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ON A VARIANT OF KAZHDAN'S PROPERTY (T) FOR SUBGROUPS OF SEMISIMPLE GROUPS

by M. B. BEKKA and N. LOUVET

1. Introduction and statement of the results.

Lubotzky and Zimmer considered in [LuZ] the following variant of Kazhdan's property (T) for a locally compact group G . Let \mathcal{R} be a set of equivalence classes of unitary representations of G containing the trivial one dimensional representation 1_G . Throughout this paper, all group representations will be assumed to be unitary and strongly continuous representations in non zero Hilbert spaces. The group G is said to have property (T; \mathcal{R}) if 1_G is isolated in \mathcal{R} . For $\mathcal{R} = \widehat{G}$, the unitary dual of G , this is just the celebrated Kazhdan's property (T) (see [HaV] for an account on this property).

There are interesting examples of (discrete) groups that are not Kazhdan groups and that satisfy property (T; \mathcal{R}) for some natural classes of representations \mathcal{R} . For instance, $SL(2, \mathbb{Z})$ has property (T; \mathcal{R}) for the set \mathcal{R} of all irreducible representations of $SL(2, \mathbb{Z})$ that factorize through a quotient by a congruence subgroup of $SL(2, \mathbb{Z})$. Another interesting example is the group $SL(2, \mathbb{Z}[1/p])$ which has the property (T; \mathcal{R}), \mathcal{R} being the set of all finite dimensional representations of $SL(2, \mathbb{Z}[1/p])$ (see [LuZ]). Such isolation properties allow for instance the construction of expanders graphs (see [Lub]).

In [LuZ], other examples were discovered. They occur as follows. Let Γ be a lattice in the direct product $G_1 \times G_2$ of two groups. Assume that Γ projects densely on G_1 and that G_2 has property (T). Then Γ has (T; \mathcal{R})

for the set \mathcal{R} of all finite dimensional representations of Γ (see below for some concrete examples of such Γ).

In this paper, we shall generalize the results in [LuZ]. It is interesting to note that our methods are elementary.

We fix, once and for all, a locally compact group G and we let H be a closed (not necessarily discrete) subgroup of G and N a closed normal subgroup of G . Throughout this paper, we shall make the following two assumptions:

- (I) HN is dense in G , and
- (II) the homogeneous space G/H has a finite invariant measure.

Denoting by p the canonical projection $G \rightarrow G/N$, assumption (I) says that $p(H)$ is dense in G/N . If π is an irreducible representation of G/N , it is clear that $(\pi \circ p)|_H$ is an irreducible representation of H and that π is determined (up to unitary equivalence) by $(\pi \circ p)|_H$. So, we may view $\widehat{G/N}$ as a subset of \widehat{H} . Our main result states that if an irreducible representation π of H is sufficiently close to the trivial representation 1_H and if N has property (T), then π is in $\widehat{G/N}$.

More precisely, the following holds:

THEOREM A. — *Let G , H , and N satisfy the above assumptions (I) and (II). Assume moreover that N has Kazhdan's property (T). Let \mathcal{U} be a neighbourhood of $1_{G/N}$ in $\widehat{G/N}$. Then the set of all $(\pi \circ p)|_H$, $\pi \in \mathcal{U}$, is a neighbourhood of 1_H in \widehat{H} .*

In other words, H has property (T; \mathcal{R}) with \mathcal{R} the set of all irreducible representations of H that do not factorize through G/N together with 1_H . Of course, in case G has Kazhdan's property (T), then taking $N = G$ shows that Theorem A is a generalization of the fact that property (T) is inherited by subgroups of cofinite volume. (see [HaV, Chapter 3, Theorem 4]).

Theorem A is a rigidity result in the following sense. It says that any irreducible representation of H that is sufficiently close to the trivial representation 1_H extends to a representation of G . One may speculate whether this is always true for irreducible lattices in a (nontrivial) product of, say, simple Lie groups. Recall that the only simple Lie groups without property (T) are the ones that are locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$.

