The Bochner-Hartogs dichotomy for weakly 1-complete Kähler manifolds

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THE BOCHNER-HARTOGS DICHOTOMY FOR
WEAKLY 1-COMPLETE KÄHLER MANIFOLDS

by T. NAPIER (*) & M. RAMACHANDRAN (**) 

Introduction.

A complex manifold $M$ for which $H^1_c(M, \mathcal{O}) = 0$ is said to have the \textit{Bochner-Hartogs property} (see Hartogs [H], Bochner [B], and Harvey and Lawson [HL]). Equivalently, for every $C^\infty$ compactly supported form $\alpha$ of type $(0, 1)$ with $\bar{\partial}\alpha = 0$ on $M$, there is a $C^\infty$ compactly supported function $\beta$ on $M$ such that $\bar{\partial}\beta = \alpha$. Andreotti and Vesentini [AV] proved that a strongly $(n-1)$-complete complex manifold of dimension $n > 1$ has the Bochner-Hartogs property, and Grauert and Riemenschneider [GR], that a strongly hyper-$(n-1)$-convex Kähler manifold of dimension $n > 1$ has the Bochner-Hartogs property (see Section 1). In [R], the second author proved that if the universal covering $M$ of a compact Kähler manifold (or a Galois covering $M$ with infinite covering group of more than quadratic growth) admits a nonconstant holomorphic function, then $M$ satisfies the following dichotomy:

\textit{(BHD) Either $M$ has the Bochner-Hartogs property or there exists a proper holomorphic mapping of $M$ onto a Riemann surface.}

A complex manifold which admits a continuous plurisubharmonic exhaustion function is said to be \textit{weakly 1-complete}. The main result of this paper (Theorem 2.5) is the following:

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THEOREM. — If $(M,g)$ is a connected noncompact weakly 1-complete Kähler manifold which has exactly one end, then $M$ satisfies (BHD).

If a complex manifold $M$ has the Bochner-Hartogs property, then every holomorphic function $f$ on a neighborhood of infinity with no relatively compact connected components extends to a holomorphic function $f_0$ on $M$ as follows. Cutting off away from infinity, one obtains a $C^\infty$ function $\lambda$ on $M$. For $\alpha = \bar{\partial}\lambda$, there is a function $\beta$ as above and the function $f_0 = \lambda - \beta$ is the desired extension. In particular, a Riemann surface cannot have the Bochner-Hartogs property and a complex manifold cannot satisfy both of the conditions in (BHD). Moreover, a manifold with the Bochner-Hartogs property has only one end because, on a manifold with more than one end, there exists a function which is locally constant, but not constant, near infinity. A related result due to Arapura, Bressler, and the second author [ABR] is that the universal covering of a compact Kähler manifold has at most one end. In fact, as shown in [NR], a complete noncompact connected Kähler manifold $M$ which satisfies $H^1(M,\mathbb{R}) = 0$ and which has bounded geometry or is weakly 1-complete has exactly one end. It was also proved in [NR] that a complete Kähler manifold with at least three ends which has bounded geometry or is weakly 1-complete admits a proper holomorphic mapping onto a Riemann surface.

Facts concerning strictly $q$-plurisubharmonic functions and Green’s functions are collected in Section 1. Section 2 contains the proof of the theorem. The main step is to prove Proposition 2.3 that (BHD) holds for a weakly 1-complete Kähler manifold on which there exists, outside a compact subset, a pair of pluriharmonic functions with linearly independent differentials (this may be thought of as a generalization of a theorem of Ohsawa [O1]). One may assume that $M$ is complete and admits a positive Green’s function since one can exhaust $M$ by domains with these properties. Hence, if $[\alpha] \in H^1_c(M,\mathcal{O})$ is a nonzero element, then one can form the $L^2$ harmonic projection $\gamma$ and a function $\beta$ such that $\gamma = \alpha - \bar{\partial}\beta$. In particular, $\beta$ is pluriharmonic outside a compact set $K$. One then forms a pluriharmonic function on a covering space, pushes down to obtain a second pluriharmonic function on $M \setminus K$, and applies Proposition 2.3.

