SOPHIE CHEMLA

Extremal projectors in the semi-classical case


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EXTREMAL PROJECTORS
IN THE SEMI-CLASSICAL CASE

by Sophie CHEMLA

1. Introduction.

Let \( \mathfrak{g} \) be a complex semi-simple finite dimensional Lie algebra, \( \mathfrak{h} \) a Cartan subalgebra of \( \mathfrak{g} \) and \( \Delta \) the root system associated to \( \mathfrak{h} \). We will write \( \Delta^+ \) (respectively \( \Delta^- \)) for the set of positive (respectively negative) roots of \( \Delta \) and put \( \rho = \frac{1}{2} \sum \gamma \). We will denote by \( B = (\alpha_1, \ldots, \alpha_l) \) the set of simple roots. Let \( \mathfrak{g}_\gamma \) be the root space associated to the root \( \gamma \). We put

\[
\mathfrak{n} = \bigoplus_{\gamma \in \Delta^+} \mathfrak{g}_\gamma, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{n}_- = \bigoplus_{\gamma \in \Delta^+} \mathfrak{g}_{-\gamma}.
\]

Let \( R(\mathfrak{h}) \) be the field of rational functions on \( \mathfrak{h}^* \). One introduces the algebra \( U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes R(\mathfrak{h}) \). Let us consider the generic Verma module \( V = U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n} \). Zhelobenko ([Z1]) showed that \( V^\mathfrak{n} = R(\mathfrak{h})1_+ \) (where \( 1_+ = 1 + U'(\mathfrak{g})\mathfrak{n} \)). The decomposition \( V = \mathfrak{n}^- V \oplus R(\mathfrak{h})1_+ \) defines a projector \( p \) onto \( R(\mathfrak{h})1_+ \) called the extremal projector. Inspired by a work of Asherova, Smirnov and Tolstoy ([AST]), Zhelobenko ([Z1]) showed that \( p \) factorizes into elementary projectors. Let \( (\gamma_1, \ldots, \gamma_m) \) be a normal ordering on the positive roots. Introduce the following notations:

\[
p_\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! f_{\alpha,k}} e_{-\alpha}^k e_\alpha^k
\]
\[
f_{\alpha,0} = 1,
\]
\[
\text{if } k > 0, \quad f_{\alpha,k} = (h_\alpha + \rho(h_\alpha) + 1) \ldots (h_\alpha + \rho(h_\alpha) + k)
\]

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(e_\delta being the root vector associated to the root \delta and h_\delta the coroot). We have \( p = p_{\gamma_1} \ldots p_{\gamma_m} \) ([Z1]). Let \( w = s_1 \ldots s_j \) be a reduced decomposition of \( w \in W \) (with \( s_k = s_{\beta_k}, \ \beta_k \) a simple root). Put \( w_i = s_1 \ldots s_i \). The roots \( \gamma_i = w_i^{-1}(\beta_i) \) (\( w_0 = 1 \)) are pairwise distinct and

\[ \Delta_w = \{ \alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0 \} = \{ \gamma_1, \ldots, \gamma_j \}. \]

Put \( n_w = \bigoplus_{\alpha \in \Delta_w} g_\alpha \). In [Z2], Zhelobenko gives an explicit description of \( V^{n_w} \).

We will establish similar results for the symmetric algebra (the so-called semi-classical case).

Let us consider the analytic manifold \((g/n)^*\). We will endow it with the following coordinate system \(((e_\alpha)_{\alpha \in \Delta_+}, (e^\lambda)_{\lambda \in \mathbb{I}, z})\). We will call \( U_\delta \) the open subset of \((g/n)^*\) defined by the equation \( h_\delta \neq 0 \). We define \( \Phi_\delta \) to be the following rational map of \( U_\delta \):

\[ \forall \lambda \in U_\delta, \ \Phi_\delta(\lambda) = \exp \left( \frac{e_{-\delta}(\lambda)}{h_\delta(\lambda)} e_\delta \right) \cdot \lambda \]

where the dot denotes natural action of \( n \) on \((g/n)^*\). By composition, \( \Phi_\delta \) defines an algebra morphism of \( A(U_\delta) \) which we call \( \pi_\delta \). We put

\[ U_w = U_{\gamma_1} \cap \ldots \cap U_{\gamma_j}. \]