Let FD denote the subset of \widehat{H} consisting of the finite dimensional representations. Recall that a group G is called minimally almost periodic if 1_G is the unique finite dimensional unitary irreducible representation of G . An immediate consequence of Theorem A is the following corollary, also noticed in [LuZ].

COROLLARY. — *If G/N is minimally almost periodic then H has property (T, FD) .*

Our Theorem A improves and makes more precise Theorem 2.2 in [LuZ] where the following was shown. Under the additional assumption that G is a direct product $M \times N$, H has property $(T; \mathcal{R})$ where \mathcal{R} is the set of all $\pi \in \widehat{H}$ such that, for all $k \geq 1$, no nontrivial subrepresentation of the symmetric power $S^k(\pi)$ factorize through $p : H \rightarrow M$. Of course, this is sufficient in order to deduce the corollary above.

Our proof of Theorem A is elementary. It is based on the following extension result which may be of independent interest.

LEMMA 1. — *Let G , H , and N satisfy the above assumptions (I) and (II). Let π be a unitary representation of H . Then the following are equivalent:*

(i) *π contains a subrepresentation that factorizes through the canonical projection $p : G \rightarrow G/N$ i.e. there exists a representation σ of G/N so that π contains $(\sigma \circ p)|_H$.*

(ii) *The induced representation $\text{Ind}_H^G \pi$ contains a non zero N -invariant vector.*

For a representation π of a locally compact group H , let $H^1(H, \pi)$ be the first cohomology group of H with coefficients in π (see [Gu1, Gu2]). It is well known that $H^1(H, \pi) = 0$ for any π if (and only if) the group H has Kazhdan's property (T) (see [HaV, Chapter 4, Theorem 7]).

Vershik and Karpushev [VeK, Theorem 2] showed that if $H^1(H, \pi) \neq 0$ for some irreducible representation π , then π is infinitesimally small, that is, there exists a net π_n in \widehat{H} such that $\lim \pi_n = 1_H$ and $\lim \pi_n = \pi$. This result has been conjectured by Guichardet [Gu2] and some partial results were previously obtained by Delorme [Del].

Using Theorem A and Vershik and Karpushev's result, we give an elementary proof of the following theorem, proved in [LuZ, Theorem 3.1] in the product case $G = M \times N$ (see also [BoW]).

THEOREM B. — *Let G, H and N be as in Theorem A and assume G/N is minimally almost periodic. Then, for any finite dimensional irreducible unitary representation π of H , we have*

$$H^1(H, \pi) = 0.$$

Here are some examples of groups H to which the above results apply. It seems that the only interesting examples occur when G is locally isomorphic to a product $M \times N$ and H is a lattice in G .

- (1) As it is well known, $H = SL(n, \mathbb{Q})$ is, via diagonal embedding, a lattice in

$$SL(n, \mathbb{A}) = SL(n, \mathbb{R}) \times SL(n, \mathbb{A}_f),$$

where \mathbb{A} is the ring of adèles of \mathbb{Q} and \mathbb{A}_f the subring of finite adèles. By the Strong Approximation Theorem (see, e. g., [Hum], 14.3), $SL(n, \mathbb{Q})$ is dense in $SL(n, \mathbb{A}_f)$. The dual space of $SL(n, \mathbb{A})$ is, as a topological space, a restricted product of the dual spaces of the factors $SL(n, \mathbb{Q}_p)$, $p \in P = \{\text{primes in } \mathbb{N}\} \cup \{\infty\}$ (see [Gu2, Corollary 11]). When $n \geq 3$, $SL(n, \mathbb{R})$ has property (T) and Theorem A gives a neat description of the topology in the neighborhood of $1_{SL(n, \mathbb{Q})}$.

The above can be generalized to $H = \mathbf{G}(\mathbb{Q})$, the rational points of a connected simple algebraic group \mathbf{G} defined over \mathbb{Q} . The group $\mathbf{G}(\mathbb{Q})$ is a lattice in

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{A}_f),$$

and when \mathbf{G} is simply connected and $\mathbf{G}(\mathbb{R})$ is non compact, it projects densely into $\mathbf{G}(\mathbb{A}_f)$ (see [Bor, 5.6.Theorem] and [Mar, Chap. II, (6.8) Theorem]).