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1. Preliminaries on $q$-plurisubharmonic functions.

Most of the facts discussed in this section are known, so the proofs are only sketched. Throughout this section, $(M, g)$ denotes a Kähler manifold of dimension $n$ and $q$ denotes a positive integer.

Let $\varphi$ be a real-valued continuous function on $M$. We will say that $\varphi$ is strictly $q$-plurisubharmonic if $\varphi$ is an element of the class $\Psi(q)$ defined by Wu [W]. We will call $\varphi$ $q$-plurisubharmonic if the function $\varphi + \psi$ is strictly $q$-plurisubharmonic for every continuous strictly $q$-plurisubharmonic function $\psi$ on $M$.

Remarks.

1. If $\varphi$ of class $C^2$, then $\varphi$ is $q$-plurisubharmonic (strictly $q$-plurisubharmonic) if and only if, for each point $x_0 \in M$, the trace of the restriction of the Levi form

$$L(\varphi) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

of $\varphi$ to any complex subspace of $T_{x_0}^{1,0}M$ of dimension $q$ is nonnegative (respectively, positive).

2. If $\varphi$ is $q$-plurisubharmonic (strictly $q$-plurisubharmonic), then $\varphi$ is $(q+1)$-plurisubharmonic (respectively, strictly $(q+1)$-plurisubharmonic).

3. A real-valued function of class $C^2$ on a complex manifold is said to be strictly $q$-convex if its Levi form has at most $q-1$ nonpositive eigenvalues at each point. A function $\psi$ on a complex space $X$ is said to be strictly $q$-convex if, for each point $x \in X$, there is a proper embedding of a neighborhood $U$ of $x$ into an open subset $V$ of some complex Euclidean space and an extension of $\psi|_U$ to a strictly $q$-convex function on $V$. It follows that if $\varphi$ is of class $C^2$ and strictly $q$-plurisubharmonic on $M$, then $\varphi$ is strictly $q$-convex on $M$ and on any analytic subset of $M$.

4. The set of smooth elements of $\Psi(q)$ is dense in the following sense:

**Proposition 1.1** (Wu [W], Proposition 1). — If $\varphi$ is a continuous strictly $q$-plurisubharmonic function on a Kähler manifold $M$ and $\alpha$ is a positive continuous function on $M$, then there exists a $C^\infty$ strictly $q$-plurisubharmonic function $\psi$ such that $|\varphi - \psi| < \alpha$ on $M$. 

In particular, it follows that the restriction of a continuous $q$-plurisubharmonic (strictly $q$-plurisubharmonic) function to a complex submanifold of dimension $q$ is subharmonic (respectively, strictly subharmonic).

5. The Kähler manifold $(M, g)$ is said to be hyper-$q$-complete if $M$ admits a $C^\infty$ strictly $q$-plurisubharmonic exhaustion function. If there exists a $C^\infty$ $q$-plurisubharmonic exhaustion function which is strictly $q$-plurisubharmonic on the complement of some compact subset of $M$, then $(M, g)$ is said to be strongly hyper-$q$-convex.

6. Standard arguments show that if $\varphi$ and $\varphi'$ are continuous $q$-plurisubharmonic functions on $M$, then $\varphi + \varphi'$, $\text{max}(\varphi, \varphi')$, and the composition $\chi(\varphi)$ of any nondecreasing convex function $\chi$ with $\varphi$, are all $q$-plurisubharmonic.

7. Hunt and Murray [HM] and Kalka [K] studied functions which satisfy a condition which they called $g$-plurisubharmonicity but which is weaker than the above notion.