We will denote by \( \mathcal{P}(U_w) \) (respectively \( A(U_w) \)) the set of regular functions (respectively analytic functions) on \( U_w \) and we will write \( \mathcal{P}(U_w)^{n_w} \) (respectively \( A(U_w)^{n_w} \)) the set of invariant functions of \( \mathcal{P}(U_w) \) (respectively \( A(U_w) \)) under the action of \( n_w \). We prove the following result:

**Theorem.** — The algebra morphism \( \pi_w = \pi_{\gamma_1} \circ \ldots \circ \pi_{\gamma_j} \) does not depend on the reduced expression of \( w \). It establishes an isomorphism between

\[ C_w = \left\{ f \in A(U_w) \mid \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots = \frac{\partial f}{\partial e_{-\gamma_j}} = 0 \right\} \]

and \( A(U_w)^{n_w} \). Moreover \( \pi_w \) sends \( C_w \cap \mathcal{P}(U_w) \) onto \( \mathcal{P}(U_w)^{n_w} \).

Let \( N_w \) be the connected simply connected group whose Lie algebra is \( n_w \). The main ingredient of the proof will be the choice of a point in each \( N_w \)-orbit lying in \( U_w \) in accordance with the following proposition:

**Proposition.** — Let \( \lambda \) be in \( U_w \). The point \( \Phi_{\gamma_j} \Phi_{\gamma_j-1} \ldots \Phi_{\gamma_1}(\lambda) \) is the unique point of the orbit \( N_w \cdot \lambda \) whose coordinates \( e_{-\gamma_1}, \ldots, e_{-\gamma_j} \) vanish.
In the appendix, we shall give a factorization for the extremal projector of the Virasoro algebra in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering.

Notations. — Along all this article \( g \) will denote a complex semi-simple finite dimensional Lie algebra and \( h, \Delta, \Delta_+, \Delta_-, n, n_- \), \( B = (\alpha_1, \ldots, \alpha_i) \) will be as above. Denote by \( W \) the Weyl group associated to these choices and \( \bar{w} \) its longest element. Let \( \gamma \) be an element of \( \Delta^+ \) and let \( h_\gamma \) be the unique element of \( \mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma} \) such that \( \gamma(h_\gamma) = 2 \). If \( e_\gamma \) is in \( \mathfrak{g}_\gamma \), then there exists a unique \( e_{-\gamma} \) such that \( (h_\gamma, e_\gamma, e_{-\gamma}) \) is a \( \text{sl}(2) \)-triple. If \( \alpha \) and \( \beta \) are two roots, we set \( [\alpha, \beta] = C_{\alpha, \beta} e_{\alpha+\beta} \) with the convention that \( C_{\alpha, \beta} \) is zero if \( \alpha + \beta \) is not a root.

The ordering \( (\gamma_1, \ldots, \gamma_m) \) on the positive roots is normal if any composite root is located between its components. Thus for all positive roots \( \gamma_i, \gamma_j, \gamma_k \), the equality \( \gamma_k = \gamma_i + \gamma_j \) implies \( i \leq k \leq j \) or \( j \leq k \leq i \). There is a one to one correspondence between normal orderings and reduced expression of \( \bar{w} ([\mathbb{Z}_2]) \). Let us recall it. Denote by \( S_i \) the reflexion with respect to a simple root \( \beta_i \). If \( \bar{w} = S_1 \cdots S_m \), then \( (\beta_1, S_1(\beta_2), \ldots, S_1 \cdots S_{i-1}(\beta_i), \ldots, S_1 \cdots S_{m-1}(\beta_m)) \) are in normal ordering.

If \( V \) is a vector space, \( S(V) \) will be the symmetric algebra of \( V \). Lastly, if \( P \) is in \( S(V) \), \( S(V)_P \) will be the localization of \( S(V) \) with respect to \( \{P^n \mid n \in \mathbb{N}\} \).

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2. Extremal equations in \((g/n)^*\).

