ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 47, nº 4 (1997), p. 1065-1078
<http://www.numdam.org/item?id=AIF_1997__47_4_1065_0>

Ann. Inst. Fourier, Grenoble
47, 4 (1997), 1065–1078

ON A VARIANT OF KAZHDAN'S PROPERTY (T) FOR SUBGROUPS OF SEMISIMPLE GROUPS

by M. B. BEKKA and N. LOUVET

1. Introduction and statement of the results.

Lubotzky and Zimmer considered in [LuZ] the following variant of Kazhdan's property (T) for a locally compact group G. Let $\mathcal{R}$ be a set of equivalence classes of unitary representations of G containing the trivial one dimensional representation 1_G. Throughout this paper, all group representations will be assumed to be unitary and strongly continuous representations in non zero Hilbert spaces. The group G is said to have property (T; $\mathcal{R}$) if 1_G is isolated in $\mathcal{R}$. For $\mathcal{R} = \widehat{G}$, the unitary dual of G, this is just the celebrated Kazhdan's property (T) (see [HaV] for an account on this property).

There are interesting examples of (discrete) groups that are not Kazhdan groups and that satisfy property (T; $\mathcal{R}$) for some natural classes of representations $\mathcal{R}$. For instance, $SL(2,\mathbb{Z})$ has property (T; $\mathcal{R}$) for the set $\mathcal{R}$ of all irreducible representations of $SL(2,\mathbb{Z})$ that factorize through a quotient by a congruence subgroup of $SL(2,\mathbb{Z})$. Another interesting example is the group $SL(2,\mathbb{Z}[1/p])$ which has the property (T; $\mathcal{R}$), $\mathcal{R}$ being the set of all finite dimensional representations of $SL(2,\mathbb{Z}[1/p])$ (see [LuZ]). Such isolation properties allow for instance the construction of expanders graphs (see [Lub]).

In [LuZ], other examples were discovered. They occur as follows. Let Γ be a lattice in the direct product $G_1 \times G_2$ of two groups. Assume that Γ projects densely on G_1 and that G_2 has property (T). Then Γ has (T; $\mathcal{R}$)

Key words: Discrete subgroups of Lie groups – Kazhdan's property – Cohomology.
Math. Classification: 22E40 – 22D10 – 22E41.

for the set $\mathcal{R}$ of all finite dimensional representations of Γ (see below for some concrete examples of such Γ).

In this paper, we shall generalize the results in [LuZ]. It is interesting to note that our methods are elementary.

We fix, once and for all, a locally compact group G and we let H be a closed (not necessarily discrete) subgroup of G and N a closed normal subgroup of G. Throughout this paper, we shall make the following two assumptions:

(I) HN is dense in G, and

(II) the homogeneous space G/H has a finite invariant measure.

Denoting by p the canonical projection $G \to G/N$, assumption (I) says that $p(H)$ is dense in G/N. If π is an irreducible representation of G/N, it is clear that $(\pi \circ p)\big|_H$ is an irreducible representation of H and that π is determined (up to unitary equivalence) by $(\pi \circ p)\big|_H$. So, we may view $\widehat{G/N}$ as a subset of $\widehat{H}$. Our main result states that if an irreducible representation π of H is sufficiently close to the trivial representation 1_H and if N has property (T), then π is in $\widehat{G/N}$.

More precisely, the following holds:

THEOREM A. — *Let G, H, and N satisfy the above assumptions (I) and (II). Assume moreover that N has Kazhdan's property (T). Let $\mathcal{U}$ be a neighbourhood of $1_{G/N}$ in $\widehat{G/N}$. Then the set of all $(\pi \circ p)\big|_H$, $\pi \in \mathcal{U}$, is a neighbourhood of 1_H in $\widehat{H}$.*

In other words, H has property (T; $\mathcal{R}$) with $\mathcal{R}$ the set of all irreducible representations of H that do not factorize through G/N together with 1_H. Of course, in case G has Kazhdan's property (T), then taking $N = G$ shows that Theorem A is a generalization of the fact that property (T) is inherited by subgroups of cofinite volume. (see [HaV, Chapter 3, Theorem 4]).

