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## THE BERGMAN KERNEL OF THE MINIMAL BALL AND APPLICATIONS

by K. OELJEKLAUS, P. PFLUG & E.H. YOUSSEFI

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### 1. Introduction.

Let  $\mathbb{B}_*$  be the domain in  $\mathbb{C}^n$ ,  $n \geq 2$ , defined by

$$\mathbb{B}_* := \{z \in \mathbb{C}^n : |z|^2 + |z \bullet z| < 1\},$$

where  $z \bullet z := \sum_{j=1}^n z_j^2$ . This is the ball of radius  $\frac{\sqrt{2}}{2}$  with respect to the norm

$$N_*(z) := \sqrt{\frac{|z|^2 + |z \bullet z|}{2}}, \quad z \in \mathbb{C}^n.$$

The norm  $N_*$  was introduced by Hahn and Pflug [HP], and was shown to be the smallest norm in  $\mathbb{C}^n$  that extends the euclidean norm in  $\mathbb{R}^n$  under certain restrictions. The automorphism group of  $\mathbb{B}_*$  is compact and its identity component is  $\text{Aut}_{\mathcal{O}}^0(\mathbb{B}_*) = S^1 \cdot SO(n, \mathbb{R})$ , where the  $S^1$ -action is diagonal and the  $SO(n, \mathbb{R})$ -action is the matrix multiplication, see [K] or [OY]. This shows that for  $n \geq 3$ , the ball  $\mathbb{B}_*$  is not biholomorphic to any Reinhardt domain. For  $n = 2$ ,  $\mathbb{B}_*$  is linearly biholomorphic to the Reinhardt triangle  $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ .

The main purpose of this note is to establish the following

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*Math. classification:* 32H10 - 32H35.

THEOREM. — *The Bergman kernel of  $\mathbb{B}_*$  is given by the formula*

$$K_{B_*}(z, w) = \frac{1}{n(n+1)V(B_*)} \frac{\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} X^{n-1-2j} Y^j (2nX - (n-2j)[X^2 - Y])}{(X^2 - Y)^{n+1}},$$

where

$$X = 1 - \langle z, w \rangle, \text{ and } Y = (z \bullet z)\overline{w \bullet w},$$

and  $V(\mathbb{B}_*)$  is the Lebesgue volume of  $\mathbb{B}_*$ .

In particular, when  $n = 2$ , the Bergman kernel of  $\mathbb{B}_*$  is

$$K_{\mathbb{B}_*}(z, w) = \frac{2}{\pi^2} \frac{3(1 - \langle z, w \rangle)^2(1 + \langle z, w \rangle) + (z \bullet z)\overline{w \bullet w}(5 - 3\langle z, w \rangle)}{((1 - \langle z, w \rangle)^2 - (z \bullet z)\overline{w \bullet w})^3}.$$

It should be noted that for  $n = 2$  this formula can be obtained from the Bergman kernel of the above mentioned Reinhardt triangle whose Bergman kernel can be found in ([JP], p. 176).

*Remark.* — To the best of our knowledge, the domain  $\mathbb{B}_*$  is the first bounded domain in  $\mathbb{C}^n$  which is neither Reinhardt nor homogeneous, and for which we have an explicit formula for its Bergman kernel.

## 2. Preparatory results.

Let  $G$  be a semi-simple complex Lie group and  $K$  a maximal compact subgroup of  $G$ . Suppose that  $G$  acts irreducibly on a finite dimensional complex vector space  $E_\Lambda$  via a representation  $(\Pi_\Lambda, E_\Lambda)$  with dominant weight  $\Lambda$  and dominant vector  $v_\Lambda$ . Assume further that  $E_\Lambda$  is furnished with a  $K$ -invariant hermitian scalar product  $[\cdot, \cdot]$ . If  $G = KAN$  is the Iwasawa decomposition of  $G$ , let  $\varrho$  denote the sum of the roots associated with the complex decomposition in the Lie algebra  $\mathfrak{g}$  of  $G$ . If  $\mathfrak{a}$  is the Lie algebra of  $A$ , then we have the following orthogonal decomposition with respect to the Cartan-Killing form

$$\mathfrak{a} = a_\Lambda \oplus a_\Lambda^\perp,$$

where  $a_\Lambda$  is the annihilator of  $\Lambda$ . If  $H_0$  is the unique vector in  $a_\Lambda^\perp$  such that  $\Lambda(H_0) = 1$ , we set

$$(2.1) \quad \sigma := 2\varrho(H_0).$$

Let  $\mathbb{M}^*$  be the intersection of the  $G$ -orbit of  $v_\Lambda$  and the unit ball in  $E_\Lambda$ .