- (2) Let $\mathbf{G} = SO(q)$ be the subgroup of $SL(n+1)$ preserving the quadratic form $q(x) = \sum_{i=1}^n x_i^2 - x_{n+1}^2$. For $n \geq 2$, the group $\mathbf{G}(\mathbb{R}) \simeq SO(n, 1)$ has real rank one. If $p \equiv 1 \pmod{4}$, the equation $x^2 + 1 = 0$ has a solution in \mathbb{Q}_p . This implies that the \mathbb{Q}_p -rank of $\mathbf{G}(\mathbb{Q}_p)$ is at least two

and $G(\mathbb{Q}_p)$ has Kazhdan's property. $H = G(\mathbb{Z}[1/p])$ is an irreducible lattice in $G(\mathbb{R}) \times G(\mathbb{Q}_p)$.

Note that in case $n \geq 4$ the equation $x^2 + y^2 + 1 = 0$ has a solution in \mathbb{Q}_p for any prime p , and the \mathbb{Q}_p -rank of $G(\mathbb{Q}_p)$ is at least two.

(3) For $n \geq 2$, let q be the quadratic form

$$q(x) = x_1^2 + \dots + x_n^2 - x_{n+1}^2 + \sqrt{2}x_{n+2}^2.$$

Let \mathcal{K} be the number field $\mathbb{Q}(\sqrt{2})$, and let $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ be the ring of integers of \mathcal{K} . Let G be the subgroup of $SL(n+2)$ preserving the form q . Then G is defined over \mathcal{K} , the group $G(\mathbb{R})$ is isomorphic to $SO(n+1, 1)$ and the real rank of $G(\mathbb{R})$ is one.

Now, let σ be the automorphism of \mathcal{K} with $\sigma(\sqrt{2}) = -\sqrt{2}$ and denote by G^σ the subgroup of $SL(n+2)$ preserving the form

$$\sigma q(x) = x_1^2 + \dots + x_n^2 - x_{n+1}^2 - \sqrt{2}x_{n+2}^2.$$

Then $G^\sigma(\mathbb{R}) \simeq SO(n, 2)$ has property (T) as it has real rank 2. Take $H = G(\mathcal{O})$. Embedded in $G^\sigma(\mathbb{R}) \times G(\mathbb{R})$ by means of

$$\begin{aligned} G(\mathcal{O}) &\rightarrow G^\sigma(\mathbb{R}) \times G(\mathbb{R}) \\ g &\rightarrow (g^\sigma, g), \end{aligned}$$

$G(\mathcal{O})$ an irreducible lattice in this product .

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2. Proof of Theorem A.

Recall the following formula, valid for any representations σ of G and π of a closed subgroup H ,

$$\text{Ind}_H^G(\sigma|_H \otimes \pi) \simeq \sigma \otimes \text{Ind}_H^G \pi,$$

(see, e.g., [Fe1, Lemma 4.2]).

Proof of Lemma 1. — Suppose the representation π of H contains a subrepresentation of the form

$$(\sigma \circ p)|_H$$

where σ is a representation of G/N , and p denote the canonical projection of G onto G/N .

Then the induced $\text{Ind}_H^G \pi$ contains the representation

$$\text{Ind}_H^G \left((\sigma \circ p)|_H \right) = (\sigma \circ p) \otimes \text{Ind}_H^G 1_H.$$

Since G/H carries a finite invariant measure, $\text{Ind}_H^G 1_H$ has invariant vectors. Therefore, $\text{Ind}_H^G \pi$ contains the representation $\sigma \circ p$. Restricting to N gives condition (ii) of the lemma.

The proof of the converse is much more involved. Let π be a representation of H , with space \mathcal{H}_π , such that $\text{Ind}_H^G \pi$ has a N -invariant vector ξ of norm one, that is, a measurable map

$$\xi : G \rightarrow \mathcal{H}_\pi$$

such that

- (1) for all $h \in H$, $\xi(xh) = \pi(h^{-1}) \cdot \xi(x)$ for almost all $x \in G$,
- (2) for all $n \in N$, $\xi(n^{-1}x) = \xi(x)$ for almost all $x \in G$,
- (3)

$$\int_{G/H} \|\xi(\dot{x})\|^2 d\dot{x} = 1$$

where $\dot{x} = xH$ and $d\dot{x}$ denotes the G -invariant measure on G/H .