Theorem A is a rigidity result in the following sense. It says that any irreducible representation of H that is sufficiently close to the trivial representation 1_H extends to a representation of G. One may speculate whether this is always true for irreducible lattices in a (nontrivial) product of, say, simple Lie groups. Recall that the only simple Lie groups without property (T) are the ones that are locally isomorphic to $SO(n,1)$ or $SU(n,1)$.

Let FD denote the subset of $\widehat{H}$ consisting of the finite dimensional representations. Recall that a group G is called minimally almost periodic if 1_G is the unique finite dimensional unitary irreducible representation of G. An immediate consequence of Theorem A is the following corollary, also noticed in [LuZ].

COROLLARY. — *If G/N is minimally almost periodic then H has property (T, FD).*

Our Theorem A improves and makes more precise Theorem 2.2 in [LuZ] where the following was shown. Under the additional assumption that G is a direct product $M \times N$, H has property (T; $\mathcal{R}$) where $\mathcal{R}$ is the set of all $\pi \in \widehat{H}$ such that, for all $k \geqslant 1$, no nontrivial subrepresentation of the symmetric power $S^k(\pi)$ factorize through $p : H \to M$. Of course, this is sufficient in order to deduce the corollary above.

Our proof of Theorem A is elementary. It is based on the following extension result which may be of independent interest.

LEMMA 1. — *Let G, H, and N satisfy the above assumptions (I) and (II). Let π be a unitary representation of H. Then the following are equivalent:*

(i) *π contains a subrepresentation that factorizes through the canonical projection $p : G \to G/N$ i.e. there exists a representation σ of G/N so that π contains $(\sigma \circ p)\big|_H$.*

(ii) *The induced representation $\mathrm{Ind}_H^G \pi$ contains a non zero N-invariant vector.*

For a representation π of a locally compact group H, let $\mathrm{H}^1(H, \pi)$ be the first cohomology group of H with coefficients in π (see [Gu1, Gu2]). It is well known that $\mathrm{H}^1(H, \pi) = 0$ for any π if (and only if) the group H has Kazhdan's property (T) (see [HaV, Chapter 4, Theorem 7]).

Vershik and Karpushev [VeK, Theorem 2] showed that if $\mathrm{H}^1(H, \pi) \neq 0$ for some irreducible representation π, then π is infinitesimally small, that is, there exists a net π_n in $\widehat{H}$ such that $\lim \pi_n = 1_H$ and $\lim \pi_n = \pi$. This result has been conjectured by Guichardet [Gu2] and some partial results were previously obtained by Delorme [Del].

Using Theorem A and Vershik and Karpushev's result, we give an elementary proof of the following theorem, proved in [LuZ, Theorem 3.1] in the product case $G = M \times N$ (see also [BoW]).

THEOREM B. — *Let G, H and N be as in Theorem A and assume G/N is minimally almost periodic. Then, for any finite dimensional irreducible unitary representation π of H, we have*

$$\mathrm{H}^1(H, \pi) = 0.$$

Here are some examples of groups H to which the above results apply. It seems that the only interesting examples occur when G is locally isomorphic to a product $M \times N$ and H is a lattice in G.

(1) As it is well known, $H = SL(n, \mathbb{Q})$ is, via diagonal embedding, a lattice in

$$SL(n, \mathbb{A}) = SL(n, \mathbb{R}) \times SL(n, \mathbb{A}_f),$$

where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ the subring of finite adeles. By the Strong Approximation Theorem (see, e. g., [Hum], 14.3), $SL(n, \mathbb{Q})$ is dense in $SL(n, \mathbb{A}_f)$. The dual space of $SL(n, \mathbb{A})$ is, as a topological space, a restricted product of the dual spaces of the factors $SL(n, \mathbb{Q}_p)$, $p \in P = \{\text{primes in } \mathbb{N}\} \cup \{\infty\}$ (see [Gu2, Corollary 11]). When $n \geqslant 3$, $SL(n, \mathbb{R})$ has property (T) and Theorem A gives a neat description of the topology in the neighborhood of $1_{SL(n,\mathbb{Q})}$.