In his work [Lo], Loeb proved that the Bergman kernel of the manifold  $\mathbb{M}^*$  with an invariant form on  $\mathbb{M}^*$  is given for  $\zeta = \Pi_\Lambda(g_1)v_\Lambda, \eta = \Pi_\Lambda(g_2)v_\Lambda, g_1, g_2 \in G$ , by

$$(2.2) \quad K_{\mathbb{M}^*}(\zeta, \eta) = \sum_{j=0}^{\infty} (2j + \sigma) T_\Lambda(j) [\zeta, \eta]^j,$$

where  $T_\Lambda(j)$  is the dimension of the representation with dominant weight  $j\Lambda$ .

Here we consider the special case  $G = SO(n + 1, \mathbb{C})$  with its natural linear representation on the complex hermitian space  $(\mathbb{C}^{n+1}, \langle \cdot, \cdot \rangle)$ , where  $\Lambda$  is the dominant weight associated to this representation and  $v_\Lambda = \frac{\sqrt{2}}{2}(1, i, 0, \dots, 0)$ . The intersection of the  $G$ -orbit of  $v_\Lambda$  and the unit ball in  $\mathbb{C}^{n+1}$  is  $\mathbb{M}^* = \mathbb{M} \setminus \{0\}$ , where

$$\mathbb{M} := \{z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : |z| < 1, z \bullet z = 0\}.$$

Then by formula (2.2), we see that the Bergman kernel of  $\mathbb{M}^*$  with respect to an  $SO(n+1, \mathbb{C})$ -invariant form  $\alpha(z) \wedge \overline{\alpha(z)}$  is given (up to a multiplicative constant) by

$$(2.3) \quad K_{\mathbb{M}^*}(\zeta, \eta) = \sum_{j=0}^{\infty} (2j + \sigma) T_\Lambda(j) \langle \zeta, \eta \rangle^j,$$

for  $\zeta, \eta \in \mathbb{M}^*$ .

LEMMA 2.1. — *If  $\alpha(z)$  is an  $SO(n + 1, \mathbb{C})$ -invariant nonzero  $n$ -form on  $\mathbb{M}^*$  (the invariant  $n$ -form  $\alpha$  is unique up to a constant), then the Bergman kernel of  $\mathbb{M}^*$  with respect to the invariant form  $\alpha(z) \wedge \overline{\alpha(z)}$  is given (up to a multiplicative constant) by*

$$\begin{aligned} K_{\mathbb{M}^*}(\zeta, \eta) &= \frac{2(n + 1) \langle \zeta, \eta \rangle + 2n - 2}{(1 - \langle \zeta, \eta \rangle)^{n+1}} \\ &= \frac{4n}{(1 - \langle \zeta, \eta \rangle)^{n+1}} - \frac{2n + 2}{(1 - \langle \zeta, \eta \rangle)^n}, \end{aligned}$$

for  $\zeta, \eta \in \mathbb{M}^*$ .

*Proof.* — Using the notations above, a calculation involving the Weyl character formula implies that

$$T_\Lambda(j) = \frac{n + 2j - 1}{n - 1} \binom{n - 2 + j}{j}, \text{ for all positive integers } j.$$

See ([FH], pp. 267-315). In addition, some computing shows that  $\sigma = 2n - 2$ . See ([FH], pp. 399-414). The lemma now follows from (2.3).  $\square$

LEMMA 2.2. — *The  $n$ -form on  $(\mathbb{C} \setminus \{0\})^{n+1}$*

$$\tilde{\alpha}(z) := \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1},$$

*induces by restriction an  $SO(n + 1, \mathbb{C})$ -invariant and holomorphic  $n$ -form  $\alpha$  on  $\mathbb{M}^*$ .*