Let ρ denote the induced representation $\text{Ind}_H^G \pi$ with space \mathcal{H}_ρ . Using a well known smoothing procedure, we are going to construct a continuous N -invariant vector in \mathcal{H}_ρ as follows. For every compact neighbourhood U of the identity in G , fix a continuous non negative function φ_U on G with support in U and $\int_G \varphi_U(g) dg = 1$.

Consider the map

$$\xi_U : G \rightarrow \mathcal{H}_\pi$$

defined by

$$\xi_U(x) = \int_G \varphi_U(xg) \xi(g^{-1}) dg.$$

Then the following holds:

- (a) ξ_U is continuous on G .

To see this, observe first that the function

$$G \rightarrow \mathbb{R}, \quad g \mapsto \|\xi(g)\|$$

is integrable over any compact subset K of G . Indeed, let the left Haar measure dh on H be so normalized that $dg = dh d\dot{g}$ holds. Then, denoting by χ_K the characteristic function of K , one has

$$\begin{aligned} \int_K \|\xi(g)\| dg &= \int_G \|\xi(g)\| \chi_K(g) dg \\ &= \int_{G/H} \left(\int_H \|\xi(gh)\| \chi_K(gh) dh \right) d\dot{g} \\ &= \int_{G/H} \|\xi(\dot{g})\| \left(\int_H \chi_K(gh) dh \right) d\dot{g} \\ &= \int_{G/H} \|\xi(\dot{g})\| \mu_H(H \cap g^{-1}K) d\dot{g}, \end{aligned}$$

where $\mu_H(H \cap g^{-1}K)$ denotes the H -measure of $H \cap g^{-1}K$ and depends only on \dot{g} . Observe that $H \cap g^{-1}K$ is non empty if and only if $\dot{g} = \dot{k}$ for some k in K . Hence, $\mu_H(H \cap g^{-1}K) \leq \mu_H(H \cap K^{-1}K)$. Note that $\mu_H(H \cap K^{-1}K) < \infty$ since $H \cap K^{-1}K$ is compact. Thus, by Cauchy-Schwarz inequality,

$$\begin{aligned} \int_K \|\xi(g)\| dg &\leq \mu_H(H \cap K^{-1}K) \int_{G/H} \|\xi(\dot{g})\| d\dot{g} \\ &\leq \mu_H(H \cap K^{-1}K) \sqrt{\text{vol}(G/H)} \left(\int_{G/H} \|\xi(\dot{g})\|^2 d\dot{g} \right)^{1/2} \\ &< \infty. \end{aligned}$$

Now, fix a compact neighbourhood V of the group unit e of G , let x be in G and y in Vx . Denote by K the compact set $x^{-1}(U \cup V^{-1}U)$. Since the support of φ_U is contained in U , one has

$$\begin{aligned} \|\xi_U(x) - \xi_U(y)\| &\leq \int_G \|\varphi_U(xg)\xi(g^{-1}) - \varphi_U(yg)\xi(g^{-1})\| dg \\ &= \int_{x^{-1}(U \cup V^{-1}U)} \Delta(g^{-1}) |\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})| \|\xi(g)\| dg \\ &\leq \sup_{g \in K} \Delta(g^{-1}) |\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})| \int_K \|\xi(g)\| dg, \end{aligned}$$

where Δ denotes the modular function on G . Let $\varepsilon > 0$. As φ_U is uniformly continuous, there exists some neighbourhood W of e contained in V such that

$$|\varphi_U(g) - \varphi_U(zg)| < \varepsilon$$

for all z in W and all g in G . Hence, for all y in Wx

$$\|\xi_U(x) - \xi_U(y)\| \leq C\varepsilon,$$

where $C = \sup_{g \in K} \Delta(g^{-1}) \int_K \|\xi(g)\| dg$ is a constant depending only on x, U , and V . Thus, ξ_U is continuous.