The above can be generalized to $H = \boldsymbol{G}(\mathbb{Q})$, the rational points of a connected simple algebraic group $\boldsymbol{G}$ defined over $\mathbb{Q}$. The group $\boldsymbol{G}(\mathbb{Q})$ is a lattice in

$$\boldsymbol{G}(\mathbb{A}) = \boldsymbol{G}(\mathbb{R}) \times \boldsymbol{G}(\mathbb{A}_f),$$

and when $\boldsymbol{G}$ is simply connected and $\boldsymbol{G}(\mathbb{R})$ is non compact, it projects densely into $\boldsymbol{G}(\mathbb{A}_f)$ (see [Bor, 5.6.Theorem] and [Mar, Chap. II, (6.8) Theorem]).

(2) Let $\boldsymbol{G} = SO(q)$ be the subgroup of $SL(n+1)$ preserving the quadratic form $q(x) = \sum_{i=1}^{n} x_i^2 - x_{n+1}^2$. For $n \geqslant 2$, the group $\boldsymbol{G}(\mathbb{R}) \simeq SO(n, 1)$ has real rank one. If $p \equiv 1 \pmod 4$, the equation $x^2 + 1 = 0$ has a solution in $\mathbb{Q}_p$. This implies that the $\mathbb{Q}_p$-rank of $\boldsymbol{G}(\mathbb{Q}_p)$ is at least two

and $G(\mathbb{Q}_p)$ has Kazhdan's property. $H = G(\mathbb{Z}[1/p])$ is an irreducible lattice in $G(\mathbb{R}) \times G(\mathbb{Q}_p)$.

Note that in case $n \geqslant 4$ the equation $x^2 + y^2 + 1 = 0$ has a solution in $\mathbb{Q}_p$ for any prime p, and the $\mathbb{Q}_p$-rank of $G(\mathbb{Q}_p)$ is at least two.

(3) For $n \geqslant 2$, let q be the quadratic form

$$q(x) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2 + \sqrt{2}x_{n+2}^2.$$

Let $\mathcal{K}$ be the number field $\mathbb{Q}(\sqrt{2})$, and let $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ be the ring of integers of $\mathcal{K}$. Let G be the subgroup of $SL(n+2)$ preserving the form q. Then G is defined over $\mathcal{K}$, the group $G(\mathbb{R})$ is isomorphic to $SO(n+1,1)$ and the real rank of $G(\mathbb{R})$ is one.

Now, let σ be the automorphism of $\mathcal{K}$ with $\sigma(\sqrt{2}) = -\sqrt{2}$ and denote by G^σ the subgroup of $SL(n+2)$ preserving the form

$${}^\sigma q(x) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2 - \sqrt{2}x_{n+2}^2.$$

Then $G^\sigma(\mathbb{R}) \simeq SO(n,2)$ has property (T) as it has real rank 2. Take $H = G(\mathcal{O})$. Embedded in $G^\sigma(\mathbb{R}) \times G(\mathbb{R})$ by means of

$$\begin{aligned} G(\mathcal{O}) &\to G^\sigma(\mathbb{R}) \times G(\mathbb{R}) \\ g &\to (g^\sigma, g), \end{aligned}$$

$G(\mathcal{O})$ an irreducible lattice in this product .

It is a pleasure to thank M. Mayer for a careful reading of a first version of this paper.

2. Proof of Theorem A.

Recall the following formula, valid for any representations σ of G and π of a closed subgroup H,

$$\mathrm{Ind}_H^G(\sigma|_H \otimes \pi) \simeq \sigma \otimes \mathrm{Ind}_H^G \pi,$$

(see, e.g., [Fe1, Lemma 4.2]).

Proof of Lemma 1. — Suppose the representation π of H contains a subrepresentation of the form

$$(\sigma \circ p)|_H$$

where σ is a representation of G/N, and p denote the canonical projection of G onto G/N.

Then the induced $\mathrm{Ind}_H^G \pi$ contains the representation

$$\mathrm{Ind}_H^G \left((\sigma \circ p)\big|_H \right) = (\sigma \circ p) \otimes \mathrm{Ind}_H^G 1_H.$$

Since G/H carries a finite invariant measure, $\mathrm{Ind}_H^G 1_H$ has invariant vectors. Therefore, $\mathrm{Ind}_H^G \pi$ contains the representation $\sigma \circ p$. Restricting to N gives condition (ii) of the lemma.