*Proof.* — Let  $A \in SO(n + 1, \mathbb{C})$ ,  $z \in \mathbb{M}^*$  and set  $w = Az$ . Denote by  $A_{jk}$  the  $n \times n$  matrix obtained from  $A$  by deleting the  $j$ th row and the  $k$ th column. Since  $A \in SO(n + 1, \mathbb{C})$ , Cramer’s rule gives that

$$(2.4) \quad a_{jk} = (-1)^{k+j} \det A_{jk}.$$

Note also that for  $z \in \mathbb{M}^*$  and  $z_j \neq 0$ , then

$$(2.5) \quad dz_j = - \sum_{l \neq j} \frac{z_l}{z_j} dz_l \quad \text{on } T_z \mathbb{M}^*,$$

where  $T_z \mathbb{M}^*$  denotes the tangent space of  $\mathbb{M}^*$  at the point  $z$ . Denote by  $A^* \alpha$  the pull-back of  $\alpha$ . Then

$$\begin{aligned} (A^* \alpha)(z) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\ &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} \sum_{k=1}^{n+1} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \cdots \wedge dz_{n+1} \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} \frac{(-1)^{k+j}}{w_j} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1} \end{aligned}$$

$$\begin{aligned}
 \text{by (2.4)} &= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} \frac{a_{jk}}{w_j} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1} \\
 \text{by (2.5)} &= \sum_{k=1}^{n+1} (-1)^k \sum_{j=1}^{n+1} \frac{a_{jk}}{w_j} dz_1 \wedge \cdots \wedge dz_{j-1} \wedge \widehat{dz_j} \wedge \left( \sum_{l \neq j} \frac{z_l}{z_j} dz_l \right) \\
 &\quad \wedge dz_{j+1} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1} \\
 &= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} (-1)^{j-k} \frac{a_{jk}}{z_j w_j} z_k dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} \\
 &= \sum_{j=1}^{n+1} \left( \frac{(-1)^{j-1}}{w_j z_j} \sum_{k=1}^{n+1} a_{jk} z_k \right) dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} \\
 &= \sum_{j=1}^{n+1} (-1)^{j-1} \frac{1}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} = \alpha(z).
 \end{aligned}$$

That the restriction of  $\alpha$  to  $\mathbb{M}^*$  is holomorphic can be seen by evaluating the form  $\alpha$  on the  $n$ -fold exterior power of the tangent space.  $\square$

### 3. Proper holomorphic mappings from $\mathbb{M}$ into $\mathbb{C}^n$ .

Consider the projection  $pr : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$  defined by

$$pr(z_1, \dots, z_{n+1}) := (z_1, \dots, z_n).$$

The restriction  $F := pr|_{\mathbb{M}}$  of  $pr$  to  $\mathbb{M}$  gives a proper holomorphic mapping of degree 2 from  $\mathbb{M}$  onto  $\mathbb{B}_*$ . Let  $W$  be the branching locus of  $F$  and  $V$  the image of  $W$  under  $F$ . Denote by  $\varphi$  and  $\psi$  the two local inverses of  $F$  defined for  $z = (z_1, \dots, z_n) \in \mathbb{B}_* \setminus V$  by

$$\begin{aligned}
 \varphi(z) &= (z, i\sqrt{z \bullet z}) \\
 \psi(z) &= (z, -i\sqrt{z \bullet z}).
 \end{aligned}$$

LEMMA 3.1. — *If  $\varphi := (\varphi_1, \dots, \varphi_{n+1})$  and  $\psi := (\psi_1, \dots, \psi_{n+1})$  are the local inverses of  $F$  defined on  $\mathbb{B}_* \setminus V$ , then*

$$(3.1) \quad \varphi^*(\alpha) = \frac{1+n}{i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n$$

$$(3.2) \quad \psi^*(\alpha) = \frac{1+n}{-i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.$$

*Proof.* — The pull back of  $\alpha$  under  $\varphi$  is

$$\begin{aligned} \varphi^*(\alpha) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\ &= \sum_{j=1}^n \frac{(-1)^{j-1}}{w_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge dz_n \wedge dw_{n+1} + \frac{(-1)^n}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_n. \end{aligned}$$