- (b) ξ_U belongs to the space \mathcal{H}_ρ of the induced representation $\rho = \text{Ind}_H^G \pi$. Indeed, let x in G and h in H . According to (1) above, for almost all $g \in G$,

$$\xi(g^{-1}xh) = \pi(h^{-1}) \cdot \xi(g^{-1}x)$$

and hence

$$\begin{aligned} \xi_U(xh) &= \int_G \varphi_U(g) \xi(g^{-1}xh) dg \\ &= \int_G \varphi_U(g) \pi(h^{-1}) \cdot \xi(g^{-1}x) dg \\ &= \pi(h^{-1}) \cdot \int_G \varphi_U(g) \xi(g^{-1}x) dg \\ &= \pi(h^{-1}) \cdot \xi_U(x). \end{aligned}$$

Moreover,

$$\begin{aligned} |(\xi_U(x), \xi_U(x))| &\leq \int_G \int_G \varphi_U(g) \varphi_U(g') |(\xi(g^{-1}x), \xi(g'^{-1}x))| dg dg' \\ &\leq \int_G \int_G \varphi_U(g) \varphi_U(g') \|\xi(g^{-1}x)\| \|\xi(g'^{-1}x)\| dg dg', \end{aligned}$$

and hence, using Cauchy–Schwarz inequality in the space $L^2(G/H)$,

$$\begin{aligned} \|\xi_U\|^2 &\leq \int_G \varphi_U(g) \int_G \varphi_U(g') \int_{G/H} \|\xi(g^{-1}x)\| \|\xi(g'^{-1}x)\| dx dg dg' \\ &\leq \int_G \varphi_U(g) \int_G \varphi_U(g') \|\rho(g)\xi\| \|\rho(g')\xi\| dg dg' \\ &\leq \left(\int_G \varphi_U(g) \|\rho(g)\xi\| dg \right)^2 = \|\xi\|^2. \end{aligned}$$

(c) ξ_U is N -invariant in \mathcal{H}_ρ .

Indeed, for $n \in N$, $x \in G$ and $\eta \in \mathcal{H}_\rho$

$$\begin{aligned} & \int_{G/H} \langle \xi_U(n\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \\ &= \int_{G/H} \left\langle \int_G \varphi_U(g) \xi(g^{-1}n\dot{x}) dg, \eta(\dot{x}) \right\rangle d\dot{x} \\ &= \int_G \varphi_U(g) \left(\int_{G/H} \langle \xi((g^{-1}ng)^{-1}\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \right) dg \\ &= \int_G \varphi_U(g) \left(\int_{G/H} \langle \rho(g)\rho(g^{-1}n^{-1}g)\xi(\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \right) dg \\ &= \int_G \varphi_U(g) \left(\int_{G/H} \langle \rho(g)\xi(\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \right) dg \\ &= \int_{G/H} \int_G \varphi_U(g) \langle \xi(g^{-1}\dot{x}), \eta(\dot{x}) \rangle d\dot{x} dg \\ &= \int_{G/H} \langle \xi_U(\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \end{aligned}$$

as N is a normal subgroup and ξ is N -invariant.

(d) ξ_U is non zero, for U sufficiently small. This is clear since $\|\xi_U - \xi\| \rightarrow 0$ when $U \rightarrow \{e\}$.

Now, for any $c_1, \dots, c_n \in \mathbb{C}$, $g_1, \dots, g_n \in G$, $k \in N$, $h \in H$, using the continuity of ξ_U , we have

$$\begin{aligned} \sum_{i=1}^n c_i \xi_U(g_i h k) &= \sum_{i=1}^n c_i \xi_U((g_i h) k (g_i h)^{-1} g_i h) \\ &= \sum_{i=1}^n c_i \rho((g_i h) k (g_i h)^{-1}) \xi_U(g_i h) \\ &= \sum_{i=1}^n c_i \xi_U(g_i h) \end{aligned}$$

and

$$\left\| \sum_{i=1}^n c_i \xi_U(g_i h) \right\| = \left\| \pi(h^{-1}) \sum_{i=1}^n c_i \xi_U(g_i) \right\| = \left\| \sum_{i=1}^n c_i \xi_U(g_i) \right\|.$$

Therefore, by density of HN in G and, again, by continuity of ξ_U ,

$$\left\| \sum_{i=1}^n c_i \xi_U(g_i g) \right\| = \left\| \sum_{i=1}^n c_i \xi_U(g_i) \right\|$$

for all g in G .