The proof of the converse is much more involved. Let π be a representation of H, with space $\mathcal{H}_\pi$, such that $\mathrm{Ind}_H^G \pi$ has a N-invariant vector ξ of norm one, that is, a measurable map

$$\xi : G \to \mathcal{H}_\pi$$

such that

(1) for all $h \in H$, $\xi(xh) = \pi(h^{-1}) \cdot \xi(x)$ for almost all $x \in G$,

(2) for all $n \in N$, $\xi(n^{-1}x) = \xi(x)$ for almost all $x \in G$,

(3)

$$\int_{G/H} \|\xi(\dot{x})\|^2 d\dot{x} = 1$$

where $\dot{x} = xH$ and $d\dot{x}$ denotes the G-invariant measure on G/H.

Let ρ denote the induced representation $\mathrm{Ind}_H^G \pi$ with space $\mathcal{H}_\rho$. Using a well known smoothing procedure, we are going to construct a continuous N-invariant vector in $\mathcal{H}_\rho$ as follows. For every compact neighbourhood U of the identity in G, fix a continuous non negative function φ_U on G with support in U and $\int_G \varphi_U(g)dg = 1$.

Consider the map

$$\xi_U : G \to \mathcal{H}_\pi$$

defined by

$$\xi_U(x) = \int_G \varphi_U(xg)\xi(g^{-1})dg.$$

Then the following holds:

(a) ξ_U is continuous on G.

To see this, observe first that the function

$$G \to \mathbb{R}, \qquad g \mapsto \|\xi(g)\|$$

is integrable over any compact subset K of G. Indeed, let the left Haar measure dh on H be so normalized that $dg = dhd\dot{g}$ holds. Then, denoting by χ_K the characteristic function of K, one has

$$\begin{aligned}\int_K \|\xi(g)\|\, dg &= \int_G \|\xi(g)\|\, \chi_K(g)dg \\ &= \int_{G/H} \Big(\int_H \|\xi(gh)\|\, \chi_K(gh)dh\Big) d\dot{g} \\ &= \int_{G/H} \|\xi(\dot{g})\| \Big(\int_H \chi_K(gh)dh\Big) d\dot{g} \\ &= \int_{G/H} \|\xi(\dot{g})\|\, \mu_H(H \cap g^{-1}K)d\dot{g},\end{aligned}$$

where $\mu_H(H\cap g^{-1}K)$ denotes the H–measure of $H\cap g^{-1}K$ and depends only on $\dot{g}$. Observe that $H \cap g^{-1}K$ is non empty if and only if $\dot{g} = \dot{k}$ for some k in K. Hence, $\mu_H(H \cap g^{-1}K) \leqslant \mu_H(H \cap K^{-1}K)$. Note that $\mu_H(H \cap K^{-1}K) < \infty$ since $H \cap K^{-1}K$ is compact. Thus, by Cauchy–Schwarz inequality,

$$\begin{aligned}\int_K \|\xi(g)\| dg &\leqslant \mu_H(H \cap K^{-1}K) \int_{G/H} \|\xi(\dot{g})\|\, d\dot{g} \\ &\leqslant \mu_H(H \cap K^{-1}K)\sqrt{\mathrm{vol}(G/H)} \left(\int_{G/H} \|\xi(\dot{g})\|^2\, d\dot{g}\right)^{1/2} \\ &< \infty.\end{aligned}$$