But for  $1 \leq j \leq n$

$$\begin{aligned} &\frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge d\varphi_{n+1} \\ &= \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge \left( - \sum_{k=1}^n \frac{z_k}{w_{n+1}} dz_k \right) \\ &= \frac{(-1)^j}{w_{n+1}} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge dz_j \\ &= \frac{(-1)^{j+n-j}}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \\ &= (-1)^n \frac{dz_1 \wedge \cdots \wedge dz_n}{w_{n+1}}. \end{aligned}$$

Thus

$$\begin{aligned} \varphi^*(\alpha) &= \left( \frac{(-1)^n}{w_{n+1}} + (-1)^n \frac{n}{w_{n+1}} \right) dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{w_{n+1}} (-1)^n dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n. \end{aligned}$$

Similarly one has that

$$\psi^*(\alpha) = \frac{1+n}{-i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.$$

□

If  $P_{\mathbb{M}^*}$  denotes the Bergman projection of  $\mathbb{M}^*$  with respect to the volume form  $\alpha(z) \wedge \overline{\alpha(z)}$ , and if  $P_{\mathbb{B}_*}$  denotes the Bergman projection of  $\mathbb{B}_*$ , then we have the following transformation rule

LEMMA 3.2. — *If  $\varphi := (\varphi_1, \dots, \varphi_{n+1})$  and  $\psi := (\psi_1, \dots, \psi_{n+1})$  are the local inverses of  $F$  defined locally on  $\mathbb{B}_* \setminus V$ , then*

$$P_{\mathbb{M}^*}(z_{n+1}(h \circ F))(z) = z_{n+1}((P_{\mathbb{B}_*} h) \circ F)(z)$$

for all  $h \in L^2(\mathbb{B}_*)$ , where  $V$  is the image of the branching locus of  $F$ .

*Proof.* — First observe that the lemma holds for all holomorphic functions  $h \in L^2(\mathbb{B}_*)$ . Indeed, by virtue of Lemma 3.1 we have that

$$\begin{aligned} \int_{\mathbb{M}^*} |z_{n+1}(h \circ F)(z)|^2 \alpha(z) \wedge \overline{\alpha(z)} &= \int_{\mathbb{M}^* \setminus W} |(z_{n+1}(h \circ F))(z)|^2 \alpha(z) \wedge \overline{\alpha(z)} \\ &= \int_{\mathbb{B}_* \setminus V} |\varphi_{n+1}(w)h(w)|^2 \varphi^*(\alpha)(w) \wedge \varphi^*(\overline{\alpha})(w) \\ &\quad + \int_{\mathbb{B}_* \setminus V} |\psi_{n+1}(w)h(w)|^2 \psi^*(\alpha)(w) \wedge \psi^*(\overline{\alpha})(w) \\ &= 2(n+1)^2 \int_{\mathbb{B}_* \setminus V} |h(w)|^2 dv(w) < +\infty. \end{aligned}$$

Thus  $z_{n+1}(h \circ F)(z) \in L^2(\mathbb{M}^*, \alpha(z) \wedge \overline{\alpha(z)})$ .

Next let  $f \in L^2(\mathbb{M}^*, \alpha(z) \wedge \overline{\alpha(z)})$  be a holomorphic function, and let  $g$  be an element of the space  $C_0^\infty(\mathbb{B}_* \setminus V)$  of all  $C^\infty$ -function with compact support in  $\mathbb{B}_* \setminus V$ . Then

$$\begin{aligned} \int_{\mathbb{M}^*} f(z) z_{n+1} \overline{\left( \frac{\partial g}{\partial w_j} \circ F \right)}(z) \alpha(z) \wedge \overline{\alpha(z)} \\ = (n+1)^2 \left[ \int_{\mathbb{B}_*} \frac{(f \circ \varphi)(w)}{\varphi_{n+1}(w)} \overline{\frac{\partial g}{\partial w_j}}(w) dv(w) + \int_{\mathbb{B}_*} \frac{(f \circ \psi)(w)}{\psi_{n+1}(w)} \overline{\frac{\partial g}{\partial w_j}}(w) dv(w) \right] \end{aligned}$$

so that by integration by parts we obtain that

$$P_{\mathbb{M}} \left( z_{n+1} \left( \frac{\partial g}{\partial w_j} \circ F \right) \right) = 0, \text{ for all } j = 1, \dots, n.$$