We consider \((g/n)^*\) as an analytic manifold. We endow it with the following coordinate system \( (e_{-\alpha})_{\alpha \in \Delta^+}, (h_{\alpha_i})_{i \in [1, d]} \). If \( \delta \) is a positive root, we will denote by \( U_\delta \) the open subset of \((g/n)^*\) defined by the equation \( h_\delta \neq 0 \). If \( U \) is an open subset for the Zariski topology, we will write \( \mathcal{A}(U) \) for the algebra of analytic functions on \( U \) and \( \mathcal{P}(U) \) for the algebra of regular functions on \( U \). We will define \( \Phi_\delta \) to be the following rational map
of $U_\delta$
\[ \forall \lambda \in U_\delta, \quad \Phi_\delta(\lambda) = \exp \left( \frac{e_\delta(\lambda)}{h_\delta(\lambda)} e_\delta \right) \cdot \lambda. \]

By composition, $\Phi_\delta$ defines an algebra morphism of $\mathcal{A}(U_\delta)$ which we call $\pi_\delta$. We will denote by $X_\delta$ the natural action of $e_\delta$ on $\mathcal{A}(U_\delta)$. Remark that $X_\delta$ is a derivation. If $f$ is in $\mathcal{P}(U_\delta)$, we have
\[ (\pi_\delta(f)) = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-\delta}}{k! h_\delta^k} X^k \cdot f \]
where $e^{-\delta}$ denotes the multiplication by $e^{-\delta}$. The operator $\pi_\delta$ is the commutative analog of the Zhelobenko's elementary projector.

Let $w = s_1 \ldots s_j$ be a reduced decomposition of $w \in W$ (with $s_k = s_{\beta_k}, \beta_k \in B$). Put $w_i = s_1 \ldots s_i$. The roots $\gamma_i = w_{i-1}(\beta_i) (w_0 = 1)$ are pairwise distinct and 
\[ \Delta_w = \{ \alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0 \} = \{ \gamma_1, \ldots, \gamma_j \}. \]

An ordering in $\Delta_w$ is called normal if it coincides with the initial segment of some normal ordering in $\Delta^+$ (that is compatible with one of the reduced expression of $\overline{w}$). Note that $(\gamma_1, \ldots, \gamma_j)$ is a normal ordering of $\Delta_w$. Put
\[ U_w = \bigcap_{\delta \in \Delta_w} U_\delta. \]
We have
\[ \mathcal{P}(U_w) = \left( \frac{S(\mathfrak{g})}{S(\mathfrak{g}) \mathfrak{n}} \right)_{h_{\gamma_1} \ldots h_{\gamma_j}} = S \left( \frac{\mathfrak{g}}{\mathfrak{n}} \right)_{h_{\gamma_1} \ldots h_{\gamma_j}}. \]
We will denote by $\mathcal{N}_w$ the connected and simply connected group whose Lie algebra is $\mathfrak{n}_w = \bigoplus_{\alpha \in \Delta_w} \mathfrak{g}_\alpha$. We will start by proving the following proposition.

**Proposition 2.1.** — Let $\lambda$ be in $U_w$. The point $\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}(\lambda)$ is the unique point of the orbit $\mathcal{N}_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish. In particular $\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}$ does not depend on the normal ordering on $\Delta_w$.

**Proof of Proposition 2.1.** — Complete $(\gamma_1, \ldots, \gamma_j)$ into a normal ordering on the positive roots $(\gamma_1, \ldots, \gamma_m)$. $\mathfrak{g}/\mathfrak{n}$ is endowed with the basis $(e_{-\gamma_1}, \ldots, e_{-\gamma_m}, h_{\alpha_1}, \ldots, h_{\alpha_l})$. Let $(e^*_{-\gamma_1}, \ldots, e^*_{-\gamma_m}, h^*_{\alpha_1}, \ldots, h^*_{\alpha_l})$ be the dual basis. We will often identify the point $a_{\gamma_1} e^*_{-\gamma_1} + \ldots + a_{\gamma_m} e^*_{-\gamma_m} + b_1 h^*_{\alpha_1} +$
... + bih^ with its coordinates (a_\gamma_1, ..., a_\gamma_m, b_1, ..., b_l). Let us see that there is a unique point in N_w \cdot \lambda whose coordinates e_{-\gamma_1}, ..., e_{-\gamma_j} vanish. Assume that there are two such points f = (0, ..., 0, a_{\gamma_{j+1}}, ..., a_{\gamma_m}, b_1, ..., b_l) and f' = (0, ..., 0, a'_{\gamma_{j+1}}, ..., a'_{\gamma_m}, b'_1, ..., b'_l). Then there exist complex numbers (t_1, ..., t_j) such that exp(t_1 e_{\gamma_1} + ... + t_j e_{\gamma_j}) \cdot f = f'. One can show easily the following equalities:

\begin{align*}
  e_{\gamma_1} \cdot e_{-\gamma_k} &= -C_{\gamma_1, -\gamma_k} e_{-\gamma_1 - \gamma_k} \\
  e_{\gamma_1} \cdot h^*_{-\alpha_\gamma} &= -h^*_{-\alpha_\gamma} (h_{\gamma}) e_{-\gamma_1}.
\end{align*}