Now, fix a compact neighbourhood V of the group unit e of G, let x be in G and y in Vx. Denote by K the compact set $x^{-1}(U \cup V^{-1}U)$. Since the support of φ_U is contained in U, one has

$$\begin{aligned}\|\xi_U(x) - \xi_U(y)\| &\leqslant \int_G \big\|\varphi_U(xg)\xi(g^{-1}) - \varphi_U(yg)\xi(g^{-1})\big\|\, dg \\ &= \int_{x^{-1}(U\cup V^{-1}U)} \Delta(g^{-1})\, \big|\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})\big|\, \|\xi(g)\|\, dg \\ &\leqslant \sup_{g\in K} \Delta(g^{-1})\, \big|\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})\big| \int_K \|\xi(g)\|\, dg,\end{aligned}$$

where Δ denotes the modular function on G. Let $\varepsilon > 0$. As φ_U is uniformly continuous, there exists some neighbourhood W of e contained in V such that

$$|\varphi_U(g) - \varphi_U(zg)| < \varepsilon$$

for all z in W and all g in G. Hence, for all y in Wx

$$\|\xi_U(x) - \xi_U(y)\| \leqslant C\varepsilon,$$

where $C = \sup_{g \in K} \Delta(g^{-1}) \int_K \|\xi(g)\| \, dg$ is a constant depending only on x, U, and V. Thus, ξ_U is continuous.

(b) ξ_U belongs to the space $\mathcal{H}_\rho$ of the induced representation $\rho = \mathrm{Ind}_H^G \pi$. Indeed, let x in G and h in H. According to (1) above, for almost all $g \in G$,

$$\xi(g^{-1}xh) = \pi(h^{-1}) \cdot \xi(g^{-1}x)$$

and hence

$$\begin{aligned}
\xi_U(xh) &= \int_G \varphi_U(g)\xi(g^{-1}xh)dg \\
&= \int_G \varphi_U(g)\pi(h^{-1}) \cdot \xi(g^{-1}x)dg \\
&= \pi(h^{-1}) \cdot \int_G \varphi_U(g)\xi(g^{-1}x)dg \\
&= \pi(h^{-1}) \cdot \xi_U(x).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\langle \xi_U(x), \xi_U(x)\rangle| &\leqslant \int_G \int_G \varphi_U(g)\varphi_U(g') \, |\langle \xi(g^{-1}x), \xi(g'^{-1}x)\rangle| \, dg dg' \\
&\leqslant \int_G \int_G \varphi_U(g)\varphi_U(g') \, \|\xi(g^{-1}x)\| \, \|\xi(g'^{-1}x)\| \, dg dg',
\end{aligned}$$

and hence, using Cauchy–Schwarz inequality in the space $\mathrm{L}^2(G/H)$,

$$\begin{aligned}
\|\xi_U\|^2 &\leqslant \int_G \varphi_U(g) \int_G \varphi_U(g') \int_{G/H} \|\xi(g^{-1}x)\| \, \|\xi(g'^{-1}x)\| \, d\dot{x} dg dg' \\
&\leqslant \int_G \varphi_U(g) \int_G \varphi_U(g') \, \|\rho(g)\xi\| \, \|\rho(g')\xi\| \, dg dg' \\
&\leqslant \left(\int_G \varphi_U(g) \, \|\rho(g)\xi\| \, dg\right)^2 = \|\xi\|^2 .
\end{aligned}$$

(c) ξ_U is N-invariant in $\mathcal{H}_\rho$.

Indeed, for $n \in N$, $x \in G$ and $\eta \in \mathcal{H}_\rho$

$$\begin{aligned}
&\int_{G/H} \langle \xi_U(n\dot{x}), \eta(\dot{x})\rangle \, d\dot{x} \\
&\quad = \int_{G/H} \Big\langle \int_G \varphi_U(g)\xi(g^{-1}n\dot{x})dg, \eta(\dot{x})\Big\rangle d\dot{x} \\
&\quad = \int_G \varphi_U(g)\Big(\int_{G/H} \langle \xi\big((g^{-1}ng)g^{-1}\dot{x}\big), \eta(\dot{x})\rangle d\dot{x}\Big) dg \\
&\quad = \int_G \varphi_U(g)\Big(\int_{G/H} \langle \rho(g)\rho(g^{-1}n^{-1}g)\xi(\dot{x}), \eta(\dot{x})\rangle d\dot{x}\Big) dg \\
&\quad = \int_G \varphi_U(g)\Big(\int_{G/H} \langle \rho(g)\xi(\dot{x})), \eta(\dot{x})\rangle d\dot{x}\Big) dg \\
&\quad = \int_{G/H}\int_G \varphi_U(g)\langle \xi(g^{-1}\dot{x}), \eta(\dot{x})\rangle d\dot{x}dg \\
&\quad = \int_{G/H} \langle \xi_U(\dot{x}), \eta(\dot{x})\rangle d\dot{x}
\end{aligned}$$

as N is a normal subgroup and ξ is N-invariant.

