation and statement of results.

Let $T_0$ be the torus with an open disk removed, pictured in Figure 1. Let $D_x$ (resp. $D_y$) denote the right handed Dehn twist about the loop $x$ (resp. the loop $y$).

Consider the manifold $M$, a once punctured torus bundle over $S^1$, given by

$$M \cong T_0 \times [0, 1]/(h(s), 0) \sim (s, 1)$$

Key words: 3-Manifold – Injective surface – Boundary slope.
where \( h = D_y^{-4} \circ D_x^4 \). Fix a basepoint \( b \in \partial T_0 \). The loops \( \alpha = b \times [0,1]/\sim \), \( \beta = \partial T_0 \) give a coordinate system for \( \partial M \); the loop \( \alpha^\mu \beta^\lambda \) in \( \partial M \) is represented by the pair \( (\mu, \lambda) \in \mathbb{Z}^2 \) and is said to have slope \( \mu/\lambda \).

Now, if \( S \subset M \) is an immersed surface, properly embedded in a neighborhood of \( \partial M \), then \( \partial S \) consists of parallel simple closed curves in \( \partial M \) parametrized by a coprime pair \( (\mu, \lambda) \).

**Theorem.** — For \( M \) as above, the coprime pair \( (\mu, \lambda) \) is realized by the boundary curves of an immersed, incompressible, \( \partial \)-incompressible surface provided \( \mu \geq 1 \) and \( |\lambda| > \mu \).

**Remarks.**

1) It suffices to prove the theorem for \( (\mu, \lambda) \) with \( \mu \geq 1 \) and \( \lambda > \mu \) since \( M \) admits an involution sending the curves \( \alpha, \beta \) to \( \alpha, -\beta \). Indeed, let \( k: T_0 \rightarrow T_0 \) be a reflection in the diagonal (see Figure 1) followed by \( D_y^{-4} \). Then \( k^2 \) is isotopic to \( D_y^{-4} \circ D_x^4 \) and the map \( (x,t) \mapsto (k(x), t + \frac{1}{2}) \) induces the desired involution on \( M \).

2) The immersed surfaces of the theorem are virtually embedded in \( M \): they lift to embedded surfaces in finite covers of \( M \) (see §3) and one obtains virtually Haken manifolds by Dehn filling on \( M \) with respect to these boundary slopes.

3) There exist manifolds with torus boundary (e.g., Seifert fibered for example) for which only finitely many slopes on \( \partial N \) are realized as the boundary curves of essential immersed surfaces.

### 3. Proof of theorem.

**3.1.** — We prove our result by constructing, for each \( (\mu, \lambda) \), a finite covering space \( \widetilde{M} \rightarrow M \) such that:

(i) The loop \( \alpha^\mu \beta^\lambda \) lifts to loops in each of the four components of \( \partial \widetilde{M} \).

(ii) In \( H_1(\widetilde{M};\mathbb{Z}) \) there exists a relation of the form \( \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0 \) where \( \gamma_i \) is a lift of \( \alpha^\mu \beta^\lambda \) to the \( i \)-th component of \( \partial \widetilde{M} \).

Property (ii) implies that \( \widetilde{M} \) contains an incompressible, \( \partial \)-incompressible surface \( S' \) whose boundary consists of the loops \( \gamma_1, \ldots, \gamma_4 \). Indeed, if we consider a triangulation for \( \widetilde{M} \) and simplicial homology, the fact that \( \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 \) is a primitive element in \( H_1(\partial \widetilde{M};\mathbb{Z}) \) that is
zero in $H_1(\tilde{M};\mathbb{Z})$ implies that these loops bound an oriented 2-complex, $K$, in $\tilde{M}$ with property that an even number of triangles meet at each interior edge, with oriented sum equal to zero. Thus by cutting and pasting along the edges of $K$, and then pulling apart at vertices, we obtain an embedded surface which can then be compressed. Now by property (i) this surface projects to an immersed incompressible surface $S$ in $M$ with boundary consisting of four parallel copies of $\alpha^\mu \beta^\lambda$.

The cover $\tilde{M}$ is obtained by constructing a cover $F \to T_0$ to which $h = D_y^{-4} \circ D_x^4$ lifts to a homeomorphism $\tilde{h}: F \to F$. Then $\tilde{M}$ is the mapping torus of the pair $(F, \tilde{h})$.

3.2. — We construct $F \to T_0$ by cutting and pasting together copies of the following two covers of $T_0$:

a) The four-fold cover $X' \to T_0$ corresponding to the kernel of the map $\theta: \pi_1(T_0) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ defined by $\theta([x]) = (1, 0)$ and $\theta([y]) = (0, 1)$.

b) The two-fold cover $Y' \to T_0$ corresponding to the kernel of the map $\theta: \pi_1(T_0) \to \mathbb{Z}/2$ defined by $\theta([x]) = 0$ and $\theta([y]) = 1$.

Note that $X'$ (resp. $Y'$) is a torus with four (resp. two) boundary circles.