Since the space

$$\mathcal{H} := \left\{ \frac{\partial g}{\partial w_j} : g \in C_0^\infty(\mathbb{B}_* \setminus V) \right\}$$

is dense in the orthogonal complement in  $L^2(\mathbb{B}_*)$  of the subspace  $L^2_h(\mathbb{B}_*)$  of all square integrable holomorphic functions on  $\mathbb{B}_*$ , the lemma follows.  $\square$

LEMMA 3.3. — *If  $\varphi$  and  $\psi$  are as before, then*

$$z_{n+1} K_{\mathbb{B}_*}(F(z), w) = (n+1)^2 \left[ \frac{K_{\mathbb{M}^*}(z, \varphi(w))}{\varphi_{n+1}(w)} + \frac{K_{\mathbb{M}^*}(z, \psi(w))}{\psi_{n+1}(w)} \right],$$

$z \in \mathbb{M}^*, w \in \mathbb{B}_*.$

*Proof.* — Let  $w \in \mathbb{B}_* \setminus V$  and let  $r > 0$  be chosen so small that  $w + r\Delta^n \subset \mathbb{B}_* \setminus V$ , where  $\Delta$  is the unit disc in  $\mathbb{C}$ . By Remark 6.1.4 in [JP],



there is a  $C^\infty$ -function  $u : \mathbb{C}^n \rightarrow [0, +\infty)$  with compact support in  $w + r\Delta^n$  such that

$$P_{\mathbb{B}_*} u = K_{\mathbb{B}_*}(\cdot, w).$$

By virtue of Lemma 3.2 we see that for  $z \in \mathbb{M}^*$ ,

$$\begin{aligned} z_{n+1}K_{\mathbb{B}_*}(F(z), w) &= z_{n+1}(P_{\mathbb{B}_*} u)(F(z)) = P_{\mathbb{M}^*}(z_{n+1}(u \circ F)(z)) \\ &= \int_{\mathbb{M}^*} \zeta_{n+1}(u \circ F)(\zeta) K_{\mathbb{M}^*}(z, \zeta) \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \\ &= (n+1)^2 \int_{\mathbb{B}_*} u(\eta) \left[ \frac{K_{\mathbb{M}^*}(z, \varphi(\eta))}{\varphi_{n+1}(\eta)} + \frac{K_{\mathbb{M}^*}(z, \psi(\eta))}{\psi_{n+1}(\eta)} \right] dv(\eta) \\ &= (n+1)^2 \left( \frac{K_{\mathbb{M}^*}(z, \varphi(w))}{\varphi_{n+1}(w)} + \frac{K_{\mathbb{M}^*}(z, \psi(w))}{\psi_{n+1}(w)} \right), \end{aligned}$$

and the lemma is proved. □

#### 4. Proof of the main result.

*Proof of the theorem.* — For  $z, w \in \mathbb{B}_* \setminus V$ , set

$$\begin{cases} s := 1 - \langle z, w \rangle, & t := \varphi_{n+1}(z) \overline{\varphi_{n+1}(w)} \\ x := \langle z, w \rangle + t & \text{and } y := \langle z, w \rangle - t. \end{cases}$$

Then using the notations in the main theorem we have  $X = s$  and  $Y = t^2$ . By Lemma 2.1 we see that for some positive constant  $C$  we have

$$\begin{aligned} K_{\mathbb{M}^*}(\varphi(z), \varphi(w)) &= C \left( \frac{4n}{(1-x)^{n+1}} - \frac{2n+2}{(1-x)^n} \right) \\ K_{\mathbb{M}^*}(\varphi(z), \psi(w)) &= C \left( \frac{4n}{(1-y)^{n+1}} - \frac{2n+2}{(1-y)^n} \right), \end{aligned}$$

so that by Lemma 3.3 we obtain that

$$\begin{aligned} K_{\mathbb{B}_*}(z, w) &= 4C(n+1)^2 \frac{f(x) - f(y)}{x - y}, \quad \text{where} \\ f(u) &= \frac{2n}{(1-u)^{n+1}} - \frac{n+1}{(1-u)^n}. \end{aligned}$$