(d) ξ_U is non zero, for U sufficiently small. This is clear since $\|\xi_U - \xi\| \to 0$ when $U \to \{e\}$.

Now, for any $c_1, \dots, c_n \in \mathbb{C}$, $g_1, \dots, g_n \in G$, $k \in N$, $h \in H$, using the continuity of ξ_U, we have

$$\begin{aligned}
\sum_{i=1}^n c_i\xi_U(g_ihk) &= \sum_{i=1}^n c_i\xi_U\big((g_ih)k(g_ih)^{-1}g_ih\big) \\
&= \sum_{i=1}^n c_i\rho\big((g_ih)k(g_ih)^{-1}\big)\xi_U(g_ih) \\
&= \sum_{i=1}^n c_i\xi_U(g_ih)
\end{aligned}$$

and

$$\left\|\sum_{i=1}^n c_i\xi_U(g_ih)\right\| = \left\|\pi(h^{-1})\sum_{i=1}^n c_i\xi_U(g_i)\right\| = \left\|\sum_{i=1}^n c_i\xi_U(g_i)\right\|.$$

Therefore, by density of HN in G and, again, by continuity of ξ_U,

$$\left\| \sum_{i=1}^{n} c_i \xi_U(g_i g) \right\| = \left\| \sum_{i=1}^{n} c_i \xi_U(g_i) \right\|$$

for all g in G.

Let $\mathcal{W}_\pi$ be the (non zero) closed subspace of $\mathcal{H}_\pi$ generated by $\xi_U(G)$. Then, for any g in G,

$$\begin{aligned} \mathcal{W}_\pi &\longrightarrow \mathcal{W}_\pi \\ \sum_{i=1}^{n} c_i \xi_U(g_i) &\mapsto \sum_{i=1}^{n} c_i \xi_U(g_i g^{-1}) \end{aligned}$$

is a unitary operator depending only on the class of g in G/N. This defines a unitary representation σ of G/N. As ξ_U is continuous, σ is continuous. Since

$$\sigma \circ p(h)\ \xi_U(g) = \xi_U(gh^{-1}) = \pi(h)\ \xi_U(g),$$

it is clear that

$$\sigma \circ p(h) = \pi(h)$$

on $\mathcal{W}_\pi$, for all h in H. □

We shall need the following lemma.

LEMMA 2. — *Let G, N and H be as in Lemma 1. Let ρ be the quasi-regular representation of G on $\mathrm{L}^2(G/H)$. The N-invariant functions in $\mathrm{L}^2(G/H)$ are constant.*

Proof. — Lemma 2 amounts to saying that the action of N (by left multiplication) on the homogeneous space G/H is ergodic. By Moore's duality theorem, ergodicity of the N-action on G/H is equivalent to ergodicity of the action of H on G/N by left multiplication (see [Zim, Corollary 2. 2. 3]). Density of HN in G implies that the subgroup $p(H)$ is dense in the group G/N, and this is equivalent with ergodicity of the action of H on G/N (see [Zim, Lemma 2. 2. 13]). □

We shall frequently use Fell's inner hull-kernel topology on the set $\mathrm{Rep}(G)$ of all equivalence classes of unitary representations of a locally compact group G. This topology is defined as follows. For π in $\mathrm{Rep}(G)$,

$\varepsilon > 0$, a compact subset K of G, and positive definite functions $\varphi_1, \dots, \varphi_n$ associated with π, let $W(\varphi_1, \dots, \varphi_n; K; \varepsilon; \pi)$ be the set of all ρ in $\operatorname{Rep}(G)$ such that there exists $\psi_1, \dots, \psi_n$, each of which is a sum of positive definite functions associated with ρ, for which

$$|\varphi_i(x) - \psi_i(x)| < \varepsilon \qquad \forall i = 1, \dots, n \quad \forall x \in K.$$