Now alter $X'$ (resp. $Y'$) by making two (resp. one) vertical cuts $\tau_1, \tau_2$ (resp. $\tau$) between boundary circles as shown in Figure 2a (resp. Figure 2b). Denote the cut surfaces by $X$ and $Y$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Figure 2}
\end{figure}

3.3. — Since $\lambda > \mu$, we have that $\lambda - \mu = 2r + s$ for $0 \leq s \leq 1$. Then $F \to T_0$ is obtained by gluing together $\mu + 2r$ copies of $X$ and $2s$ copies...
of $Y$ as indicated in steps i) – v) below. The cover $F$ for $(\mu, \lambda) = (2, 7)$, with edges glued as numbered, is illustrated in Figure 3.

\begin{center}
\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Figure 3}
\end{figure}
\end{center}

i) First arrange $\mu$ copies of $X$, indexed $X_1, \ldots, X_\mu$, in a row, followed by the remaining $2r$ copies of $X$, indexed $X_1^2, \ldots, X_{2r}^2$ in a second row. The $2s$ copies of $Y$, indexed $Y_1, \ldots, Y_{2s}$ form the third row.

Note that row 1 is never empty whereas either row 2 or row 3 (but not both) can be.

ii) In the first row, glue the right edge of $\tau_1$ (resp. left edge of $\tau_2$) in $X_i$ to the left edge of $\tau_1$ (resp. right edge of $\tau_2$) in $X_{i+1}$ for $i = 1, \ldots, \mu - 1$. In the second row, perform the same gluing as in the first row, first on the $X_1^2, \ldots, X_r^2$ for $i = 1, \ldots, r - 1$ and then on $X_{r+1}^2, \ldots, X_{2r}^2$ for $i = r + 1, \ldots, 2r - 1$.

The gluing pattern for the remaining edges depends on the values of $r$ and $s$. The three distinct cases are treated in iii) – v) below.

iii) \textit{The case $r > 0$ and $s = 1$}. There are sixteen remaining edges to glue in this case. Glue the left edge of $\tau_1$ in $X_1$ to the right edge of $\tau_2$ in $X_1^2$; the right edge of $\tau_2$ in $X_1$ to the left edge of $\tau_1$ in $X_{r+1}^2$; the right edge of $\tau_1$
in $X_\mu$ to the left edge of $\tau_1$ in $X_1^2$; the left edge of $\tau_2$ in $X_\mu$ to the right edge of $\tau_2$ in $X_{r+1}^2$.

Finally, attach $Y_1$ to $X_r$ and $Y_2$ to $X_{2r}$ by, in each case, gluing the left edge (resp. right edge) of $\tau$ to the right edge of $\tau_1$ (resp. left edge of $\tau_2$).

iv) The case $r > 0$ and $s = 0$. Glue the copies of $X$ as in iii). Finish by gluing the right edge of $\tau_1$ to the left edge of $\tau_2$, first in $X_1^2$ and then in $X_{2r}^2$.

v) The case $r = 0$ and $s = 1$. There are eight edges to glue. Glue the left edge of $\tau_1$ (resp. right edge of $\tau_2$) in $X_1$ to the right edge of $\tau$ in $Y_1$ (resp. left edge of $\tau$ in $Y_2$).

Finally, pair the right edge of $\tau_1$ (resp. left edge of $\tau_2$) in $X_\mu$ with the left edge of $\tau$ in $Y_1$ (resp. right edge of $\tau$ in $Y_2$).

Note that the surface $F$ is indeed a cover of $T_0$; some of its properties are given in:

**Lemma 1.** — The surface $F$ is a $4\lambda$-fold cover of $T_0$. It is of genus $(2\lambda - 1)$ with four boundary components, each of which projects $\lambda$ to 1 onto $\beta = \partial T_0$.

**3.4.** — Now the loop $x$ (resp. $y$) in $T_0$ is covered by loops in $F$ which project $d$ to 1 onto $x$ for $d \in 1, 2, 4$ (resp. 2 to 1 onto $y$). Thus $D_x^4$ and $D_y^{-4}$ lift to appropriate powers of Dehn twist homeomorphisms in $F$, whence $h = D_y^{-4} \circ D_x^4$ lifts to a homeomorphism $\tilde{h} : F \to F$, which fixes $\partial F$ pointwise. Denote by $\tilde{M}$ the mapping torus of $(F, \tilde{h})$:

$$\tilde{M} = F \times [0, 1]/(\tilde{h}(s), 0) \sim (s, 1).$$

From the construction of $\tilde{M}$, it is clear that we can choose on each boundary torus, $T_i$, of $\tilde{M}$ a pair of loops $\tilde{\alpha}_i, \tilde{\beta}_i$ which cover $\alpha, \beta$ in $\partial M$. The loop $\tilde{\alpha}_i$ (resp. $\tilde{\beta}_i$) projects 1 to 1 onto $\alpha$ (resp. $\lambda$ to 1 onto $\beta$). We will index the four tori in all cases so that the labelling matches that of $X_1$ in Figure 3.