The following result is contained implicitly in the work of Greene and Wu [GW], Ohsawa [O2], and Demailly [D2]:

**Theorem 1.2** (Demailly, Greene-Wu, Ohsawa). — *Let $X$ be an analytic subset of dimension $m \leq q$ in the Kähler manifold $M$ and let $Y$ be the union of the singular set $X_{\text{sing}}$ with all irreducible components of $X$ which are noncompact or which have dimension strictly less than $q$. Then there exist neighborhoods $V$ of $X$ and $W$ of $Y$ in $M$ and a $C^\infty$ strictly $(q+1)$-plurisubharmonic function $\varphi$ on $V$ such that $\varphi|_X$ exhausts $X$, $W \subset V$, and $\varphi|_W$ is strictly $q$-plurisubharmonic.*

The proof is an easy modification of Demailly's [D2] proof of the analogous result for strictly $q$-convex functions, but we include a sketch here for completeness. Similarly, as in [D2] and in the work of Colțoiu [C], a hyper-$q$-complete submanifold admits a hyper-$q$-complete neighborhood, but we won't use this fact and the proof will not be sketched.

By a theorem of Richberg [Ri], a $C^\infty$ strictly plurisubharmonic function on an analytic subset of a complex space extends to a $C^\infty$ strictly plurisubharmonic function on a neighborhood. Demailly proved a version of this theorem for strictly $q$-convex functions in which the function is approximated by a strictly $q$-convex function on a neighborhood (see also [P]). A natural modification of Richberg's proof shows that if a
function on an analytic subset of a Kähler manifold admits local $C^\infty$ strictly $q$-plurisubharmonic extensions, then it admits a $C^\infty$ strictly $q$-plurisubharmonic extension to a neighborhood. We will only need this fact for submanifolds and the proof is simple in this case.

**PROPOSITION 1.3** (Richberg). — If $\varphi$ is a $C^\infty$ strictly $q$-plurisubharmonic function on a complex submanifold $N$ of $M$ (relative to the Kähler metric $g|_N$), then there exists a $C^\infty$ strictly $q$-plurisubharmonic function $\psi$ on a neighborhood of $N$ in $M$ such that $\psi|_N = \varphi$.

**Sketch of the proof.** — Let $m = \dim N$ and suppose $(U, (z_1, \ldots, z_n))$ is a holomorphic coordinate neighborhood in which $N$ is the zero set of $w = (z_1, \ldots, z_{n-m})$. If $\varphi'$ is a function on a relatively compact polydisk $D$ in $U$ obtained by composing $\varphi$ with the associated projection mapping and $C > 0$ is sufficiently large, then the function $\varphi' + C|w|^2$ is strictly $q$-plurisubharmonic on a neighborhood of $N \cap D$. By using a partition of unity, one may patch these local extensions to obtain the desired function $\psi$. Q. E. D.

The main point of Demailly's [D2] proof of Ohsawa's theorem [O2] is essentially the following version of the theorem of Greene and Wu [GW] on the existence of subharmonic exhaustion functions (see [D2], Proof of Theorem 2):

**PROPOSITION 1.4** (Demailly, Greene-Wu). — Suppose $X$ is a complex space with no compact irreducible components, $Y$ is an analytic subset which contains the singular set $X_{\text{sing}}$, $U$ is a neighborhood of $Y$ in $X$, and $h$ is a Hermitian metric on $X \setminus Y$. Then there exists a $C^\infty$ nonnegative function $\varphi$ on $X$ such that

(i) $\varphi$ is positive and exhaustive on $X \setminus U$,

(ii) $\varphi$ vanishes on a neighborhood of $Y$, and

(iii) $\varphi$ is subharmonic (with respect to $h$) on $X \setminus Y$ and strictly subharmonic on the subset \( \{ x \in X \mid \varphi(x) > 0 \} \) of $X \setminus Y$.

**Proof of Theorem 1.2 ([D2]).** — The statement of the theorem makes sense and is trivial for $q = 0$. We proceed by induction on $q$. Assume that $q > 0$ and that the theorem holds for nonnegative integers less than $q$. Let $A$ be the union of $X_{\text{sing}}$ and all of the irreducible components of $X$ of dimension less than $q$, let $B$ be the union of all of the noncompact irreducible components of $X$ of dimension $q$, and let $C$ be the closure
of $X \setminus (A \cup B)$.