The subsets $W(\varphi_1, \dots, \varphi_n; K; \varepsilon; \pi)$ form a basis of neighbourhoods of π (see [Fel, Section 2]). This topology may also be described in terms of weak containment. Recall that π is weakly contained in a set $\mathcal{S}$ of representations of G if every positive definite function associated with π is the limit, uniformly on compact subsets of G, of sums of positive definite functions associated with representations from $\mathcal{S}$. It is clear that a net π_n of unitary representations of G converges to π if and only if, for every subnet $\pi_{n'}$ of π_n, π is weakly contained in the set $\{\pi_{n'}\}$. Restricted to $\widehat{G}$, this is just the usual Fell-Jacobson topology on $\widehat{G}$ (see also [Dix, Chap.18]). We are now in position to prove Theorem A.

Proof of Theorem A. — Let π_n be a net of irreducible representations of H converging to 1_H in $\widehat{H}$. Then, by continuity of inducing (see [Fel, Theorem 4.1]),

$$\operatorname{Ind}_H^G \pi_n \to \operatorname{Ind}_H^G 1_H$$

in $\operatorname{Rep}(G)$. Since H has finite covolume, 1_G is contained in $\operatorname{Ind}_H^G 1_H$ and this implies

$$\operatorname{Ind}_H^G \pi_n \to 1_G.$$

As N has Kazhdan's property, we may assume that $\operatorname{Ind}_H^G \pi_n$ has N-invariant vectors for all n. Hence, by Lemma 1, there are (irreducible) representations σ_n of G/N such that $\pi_n = (\sigma_n \circ p)\big|_H$ where $p : G \to G/N$ is the canonical projection. The proof will be finished if we show that

$$\sigma_n \circ p \to 1_G$$

in $\widehat{G}$.

Since H has finite covolume, one has

$$(*)\ \operatorname{Ind}_H^G \pi_n = \operatorname{Ind}_H^G (\sigma_n \circ p)\big|_H = (\sigma_n \circ p) \otimes \rho = (\sigma_n \circ p) \oplus \big((\sigma_n \circ p) \otimes \rho^0\big),$$

where $\rho = \operatorname{Ind}_H^G$ and ρ^0 is the restriction of ρ to the orthogonal of the constants in $\mathrm{L}^2(G/N)$.

Now, the restriction to N of $(\sigma_n \circ p) \otimes \rho^0$ is a multiple of $\rho^0\big|_N$. Since N has property (T), Lemma 2 above implies that $\rho^0\big|_N$ does not weakly contain the trivial representation 1_N. So, $(\sigma_n \circ p) \otimes \rho^0$ cannot converge to 1_G. As

$$\mathrm{Ind}_H^G \pi_n \to 1_G,$$

we conclude from $(*)$ that

$$\sigma_n \circ p \to 1_G.$$

□

3. Proof of Theorem B.

The finite dimensional representation π decomposes as a finite sum

$$\pi = \sum_{i=1}^{n} \pi_i$$

of irreducible subrepresentations π_i. Since, in an obvious way,

$$\mathrm{H}^1(H, \pi) \cong \bigoplus_{i=1}^{n} \mathrm{H}^1(H, \pi_i),$$

we may assume that π is irreducible.

We first deal with the case where π is the trivial representation 1_H. Then, $\mathrm{H}^1(H, \pi)$ is the group of all (continuous) homomorphisms from H to the additive group of the complex numbers $\mathbb{C}$. Let $\overline{[H, H]}$ denote the closure of the commutator subgroup of H. The corollary to Theorem A implies that the dual group of the abelian group $H/\overline{[H, H]}$ is discrete, since the trivial character is isolated. So, by duality theory, $H/\overline{[H, H]}$ is compact. Hence, $\mathrm{H}^1(H, 1_H) = 0$.

Suppose now that $\pi \neq 1_H$ and assume, by contradiction, that

$$\mathrm{H}^1(H, \pi) \neq 0.$$

Then, by Vershik-Karpushev theorem (see [VeK, Theorem 2]) there exists a net π_n in $\widehat{H}$ such that

$$\pi_n \to \pi \qquad \text{and} \qquad \pi_n \to 1_H.$$

By Theorem A, we may assume that $\pi_n = (\sigma_n \circ p)\big|_H$ for irreducible representations σ_n of G/N.