On the other hand,

$$\frac{f(x) - f(y)}{x - y} = n \frac{(s+t)^{n+1} - (s-t)^{n+1}}{t(s^2 - t^2)^{n+1}} - \frac{n+1}{2} \frac{(s+t)^n - (s-t)^n}{t(s^2 - t^2)^n},$$

and

$$\begin{aligned} \frac{(s+t)^{n+1} - (s-t)^{n+1}}{t} &= \frac{(s+t)^{n+1} - s^{n+1}}{t} + \frac{(s-t)^{n+1} - s^{n+1}}{-t} \\ &= \sum_{j=1}^{n+1} \binom{n+1}{j} s^{n+1-j} t^{j-1} \\ &\quad + \sum_{j=1}^{n+1} \binom{n+1}{j} s^{n+1-j} (-t)^{j-1} \\ &= \sum_{j=1}^{n+1} \binom{n+1}{j} s^{n+1-j} [t^{j-1} + (-t)^{j-1}] \\ &= \sum_{k=0}^n \binom{n+1}{k+1} s^{n-k} [t^k + (-t)^k] \\ &= 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-2k} t^{2k}. \end{aligned}$$

Similarly we have that

$$\frac{(s+t)^n - (s-t)^n}{t} = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} s^{n-1-2k} t^{2k}.$$

Therefore,

$$\begin{aligned} \frac{f(x) - f(y)}{x - y} &= \frac{2n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-2k} t^{2k}}{(s^2 - t^2)^{n+1}} \\ &\quad - (n+1) \frac{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} s^{n-1-2k} t^{2k}}{(s^2 - t^2)^n}. \end{aligned}$$

But  $\binom{n}{2k+1} = \frac{n-2k}{n+1} \binom{n+1}{2k+1}$ . Thus

$$\begin{aligned} \frac{f(x) - f(y)}{x - y} &= \frac{2n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-2k} t^{2k}}{(s^2 - t^2)^{n+1}} \\ &\quad - \frac{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2k) \binom{n+1}{2k+1} s^{n-1-2k} t^{2k} (s^2 - t^2)}{(s^2 - t^2)^{n+1}} \\ &= \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-1-2k} t^{2k} [2ns - (n-2k)(s^2 - t^2)]}{(s^2 - t^2)^{n+1}}. \end{aligned}$$

It follows that

$$K_{\mathbb{B}_*}(z, w) = 4C(n + 1)^2 \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n + 1}{2k + 1} X^{n-1-2k} Y^k [2nX - (n - 2k)(X^2 - Y)]}{(X^2 - Y)^{n+1}},$$

where  $X$  and  $Y$  are as in the statement of the theorem. To compute the constant we use the formula

$$1 = \int_{\mathbb{B}_*} K_{\mathbb{B}_*}(0, w) dV(w).$$

□

### 5. Applications.

**THEOREM 5.1.** — *Let  $D \subset \mathbb{C}^n$  be a pseudoconvex domain with  $C^2$ -boundary and let  $f : D \rightarrow \mathbb{B}_*$  be a proper holomorphic mapping. Then  $A_f \subset V(f)$  where*

$$A = A_f := \{z \in D : f(z) \bullet f(z) = 0\}, V(f) := \{z \in D : \det f'(z) = 0\}.$$

*Proof.* — Observe that  $A$  is an analytic subset of  $D$ . Assume that there exists a point  $a \in A$  with  $\det f'(a) \neq 0$ . To get a contradiction it suffices to show that:

$$\text{if } z^\nu \in A, z^\nu \rightarrow z^0 \in \partial D, \text{ then } \det f'(z^\nu) \rightarrow 0.$$

We choose a ball  $B(z^0, s)$ ,  $0 < \eta < 1$  so that  $\eta(n + 2) > n + 1$ , and a defining function  $r$  of  $D \cap B(z^0, s)$  such that  $\tilde{r} := -(-r)^\eta$  is plurisubharmonic on  $D \cap B(z^0, s)$ ; this can be achieved using a result of Diederich-Fornaess [DF]. Moreover, we may assume that  $f(z^\nu) \rightarrow w^0 \in \mathbb{H} \cap \partial\mathbb{B}_*$ , where  $\mathbb{H} := \{\zeta \in \mathbb{C}^n : \zeta \bullet \zeta = 0\}$ .