Applying the induction hypothesis to $A$ and cutting off away from $A$, we obtain a $C^\infty$ nonnegative function $\alpha$ on $M$ and a neighborhood $U$ of $A$ in $M$ such that $\alpha$ is strictly $q$-plurisubharmonic on $U$ and exhaustive on $\bar{U}$.

Next, by Proposition 1.4, there exists a nonnegative $C^\infty$ function $\psi$ on $X$ such that $\text{supp } \psi \subset B \setminus A = B \setminus (A \cup C)$, $\psi > 2$ on $B \setminus U$, $\psi$ exhausts $B \setminus U$, and $\psi$ is strictly $q$-plurisubharmonic on the subset $N \equiv \psi^{-1}((0, \infty))$ of $B \setminus (A \cup C)$. By Richberg's theorem (Proposition 1.3), there exists a $C^\infty$ strictly $q$-plurisubharmonic function $\tau$ on a neighborhood $V_1$ of $N$ in $M$ such that $\tau = \psi$ on $N$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ nondecreasing convex function such that $\chi(t) = 0$ if $t < 1$, $\chi(t) > t$ if $t > 2$, and $\chi'(t) > 0$ if $t > 1$. Since $\psi^{-1}([0, 1))$ is a neighborhood of the boundary of $N$ in $X$, we may assume (shrinking $V_1$ if necessary) that there is a neighborhood $V_2$ of $X \setminus N$ in $M$ such that $\tau < 1$ on $V_1 \cap V_2$. Thus the function $\chi(\tau)$ may be extended to a nonnegative $C^\infty$ $q$-plurisubharmonic function $\beta$ which is defined on the neighborhood $V_1 \cup V_2$ of $X$ and vanishes on a neighborhood of $A \cup C$. Moreover, since $\psi > 2$ on $B \setminus U$, $\beta$ is strictly $q$-plurisubharmonic and positive on a neighborhood of $B \setminus U$ in $M$ and $\beta$ exhausts $B \setminus U$.

Since $\dim C < q+1$, every $C^\infty$ function on the subset $C \setminus Y$ of $C \setminus X_{\text{sing}}$ is strictly $(q+1)$-plurisubharmonic. As in the construction of $\beta$, we may form a nonnegative $C^\infty$ $(q+1)$-plurisubharmonic function $\gamma$ on a neighborhood of $X$ such that $\gamma$ is positive and strictly $(q+1)$-plurisubharmonic on a neighborhood of $C \setminus U$ in $M$, $\gamma$ exhausts $C \setminus U$, and $\gamma$ vanishes on a neighborhood of $A \cup B = Y$.

It is now easy to check that if $\lambda$ is a $C^\infty$ increasing convex function on $\mathbb{R}$ and $\lambda'(t) \to \infty$ sufficiently fast as $t \to \infty$, then the function $\varphi \equiv \alpha + \lambda(\beta) + \lambda(\gamma)$ has the required properties on some neighborhood $V$ of $X$ and some neighborhood $W$ of $Y = A \cup B$. Q. E. D.

A $C^\infty$ strictly $q$-convex function $\varphi$ on a complex space $X$ of pure dimension $q$ has no local maximum points. Wu's approximation theorem (Proposition 1.1) implies that the same is true of the restriction of a continuous strictly $q$-plurisubharmonic function on $M$ to an analytic subset $X$ of pure dimension $q$. Similarly, we have the following:

**Proposition 1.5 (Maximum principle).** If the restriction $\varphi|_X$ of a continuous $q$-plurisubharmonic function $\varphi$ on $M$ to a connected analytic subset $X$ of pure dimension $q$ assumes its maximum value $m = \varphi(x_0)$ at some point $x_0 \in X$, then $\varphi|_X$ is constant.
Proof. — Since \( \varphi \mid_{X_{\text{reg}}} \) is subharmonic, it suffices to show that 
\( \varphi(x) = m \) at some point \( x \in X_{\text{reg}} \). Assume that this is not the case. 
Taking successive singular sets \( X_{\text{sing}}, (X_{\text{sing}})_{\text{sing}}, \ldots \), we may assume that, 
for some nowhere dense analytic subset \( Y \) of \( X \), we have \( x_0 \in Y_{\text{reg}} \) and 
\( \varphi < m \) on \( X \setminus Y \). Every \( C^\infty \) function on \( Y_{\text{reg}} \) is strictly \( q \)-plurisubharmonic since \( \dim Y < q \). So, by applying Proposition 1.3, we may form 
a relatively compact neighborhood \( U \) of \( x_0 \) in \( M \) and a \( C^\infty \) strictly \( q \)-plurisubharmonic function \( \psi \) on a neighborhood of \( U \) such that \( \psi(x_0) = 0 \) 
and \( \psi < 0 \) on \( (U \setminus \{x_0\}) \cap Y \). In particular, on some neighborhood \( V \) 
of \( Y \cap \partial U \), we have \( \psi < 0 \) and hence \( \varphi + \epsilon \psi < m \) for every \( \epsilon > 0 \). We 
also have \( \varphi + \epsilon \psi < m \) on \( (X \cap \partial U) \setminus V \) provided \( \epsilon \) is sufficiently small. 
Therefore \( \varphi + \epsilon \psi < m = (\varphi + \epsilon \psi)(x_0) \) on \( X \cap \partial U \). This contradicts the 
maximum principle for continuous strictly \( q \)-plurisubharmonic functions, 
so the proposition follows. Q. E. D.

In this paper, the main tool for obtaining the Bochner-Hartogs property is the following result of 
Grauert and Riemenschneider [GR] (who proved a version of the vanishing theorem for higher cohomology groups) 
and of Siu [S], Lemma 5.10 (who proved a version in the more general setting of harmonic maps into manifolds satisfying a certain curvature condition).

**Theorem 1.6 (Grauert-Riemenschneider, Siu). —** Suppose \( \Omega \) is a 
relatively compact domain in a Kähler manifold \( M \) and \( \Omega \) has a \( C^\infty \) \((n-1)\)- 
plurisubharmonic defining function \( \varphi \) whose differential is nonzero at every 
point in \( \partial \Omega \).

(a) If \( \beta \) is a \( C^\infty \) function on \( \overline{\Omega} \) which is harmonic on \( \Omega \) and which 
satisfies the tangential Cauchy-Riemann equations \( \bar{\partial}_b \beta = 0 \) on \( \partial \Omega \), then \( \beta \) 
is pluriharmonic on \( \Omega \).

(b) If \( \varphi \) is strictly \((n-1)\)-plurisubharmonic on a neighborhood of some 
point in \( \partial \Omega \), then \( H^1_c(\Omega, \mathcal{O}) = 0 \).

**Sketch of the proof. —** Suppose first that \( \beta \) is a function as in (a). 
Let \( \gamma = \bar{\partial} \beta \) and let \( \eta = \ast \gamma \), where * denotes the Hodge star operator. 
Then \( \eta \) may be thought of as a \( C^\infty \) form of type \((n,n-1)\) or as a 
form of type \((0,n-1)\) with values in the canonical bundle \( K_M \) on \( \overline{\Omega} \). A 
computation in normal coordinates then shows that \( \eta \) lies in the domain 
of the adjoint operator \( \bar{\partial}^* \). The Bochner-Kodaira formula is then (see [GR]
or [S], formula 2.1.4)\[\|\bar{\partial} \eta\|_{L^2(\Omega)}^2 + \|\bar{\partial}^* \eta\|_{L^2(\Omega)}^2 = \|\nabla \eta\|_{L^2(\Omega)}^2 + \int_{\partial \Omega} |\gamma|^2 \cdot \tau \, d\sigma,\]

where $d\sigma$ is the volume element on $\partial \Omega$ and $\tau$ is the trace of the restriction of the Levi form $L(\varphi)$ of $\varphi$ to $T^{1,0}(\partial \Omega)$. Here, the curvature terms drop out because $\eta$ is of type $(n, n-1)$ and we have used the fact that $\bar{\partial} \beta \wedge \bar{\partial} \varphi$ vanishes at each point of $\partial \Omega$. Since $\beta$ is harmonic, the left-hand side is equal to zero. Since $\varphi$ is $(n-1)$-plurisubharmonic, we have $\tau \geq 0$ and hence $\nabla \eta = 0$. A computation in normal coordinates now shows that $\beta$ is pluriharmonic (see [S], Proof of Lemma 5.6(d)) and (a) is proved.