Let \mathcal{W}_π be the (non zero) closed subspace of \mathcal{H}_π generated by $\xi_U(G)$. Then, for any g in G ,

$$\begin{aligned} \mathcal{W}_\pi &\longrightarrow \mathcal{W}_\pi \\ \sum_{i=1}^n c_i \xi_U(g_i) &\mapsto \sum_{i=1}^n c_i \xi_U(g_i g^{-1}) \end{aligned}$$

is a unitary operator depending only on the class of g in G/N . This defines a unitary representation σ of G/N . As ξ_U is continuous, σ is continuous. Since

$$\sigma \circ p(h) \xi_U(g) = \xi_U(gh^{-1}) = \pi(h) \xi_U(g),$$

it is clear that

$$\sigma \circ p(h) = \pi(h)$$

on \mathcal{W}_π , for all h in H . □

We shall need the following lemma.

LEMMA 2. — *Let G , N and H be as in Lemma 1. Let ρ be the quasi-regular representation of G on $L^2(G/H)$. The N -invariant functions in $L^2(G/H)$ are constant.*

Proof. — Lemma 2 amounts to saying that the action of N (by left multiplication) on the homogeneous space G/H is ergodic. By Moore’s duality theorem, ergodicity of the N -action on G/H is equivalent to ergodicity of the action of H on G/N by left multiplication (see [Zim, Corollary 2. 2. 3]). Density of HN in G implies that the subgroup $p(H)$ is dense in the group G/N , and this is equivalent with ergodicity of the action of H on G/N (see [Zim, Lemma 2. 2. 13]). □

We shall frequently use Fell’s inner hull-kernel topology on the set $\text{Rep}(G)$ of all equivalence classes of unitary representations of a locally compact group G . This topology is defined as follows. For π in $\text{Rep}(G)$,

$\varepsilon > 0$, a compact subset K of G , and positive definite functions $\varphi_1, \dots, \varphi_n$ associated with π , let $W(\varphi_1, \dots, \varphi_n; K; \varepsilon; \pi)$ be the set of all ρ in $\text{Rep}(G)$ such that there exists ψ_1, \dots, ψ_n , each of which is a sum of positive definite functions associated with ρ , for which

$$|\varphi_i(x) - \psi_i(x)| < \varepsilon \quad \forall i = 1, \dots, n \quad \forall x \in K.$$

The subsets $W(\varphi_1, \dots, \varphi_n; K; \varepsilon; \pi)$ form a basis of neighbourhoods of π (see [Fe1, Section 2]). This topology may also be described in terms of weak containment. Recall that π is weakly contained in a set \mathcal{S} of representations of G if every positive definite function associated with π is the limit, uniformly on compact subsets of G , of sums of positive definite functions associated with representations from \mathcal{S} . It is clear that a net π_n of unitary representations of G converges to π if and only if, for every subnet $\pi_{n'}$ of π_n , π is weakly contained in the set $\{\pi_{n'}\}$. Restricted to \widehat{G} , this is just the usual Fell-Jacobson topology on \widehat{G} (see also [Dix, Chap.18]). We are now in position to prove Theorem A.

Proof of Theorem A. — Let π_n be a net of irreducible representations of H converging to 1_H in \widehat{H} . Then, by continuity of inducing (see [Fe1, Theorem 4.1]),

$$\text{Ind}_H^G \pi_n \rightarrow \text{Ind}_H^G 1_H$$

in $\text{Rep}(G)$. Since H has finite covolume, 1_G is contained in $\text{Ind}_H^G 1_H$ and this implies

$$\text{Ind}_H^G \pi_n \rightarrow 1_G.$$

As N has Kazhdan's property, we may assume that $\text{Ind}_H^G \pi_n$ has N -invariant vectors for all n . Hence, by Lemma 1, there are (irreducible) representations σ_n of G/N such that $\pi_n = (\sigma_n \circ p)|_H$ where $p : G \rightarrow G/N$ is the canonical projection. The proof will be finished if we show that

$$\sigma_n \circ p \rightarrow 1_G$$

in \widehat{G} .