From these equalities, one deduces easily that the term in e_{-\gamma_1} of exp(t_1 e_{\gamma_1} + ... + t_j e_{\gamma_j}) \cdot (0, ..., 0, a_{\gamma_{j+1}}, ..., a_{\gamma_m}, b_1, ..., b_l) = -t_1 f(h_{\gamma_1}). As f is in U_w, we get t_1 = 0. We reproduce the same reasoning to show that t_2, t_3, ..., t_j are zero. So that we have proved that the two points f and f' coincide. It is not difficult to deduce from the normal ordering property that \Phi_{\gamma_1} sends the point (x_{\gamma_1}, ..., x_{\gamma_m}, y_1, ..., y_l) to a point (x'_{\gamma_1}, ..., x'_{\gamma_{j-1}}, 0, x'_{\gamma_{j+1}}, ..., x'_{\gamma_m}, y_1, ..., y_l) and that it sends the point (0, ..., 0, x_{\gamma_1}, ..., x_{\gamma_m}, y_1, ..., y_l) to a point (0, ..., 0, x'_{\gamma_{j+1}}, ..., x'_{\gamma_m}, y_1, ..., y_l). So that \Phi_{\gamma_1} \Phi_{\gamma_{j-1}} ... \Phi_{\gamma_1} (\lambda) is the unique point of N_w \cdot \lambda whose coordinates e_{-\gamma_1}, ..., e_{-\gamma_j} vanish. This finishes the proof of Proposition 2.1.

As a consequence of the previous proposition, we may write \Phi_w for the operator \Phi_{\gamma_1} \Phi_{\gamma_{j-1}} ... \Phi_{\gamma_1}. The algebra homomorphism defined by \Phi_w on A(U_w) will be denoted by \pi_w. Using Proposition 2.1, we will give a geometric proof of the following result.

**Theorem 2.2.** — 1) If \bar{n}_w denotes the linear hull of (e_{-\alpha})_{\alpha \in \Delta_w}, one has Ker\pi_w = \bar{n}_w A(U_w).

2) The operator \pi_w is the projector onto A(U_w)^{n_w} with kernel \bar{n}_w A(U_w) and its restriction to \mathcal{P}(U_w) is the projector onto \mathcal{P}(U_w)^{n_w} with kernel \bar{n}_w \mathcal{P}(U_w).

3) The operator \pi_w establishes an isomorphism \Pi_w between

\[ C_w = \left\{ f \in A(U_w) \mid \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots = \frac{\partial f}{\partial e_{-\gamma_j}} = 0 \right\} \]

and A(U_w)^{n_w}. Moreover \Pi_w sends C_w \cap \mathcal{P}(U_w) onto \mathcal{P}(U_w)^{n_w}. If f is in A(U_w)^{n_w}, \Pi_w^{-1}(f) is the restriction of f to the subvariety of equations e_{-\gamma_1} = \ldots = e_{-\gamma_j} = 0.
Proof of Theorem 2.2. — From the previous proposition, the inclusion \( \bar{\pi}_w \mathcal{A}(U_w) \subset \ker \pi_w \) is clear. Moreover, a standard reasoning shows that

\[
\mathcal{A}(U_w) = \mathcal{C}_w \oplus \bar{\pi}_w \mathcal{A}(U_w).
\]

Then one sees easily that \( \ker \pi_w \cap \mathcal{C}_w = \{0\} \). So that we have \( \bar{\pi}_w \mathcal{A}(U_w) = \ker \pi_w \).