Suppose now that $\alpha$ is a $C^\infty$ compactly supported form of type $(0, 1)$ with $\bar{\partial} \alpha = 0$. Let $\beta$ be the $C^\infty$ function on $\Omega$ which vanishes on $\partial \Omega$ and which satisfies $\frac{1}{2} \Delta \beta = \bar{\partial}^* \bar{\partial} \beta = \bar{\partial}^* \alpha$ on $\Omega$ (where $\Delta = -(d^*d + dd^*)$ is the Laplacian), let $\gamma = \alpha - \bar{\partial} \beta$, and let $\eta = \bar{\partial} \gamma$. Then
\[
\bar{\partial}^* \eta = - * \partial * (\bar{\partial} \gamma) = * \bar{\partial} \gamma = 0 \quad \text{and} \quad \delta \eta = -*[\bar{\partial}^* \alpha - \bar{\partial}^* \bar{\partial} \beta] = 0.
\]
Moreover, since $\alpha$ has compact support, $\gamma = - \bar{\partial} \beta$ near $\partial \Omega$; and, since $\beta$ vanishes on $\partial \Omega$, $\bar{\partial} \beta \equiv 0$. Applying the Bochner-Kodaira formula to $\eta$ as in the proof of (a), we get $\nabla \eta = 0$ and it follows that $\partial \gamma = 0$ (and hence $d\gamma = 0$). Thus $\gamma$ is a holomorphic 1-form on $\Omega$.

On the other hand, since $\tau > 0$ at some point, the Bochner-Kodaira formula implies that $|\gamma| = |\bar{\partial} \beta| = 0$ on some nonempty open subset in $\partial \Omega$. If $U$ is a connected neighborhood of a boundary point and $f$ is a $C^\infty$ function on $U \cap \overline{\Omega}$ which is holomorphic on $U \cap \Omega$ and which vanishes on $U \cap \partial \Omega$, then one may extend $f$ to a continuous function $h$ on $U$ which vanishes outside $\Omega$. But then $\bar{\partial} h = 0$ in the weak sense, so $h$ is holomorphic. It follows that $h$, and therefore $f$, must vanish identically. Letting $f$ be a coefficient of the holomorphic 1-form $\gamma$ with respect to some local holomorphic frame, we see that $\gamma$ vanishes on a nonempty open subset of $\Omega$ and hence on all of $\Omega$. Thus $\beta$ is holomorphic outside the support $K$ of $\alpha$. Since $\beta$ vanishes on $\partial \Omega$, the above discussion (with $f = \beta$) implies that $\beta$ vanishes on each connected component of $\Omega \setminus K$ which is not relatively compact in $\Omega$. In other words, $\beta$ has compact support and $\alpha = \bar{\partial} \beta$. Thus (b) is proved. Q. E. D.

Since the proof of the main theorem involves related arguments on a complete Kähler manifold, we close this section with a discussion of Green’s functions on Riemannian manifolds. A connected noncompact Riemannian manifold $N$ which admits a positive symmetric Green’s function $G(x, y)$ is said to be hyperbolic (otherwise, $N$ is called parabolic). We normalize $G$ so
that, for each point $x_0 \in N$,
\[
\Delta_{\text{distr}} G(\cdot, x_0) = -\delta_{x_0}
\]
where $\delta_{x_0}$ is the Dirac function at $x_0$ and $\Delta = -(d^*d + dd^*)$ is the Laplacian. We will use the same notation for the corresponding integral operator $G$ given by
\[
(G\alpha)(x) = \int_N G(x, y)\alpha(y) \, dV(y) \quad \forall \, x \in N
\]
for each suitable function $\alpha$ on $N$. If $\alpha$ is a $C^\infty$ compactly supported function, then $\beta = -G\alpha$ is a $C^\infty$ bounded function with finite energy (i.e. $\int_N |\nabla \beta|^2 \, dV < \infty$) and $\Delta \beta = \alpha$. Moreover, $\beta(x_\nu) \to 0$ if $\{x_\nu\}$ is a sequence in $N$ with $x_\nu \to \infty$ and $G(\cdot, x_\nu) \to 0$. Such a sequence $\{x_\nu\}$ always exists and will be called a regular sequence.