Assuming that  $f$  is a mapping with multiplicity  $m$ , we know by Pinchuk [Pi2] that

$$mK_D(z, z) \geq |\det f'(z)|^2 K_{\mathbb{B}_*}(f(z), f(z)), z \in D.$$

It is well known that  $K_D(z, z) \leq C_1 \text{dist}(z, \partial D)^{-(n+1)}$ ,  $z \in D$ . Hence we get

$$|\det f'(z)|^2 \leq C_2 (K_{\mathbb{B}_*}(f(z), f(z)))^{-1} \text{dist}(z, \partial D)^{-(n+1)}.$$

Now we apply the theorem to obtain that

$$|\det f'(z^\nu)|^2 \leq C_3(1 - |f(z^\nu)|^2)^{n+2} / \text{dist}(z^\nu, \partial D)^{n+1}, \nu \gg 1.$$

Fix  $s' < s$  and define on  $D$  the following function:

$$v(z) := \begin{cases} \max\{\tilde{r}(z), |z - z^0|^2 - s'^2\} & \text{if } z \in D \cap \overline{B(z^0, s')}, \\ |z - z^0|^2 - s'^2, & \text{if } z \in D \setminus B(z^0, s'). \end{cases}$$

It is clear that  $v$  is plurisubharmonic on  $D$  and that  $v(z) = \tilde{r}(z)$  for  $z \in D \cap B(z^0, s'')$ ,  $0 < s'' < s'$  sufficiently small.

For  $w \in \mathbb{B}_*$  we put  $\rho(w) := \max\{v(z) : z \in D, f(z) = w\}$ . Obviously,  $\rho$  is plurisubharmonic on  $\mathbb{B}_*$ . In particular, for  $\nu \gg 1$  we have  $\rho(f(z^\nu)) \geq v(z^\nu) = \tilde{r}(z^\nu)$ .

Exploiting that  $\mathbb{B}_*$  is balanced and the Hopf-Lemma on  $\mathbb{H} \cap \mathbb{B}_*$  leads to the following estimate:  $\rho(f(z^\nu)) \leq C_4(|f(z^\nu)|^2 - 1)$ ,  $\nu \gg 1$ ;  $C_4 > 0$  independent of  $z^\nu$ . Therefore

$$|\det f'(z^\nu)|^2 \leq C_5(-r(z^\nu))^{n(n+2)} / \text{dist}(z^\nu, \partial D)^{n+1} \rightarrow 0, \text{ if } \nu \rightarrow \infty,$$

which leads to that contradiction we mentioned at the beginning of the proof. □

**COROLLARY 5.2.** — *There are no unbranched proper holomorphic mappings from  $D$  onto  $\mathbb{B}_*$  for any bounded pseudoconvex domain with a  $C^2$ -boundary; in particular, such a  $D$  is never biholomorphically equivalent to  $\mathbb{B}_*$ .*

Moreover, if  $D$  is assumed to be strongly pseudoconvex we get even more:

**THEOREM 5.3.** — *Let  $D \subset \mathbb{C}^n$  be a strongly pseudoconvex domain with  $C^2$ -boundary. If  $f : D \rightarrow \mathbb{B}_*$  is a proper holomorphic mapping, then  $A = V(f)$ .*

*Proof.* — Assume the inclusion  $V(f) \subset A$  is not correct. Then, by the maximum principle, there is a sequence  $z^\nu \in V(f)$ ,  $z^\nu \rightarrow z^0 \in \partial D$  such that  $|f(z^\nu) \bullet f(z^\nu)| > C > 0$ . Without loss of generality we assume that  $f(z^\nu) \rightarrow w^0$ . Since  $|w^0 \bullet w^0| > 0$  we conclude that  $w^0$  is a strongly pseudoconvex boundary point of  $\mathbb{B}_*$ . By Theorem 3 of [Ber] there is a neighborhood  $U = U(z^0)$  such that  $f$  extends to a continuous mapping on  $U \cap \bar{D}$ . Then using Theorem 3' of [Pi1] we obtain that  $f \in C^1(\bar{D} \cap U)$ . Finally using Theorem 1 of [Pi2] we finally get the contradiction to the fact that  $z^0 \in \overline{V(f)}$ . □