Let us now show that \( \Im \pi_w = \mathcal{A}(U_w)^{n_w} \) and that \( \pi_w \) is a projector. Let \( \alpha \) be in \( \Delta_w \). For any \( f \) in \( \mathcal{A}(U_w) \) and any \( \lambda \) in \( U_w \), we have

\[
(X_\alpha \circ \pi_w)(f)(\lambda) = \frac{d}{dt} f(\Phi_{\gamma_1} \cdots \Phi_{\gamma_1} \exp(-te_\alpha)\lambda)|_{t=0}.
\]

But for any \( t \), \( \Phi_{\gamma_j} \cdots \Phi_{\gamma_1} \exp(-te_\alpha)\lambda \) is the unique point of \( N_w \cdot \lambda \) whose coordinates \( e_{-\gamma_1}, \ldots, e_{-\gamma_j} \) vanish. So that \( X_\alpha \circ \pi_w = 0 \). We have thus proved the inclusion \( \Im \pi_w \subset \mathcal{A}(U_w)^{n_w} \). Now it is clear that \( \pi_w \) is a projector: check that \( \pi_w \circ \pi_w = \pi_w \) on coordinates using the formula (*)

The reverse inclusion \( \mathcal{A}(U_w)^{n_w} \subset \Im \pi_w \) will be a consequence of the following lemma.

**Lemma 2.3.** — Let \( k \) be in \( [1, j] \) and let \( f \) be in \( \mathcal{A}(U_w) \). If \( X_{\gamma_k} f = 0 \), then \( \pi_{\gamma_k} f = f \).

**Proof of Lemma 2.3.** — We first remark that \( \pi_{\gamma_k}(e_{-\gamma_1}), \ldots, \pi_{\gamma_k}(e_{-\gamma_{k-1}}), e_{-\gamma_k}, \pi_{\gamma_k}(e_{-\gamma_{k+1}}), \ldots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l} \) is a coordinate system in \( U_w \). Indeed, one may see by induction that for any \( i \leq k - 1 \) (respectively \( i \geq k + 1 \) ), \( e_{-\gamma_i} \) may be expressed as a regular function of \( \pi_{\gamma_k}(e_{-\gamma_i}), \ldots, \pi_{\gamma_k}(e_{-\gamma_i}), h_{\alpha_1}, \ldots, h_{\alpha_l} \) (respectively \( \pi_{\gamma_k}(e_{-\gamma_i}), \ldots, \pi_{\gamma_k}(e_{-\gamma_i}), h_{\alpha_1}, \ldots, h_{\alpha_l} \)). We put \( (\epsilon_1, \ldots, \epsilon_{m+1}) = (\pi_{\gamma_k}(e_{-\gamma_i}), \ldots, \pi_{\gamma_k}(e_{-\gamma_i}), e_{-\gamma_k}, \pi_{\gamma_k}(e_{-\gamma_{k+1}}), \ldots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l}). \)

In these coordinates, we have \( X_{\gamma_k} = h_{\gamma_k} \frac{\partial}{\partial \epsilon_k} \). So that if \( X_{\gamma_k} f = 0 \), then \( f \) does not depend on \( \epsilon_k \) and it becomes clear that there exists \( g \) such that \( f = \pi_{\gamma_k} g \). As \( \pi_{\gamma_k} \) is a projector, we have \( \pi_{\gamma_k} f = \pi_{\gamma_k} \pi_{\gamma_k} g = \pi_{\gamma_k} g \), which finishes the proof of the lemma.

It is clear from the proof that \( \pi_w \) sends \( \mathcal{C}_w \cap \mathcal{P}(U_w) \) onto \( \mathcal{P}(U_w)^{n_w} \).

In particular \( \pi_{\bar{w}}|\mathcal{P}(U_{\bar{w}}) \) is the projector onto \( S(\mathfrak{h})_{\gamma_1 \cdots \gamma_m} \) with kernel \( n_- \mathcal{P}(U_{\bar{w}}) \). By analogy to Asherova, Tolstoy, Smirnov and Zhelobenko's work, we will call it the extremal projector.

Proposition 2.1 gives a geometric interpretation of the projector \( \pi_w \).
3. Appendix: Extremal projector
for the Virasoro algebra in the semi-classical case.