Since H has finite covolume, one has

$$(*) \quad \text{Ind}_H^G \pi_n = \text{Ind}_H^G (\sigma_n \circ p)|_H = (\sigma_n \circ p) \otimes \rho = (\sigma_n \circ p) \oplus ((\sigma_n \circ p) \otimes \rho^0),$$

where $\rho = \text{Ind}_H^G 1_H$ and ρ^0 is the restriction of ρ to the orthogonal of the constants in $L^2(G/N)$.

Now, the restriction to N of $(\sigma_n \circ p) \otimes \rho^0$ is a multiple of $\rho^0|_N$. Since N has property (T), Lemma 2 above implies that $\rho^0|_N$ does not weakly contain the trivial representation 1_N . So, $(\sigma_n \circ p) \otimes \rho^0$ cannot converge to 1_G . As

$$\text{Ind}_H^G \pi_n \rightarrow 1_G,$$

we conclude from (*) that

$$\sigma_n \circ p \rightarrow 1_G. \quad \square$$

3. Proof of Theorem B.

The finite dimensional representation π decomposes as a finite sum

$$\pi = \sum_{i=1}^n \pi_i$$

of irreducible subrepresentations π_i . Since, in an obvious way,

$$H^1(H, \pi) \cong \bigoplus_{i=1}^n H^1(H, \pi_i),$$

we may assume that π is irreducible.

We first deal with the case where π is the trivial representation 1_H . Then, $H^1(H, \pi)$ is the group of all (continuous) homomorphisms from H to the additive group of the complex numbers \mathbb{C} . Let $[\overline{H, H}]$ denote the closure of the commutator subgroup of H . The corollary to Theorem A implies that the dual group of the abelian group $H/[\overline{H, H}]$ is discrete, since the trivial character is isolated. So, by duality theory, $H/[\overline{H, H}]$ is compact. Hence, $H^1(H, 1_H) = 0$.

Suppose now that $\pi \neq 1_H$ and assume, by contradiction, that

$$H^1(H, \pi) \neq 0.$$

Then, by Vershik-Karpushev theorem (see [VeK, Theorem 2]) there exists a net π_n in \widehat{H} such that

$$\pi_n \rightarrow \pi \quad \text{and} \quad \pi_n \rightarrow 1_H.$$

By Theorem A, we may assume that $\pi_n = (\sigma_n \circ p)|_H$ for irreducible representations σ_n of G/N .

Let $\bar{\pi}$ denote the conjugate representation of π . Then, by continuity of tensoring (see [Fe2, Theorem 1]),

$$\pi_n \otimes \bar{\pi} \rightarrow \pi \otimes \bar{\pi}.$$

Hence,

$$(\sigma_n \circ p) \otimes \text{Ind}_H^G \bar{\pi} = \text{Ind}_H^G (\pi_n \otimes \bar{\pi}) \rightarrow \text{Ind}_H^G (\pi \otimes \bar{\pi}).$$

Restricting to N , this implies that $\text{Ind}_H^G \bar{\pi}|_N$ weakly contains the representation $\text{Ind}_H^G (\pi \otimes \bar{\pi})|_N$.

Since π is finite dimensional, it is well known that $\pi \otimes \bar{\pi}$ has an invariant vector. Therefore, as H has finite covolume, $\text{Ind}_H^G \bar{\pi}|_N$ weakly contains 1_N . We conclude that $\text{Ind}_H^G \bar{\pi}$ has N -invariant vectors. But then, Lemma 1 implies that π factorizes to a representation σ of G/N , thus, $\pi = (\sigma \circ p)|_H$. As G/N is minimally almost periodic, this forces $\sigma = 1_{G/N}$ and hence $\pi = 1_H$, a contradiction. \square

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