Let $\bar{\pi}$ denote the conjugate representation of π. Then, by continuity of tensoring (see [Fe2, Theorem 1]),

$$\pi_n \otimes \bar{\pi} \longrightarrow \pi \otimes \bar{\pi}.$$

Hence,

$$(\sigma_n \circ p) \otimes \mathrm{Ind}_H^G \bar{\pi} = \mathrm{Ind}_H^G (\pi_n \otimes \bar{\pi}) \longrightarrow \mathrm{Ind}_H^G (\pi \otimes \bar{\pi}).$$

Restricting to N, this implies that $\mathrm{Ind}_H^G \bar{\pi}\big|_N$ weakly contains the representation $\mathrm{Ind}_H^G (\pi \otimes \bar{\pi})\big|_N$.

Since π is finite dimensional, it is well known that $\pi \otimes \bar{\pi}$ has an invariant vector. Therefore, as H has finite covolume, $\mathrm{Ind}_H^G \bar{\pi}\big|_N$ weakly contains 1_N. We conclude that $\mathrm{Ind}_H^G \bar{\pi}$ has N-invariant vectors. But then, Lemma 1 implies that π factorizes to a representation σ of G/N, thus, $\pi = (\sigma \circ p)\big|_H$. As G/N is minimally almost periodic, this forces $\sigma = 1_{G/N}$ and hence $\pi = 1_H$, a contradiction. □

BIBLIOGRAPHY

[Bor] A. BOREL, Some finiteness properties of adèles groups over numbers fields, Publ. Math. IHES, 16 (1963), 1–30.

[BoW] A. BOREL and N. WALLACH, Continuous cohomology, discrete subgroups and representations of reductive groups, Annals of Math. Studies, Princeton University Press, 1980.

[Del] P. DELORME, 1–cohomologie des représentations unitaires des groupes de Lie semi–simples et résolubles Produits tensoriels continus de représentations, Bull. Soc. Math. France, 105 (1977), 281–336

[Dix] J. DIXMIER, Les C^*-algèbres et leurs représentations, Gauthier-Villars, 1969.

[Fe1] J.M.G. FELL, Weak containment and induced representations of groups, Canad. J. Math., 14 (1962), 237–268

[Fe2] J.M.G. FELL, Weak containment and Kronecker products of group representations, Pac. J. Math., 13 (1963), 503–510.

[Gu1] A. GUICHARDET, Symmetric Hilbert spaces and related topics, Lecture Notes in Math., 261, Springer, 1972.

[Gu2] A. GUICHARDET, Cohomologie des groupes localement compacts et produits tensoriels continus de représentations, J. Multivariate Anal., 6 (1976), 138–158.

[Gu3] A. GUICHARDET, Tensor products of C^*–algebras, Part II, Lecture Notes Series, 13, Aarhus Universitet, 1969.

[Hum] J.E. HUMPHREYS, Arithmetic groups, Lecture Notes in Math., 789, Springer, 1980.

[HaV] P. de la HARPE et A. VALETTE, La propriété (T) de Kazhdan pour les groupes localement compacts, Astérisque 75, Soc. Math. de France, 1989.

[Lub] A. LUBOTZKY, Discrete groups, expanding graphs and invariant measures, Progress in Math. 125, Birkhäuser, 1994.

[LuZ] A. LUBOTZKY and R.J. ZIMMER, Variants of Kazhdan's property for subgroup of semisimple groups, Israel J. of Math., 66 (1989), 289–298.

[Mar] G.A. MARGULIS, Discrete subgroups of semisimple Lie groups, Springer, 1991.

[VeK] A.M. VERSHIK and S.I. KARPUSHEV, Cohomology of groups in unitary representations, the neighbourhood of the identity and conditionally positive definite functions, Math. USSR Sbornik, 47 (1984), 513–526.

[Zim] R.J. ZIMMER, Ergodic theory and semisimple groups, Birkhäuser, Boston, 1984.

Manuscrit reçu le 21 novembre 1996,
accepté le 25 avril 1997.

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