We recall that a bounded domain  $\Omega$  is said to satisfy condition (Q) if the Bergman projection of  $\Omega$  maps  $C_0^\infty(\Omega)$  into the space  $\mathcal{O}(\bar{\Omega})$  of all holomorphic functions on a neighborhood of  $\bar{\Omega}$ . It was proved recently in [Th] that  $\Omega$  satisfies condition Q is if and only if that for every compact subset  $L$  of  $\Omega$ , there is an open neighborhood  $U = U(L)$  of  $\bar{\Omega}$  such that the Bergman kernel  $K_\Omega(z, w)$  of  $\Omega$  extends to be holomorphic on  $U$  as a function of  $z$  for each  $w \in L$ , and  $K_\Omega$  is continuous on  $U \times L$ .

LEMMA 5.4. — *The ball  $\mathbb{B}_*$  satisfies condition (Q).*

*Proof.* — For  $z, w \in \mathbb{B}_*$ , we have

$$\begin{aligned} \left| (1 - \langle z, w \rangle)^2 - (z \bullet z)(\overline{w \bullet w}) \right| &\geq |1 - \langle z, w \rangle|^2 - |z \bullet z||w \bullet w| \\ &\geq (1 - |z||w|)^2 - |z \bullet z||w \bullet w| \\ &\geq \left( 1 - |z||w| - \sqrt{|z \bullet z|} \sqrt{|w \bullet w|} \right)^2 \\ &\geq \left( 1 - \sqrt{|z|^2 + |z \bullet z|} \sqrt{|w|^2 + |w \bullet w|} \right)^2 \end{aligned}$$

where the last inequality holds because of Cauchy-Schwarz’s inequality. Therefore for some positive constant  $C$  we have

$$|K_{\mathbb{B}_*}(z, w)| \leq \frac{C}{\left( 1 - \sqrt{|z|^2 + |z \bullet z|} \sqrt{|w|^2 + |w \bullet w|} \right)^{2n+4}}, \text{ for all } z, w \in \mathbb{B}_*.$$

This shows that  $\mathbb{B}_*$  satisfies condition (Q). □

THEOREM 5.5. — *Let  $D \subset \mathbb{C}^n$  be an arbitrary bounded circular domain which contains the origin.*

- (1) *If  $f : \mathbb{B}_* \rightarrow D$  is a proper holomorphic mapping, then  $f$  extends holomorphically to a neighborhood of  $\bar{\mathbb{B}}_*$ .*
- (2) *If  $D$  is smooth then there is no proper holomorphic mapping from  $\mathbb{B}_*$  into  $D$ .*

*Proof.* — Since, by Lemma 5.4,  $\mathbb{B}_*$  satisfies condition Q, part (1) of the theorem becomes a consequence of Theorem 2 of [Bel]. To see that part (2) of theorem holds, it is enough to notice that if there is proper holomorphic mapping  $f : \mathbb{B}_* \rightarrow D$ , and if  $\varrho$  is a defining function of  $D$ , then  $\varrho \circ f$  is a defining function for  $\mathbb{B}_*$ , which will imply that  $\mathbb{B}_*$  is smooth and thus leads to a contradiction. □

THEOREM 5.6. — *Let  $L$  be a compact subset of  $\mathbb{B}_*$  and let  $\zeta$  be a boundary point of  $\mathbb{B}_*$ . Then every holomorphic function  $f$  in a*

neighborhood of  $L$  is the uniform limit of functions in the complex span of the functions

$$\frac{\partial}{\partial \zeta^\beta} K_{\mathbb{B}_*}(\cdot, \zeta), \quad \beta \in \mathbb{N}_0^n.$$

*Proof.* — Since  $\mathbb{B}_*$  is a Runge domain and satisfies condition (Q) the proposition follows from Theorem 2.5 of [Th].  $\square$

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