**3.5.** — It is clear from the preceding paragraph that the loops $\tilde{\alpha}_i^\mu \tilde{\beta}_i$ in $\partial \tilde{M}$ project homeomorphically to $\alpha^\mu \beta^\lambda$ in $\partial M$. Thus property (i) of §3.1 is verified and all that remains to show is that property (ii) holds.
3.6. Lemma 2. — Let $\gamma_1$ denote the loop $\tilde{\alpha_1} \tilde{\beta_1}$. Then, in $H_1(\widetilde{M}; \mathbb{Z})$,

$\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0$.

Hence there is a properly embedded incompressible surface $S' \subset \widetilde{M}$ with $\partial S'$ the collection of boundary curves $\tilde{\alpha_i} \tilde{\beta_i}$ which projects to an immersed incompressible surface $S$ in $M$ with boundary consisting of four parallel copies of $\alpha^\mu \beta^\lambda$.

**Proof.** — It suffices to show that in $H_1(\widetilde{M}; \mathbb{Z})$ the following relation holds:

\[(\mu\tilde{\alpha_1} + \tilde{\beta_1}) - (\mu\tilde{\alpha_2} + \tilde{\beta_2}) + (\mu\tilde{\alpha_3} + \tilde{\beta_3}) - (\mu\tilde{\alpha_4} + \tilde{\beta_4}) = 0.\]

Note that

(a) $\tilde{\beta_1} + \tilde{\beta_2} + \tilde{\beta_3} + \tilde{\beta_4} = 0$ the relation being given by the fiber surface $F$,

and

(b) $\mu_1(\tilde{\alpha_1} - \tilde{\alpha_2} + \tilde{\alpha_3} - \tilde{\alpha_4}) + 2(\tilde{\beta_1} + \tilde{\beta_3}) = 0$.

Relations (a) and (b) imply (*)

One sees (b) as follows: Consider the loops $\tilde{y}_1, \tilde{y}_2$ on $X$, pictured in Figure 2, and let $\tilde{y}_{1,i}$ and $\tilde{y}_{2,i}$ denote the corresponding loops in $X_i \subset F$, $i = 1, \ldots, \mu$ (see §3.3). Now on each of these $X_i$, let $\gamma_i \subset X_i$ (resp. $\delta_i \subset X_i$) denote a horizontal, properly embedded arc between $\tilde{\beta_1}$ and $\tilde{\beta_2}$ crossing $\tilde{y}_{2,i}$ (resp. between $\tilde{\beta_3}$ and $\tilde{\beta_4}$ crossing $\tilde{y}_{1,i}$). Then the disks $\gamma_i \times I \subset F \times I$ and $\delta_i \times I \subset F \times I$ provide the relations

$\tilde{\alpha_1} - \tilde{\alpha_2} + (h(\gamma_i) \ast \gamma_i^{-1}) = 0$,

$\tilde{\alpha_3} - \tilde{\alpha_4} + (h(\delta_i) \ast \delta_i^{-1}) = 0$,

where $h(\gamma_i) \ast \gamma_i^{-1}$ (resp. $h(\delta_i) \ast \delta_i^{-1}$) denotes path composition and is equal to $2\tilde{y}_{2,i}$ (resp. $-2\tilde{y}_{2,i}$).

Combining the $2\mu$ relations thus obtained gives:

$\mu(\tilde{\alpha_1} - \tilde{\alpha_2} + \tilde{\alpha_3} - \tilde{\alpha_4}) + 2(\tilde{y}_2, 1 + \cdots + \tilde{y}_2, \mu) - 2(\tilde{y}_{1,1} + \cdots + \tilde{y}_{1,\mu}) = 0$

and hence

$\mu(\tilde{\alpha_1} - \tilde{\alpha_2} + \tilde{\alpha_3} - \tilde{\alpha_4}) + 2(\tilde{\beta_1} + \tilde{\beta_3}) = 0$

since the sum of loops about the boundary circles contributing to $\tilde{\beta_1}$ and $\tilde{\beta_3}$ in the $X_i^2$ and $Y_i$ are all homologous to zero. \qed

1) A once-punctured torus bundle will have infinitely many slopes realized by immersed incompressible surfaces if its characteristic homeomorphism is of the form $D_x^{r_1} \circ D_y^{s_1} \circ \cdots \circ D_x^{r_k} \circ D_y^{s_k}$ where $2 \mid s_i$ and $4 \mid r_i$, $i = 1, \ldots, k$, provided that $s_1 + \cdots + s_k \neq 0$. Hence the same will be true for any once-punctured torus bundle whose monodromy (in $\text{SL}_2(\mathbb{Z})$) has a power that is conjugate to the monodromy of one of the above bundles.

2) The cut and paste techniques in §3 can also be used to produce families of once-punctured surface bundles over $S^1$ of any given genus $g > 1$ having infinitely many slopes realized by immersed incompressible surfaces.

BIBLIOGRAPHY