In this section, we shall give a factorization of the Virasoro algebra extremal projector in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering. Recall that the Virasoro algebra Vir is the infinite dimensional Lie algebra generated by \{e_i \mid i \in \mathbb{Z}\} \cup \{c\} with commutation rules

\[[e_i, e_j] = (j - i) e_{i+j} + \frac{(j^3 - j)}{12} \delta_{i+j,0} c, \quad [e_i, c] = 0.\]

Vir admits the following triangular decomposition:

\[\text{Vir} = \text{Vir}_+ \oplus \text{Vir}_0 \oplus \text{Vir}_-.\]

where

\[\text{Vir}_+ = \bigoplus_{i \geq 1} c e_i, \quad \text{Vir}_0 = c e_0 \oplus c, \quad \text{Vir}_- = \bigoplus_{i \leq -1} c e_i.\]

We will also use the notation

\[\text{Vir}_{r,+} = \bigoplus_{i \geq r} c e_i \quad \text{and} \quad \text{Vir}_{r,-} = \bigoplus_{i \leq -r} c e_i.\]

Vir_{r,+} and Vir_{r,-} are Lie subalgebras of Vir.

Let \(R(\text{Vir}_0)\) be the field of fractions of \(S(\text{Vir}_0)\). We introduce the algebra

\[S'(\text{Vir}) = S(\text{Vir}) \otimes_{S(\text{Vir}_0)} R(\text{Vir}_0) = S'\left(\frac{\text{Vir}}{\text{Vir}_-}\right).\]

There is a natural action of \(\text{Vir}_-\) on \(S'\left(\frac{\text{Vir}}{\text{Vir}_-}\right)\). Through this action, for any negative \(i\), \(e_i\) defines a derivation \(X_i\) of \(S'\left(\frac{\text{Vir}}{\text{Vir}_-}\right)\). Set

\[T_r = \left(\frac{S'(\text{Vir})}{S'(\text{Vir})\text{Vir}_-}\right)^{\text{Vir}_{r,-}}.\]

The result and the proof of the following lemma is left to the reader.

**Lemma 3.1.**

\[T_r = \bigoplus_{k_1, \ldots, k_r \in \mathbb{N}} R(\text{Vir}_0) e_1^{k_1} \ldots e_r^{k_r-1}.\]
As a consequence of Lemma 3.1, we have the following decomposition:
\[ S' (\text{Vir}/\text{Vir}_-) = T_r \oplus \text{Vir}_{r,+} S' (\text{Vir}/\text{Vir}_-) . \]

The proof of the next lemma is an easy computation.

**Lemma 3.2.** — For any \( i > 1 \), the operator

\[
\pi_i = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \left( 2ic_0 + \frac{(i^3 - i)c}{12} \right)^k} e_i^k X_{-i}^k
\]

is an algebra morphism and satisfies the relations

\[ X_{-i} \circ \pi_i = 0 \text{ and } \pi_i \circ e_i = 0 \]

(where \( e_i \) denotes multiplication by \( e_i \)).

It is not hard to see that the operator \( \Pi_r = \prod_{i=r}^{\infty} \pi_i \) is well defined. Actually \( \prod_{i=r}^{\infty} \pi_i(e_1 \ldots e_k) = \prod_{r \leq i \leq k} \pi_i(e_1 \ldots e_k) = 0 \) (by Lemma 3.2).

**Theorem 3.3.** — The operator \( \Pi_r \) satisfies the relations

\[ \forall i \geq r, X_{-i} \circ \Pi_k = 0, \Pi_k \circ e_i = 0. \]

It is the projector onto \( T_r \) with kernel \( \text{Vir}_{r,+} S' (\text{Vir}/\text{Vir}_-) \).

In particular, \( \Pi_1 \) is the extremal projector.

**Proof of Theorem 3.3.** — The relations of the theorem are easy to check and they prove that \( \Pi_r \) is a projector. To prove that the kernel of \( \Pi_r \) is \( \text{Vir}_{r,+} \), we proceed as in the semi-simple case. The inclusion \( \text{Im} \Pi_r \subset T_r \) is a consequence of the theorem. To prove the reverse inclusion, remark that if \( x \) is in \( T_r \), then \( \Pi_r x = x \), so that \( x \) is in \( \text{Im} \Pi_r \).
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Sophie CHEMLA,
Institut de Mathématiques
Université Paris VI
Case 82
4, place Jussieu
75252 Paris Cedex 05 (France).
schemla@math.jussieu.fr