2. Proof of the main result.

We begin with two lemmas. The first is a special case of a result of Nishino [Ni] who proved it without the assumption that $M$ is Kähler. In the Kähler case, one may prove it using arguments contained in the proof of [NR], Theorem 4.6.

**Lemma 2.1.** — Suppose $(M, \omega)$ is a connected weakly 1-complete Kähler manifold and there exists a proper holomorphic mapping of some nonempty open subset of $M$ onto a Riemann surface. Then $M$ also admits a proper holomorphic mapping onto a Riemann surface.

The second lemma often helps one obtain a holomorphic mapping to a Riemann surface. An elementary proof is given here. One may also prove this fact by using holomorphic equivalence relations (see, for example, [Ka]).

**Lemma 2.2.** — If $\omega_1$ and $\omega_2$ are two linearly independent closed holomorphic 1-forms satisfying $\omega_1 \wedge \omega_2 \equiv 0$ on a connected complex manifold $M$, then the meromorphic function $h \equiv \omega_1/\omega_2$ has no points of indeterminacy in $M$ and is locally constant on the analytic set $S \equiv \{x \in M \mid (\omega_1)_x = 0\} \cup \{x \in M \mid (\omega_2)_x = 0\}$.

**Proof.** — Let $I$ be the set of points of indeterminacy of $h$ and, for each $\zeta \in \mathbb{P}^1$, let
\[
F_\zeta = (h|_{(M \setminus I)})^{-1}(\zeta) \supset I
\]
be the fiber over \(\zeta\) (see [Gu]). Since the problem is local, we may assume that there exist holomorphic functions \(f_1\) and \(f_2\) on \(M\) such that \(df_i = \omega_i\) for \(i = 1, 2\). One may see that \(f_1\) and \(f_2\) are locally constant on \(F_\zeta\) for each \(\zeta \in \mathbb{P}^1\) as follows. Near each smooth point \(x\) of \(F_\zeta\) at which \(df_2 \neq 0\), we may choose holomorphic coordinates \(z = (z_1, \ldots, z_n)\) in which \(z_1 = f_2\) on a connected neighborhood of \(x\). Since \(df_1 \wedge dz_1 = 0\), we have then \(f_1 = f_1(z_1)\) and hence \(h = f_1(z_1)\). In particular, \(h\) is constant along each fiber of \(z_1\) and hence each of these fibers (being of codimension 1) must be an open set in \(F_\zeta\). Therefore, if \(v\) is a vector tangent to \(F_\zeta\) at \(x\), then \(\omega_2(v) = df_2(v) = 0\). This is also the case if \(df_2 = 0\) at \(x\). Thus \(f_2\) is locally constant on \(F_\zeta\) for each \(\zeta \in \mathbb{P}^1\), and, by symmetry, the same is true of \(f_1\). It follows that, if \(x_0 \in I\), then \(f_2^{-1}(f_2(x_0))\) contains the connected component \(H_\zeta\) of \(F_\zeta\) containing \(x_0\) for each \(\zeta \in \mathbb{P}^1\). But \(\{H_\zeta\}_{\zeta \in \mathbb{P}^1}\) is an infinite collection of distinct analytic sets of pure dimension \(n-1\), so this is impossible. Therefore \(I = \emptyset\).

It remains to show that \(h\) is locally constant on the analytic set \(S\), which we may assume to be irreducible. Since \(f_1\) or \(f_2\) is then constant on \(S\), we may assume that \(S\) lies in some irreducible component \(T\) of the zero set of \(f_1 \cdot f_2\) and it suffices to show that \(h\) is constant on \(T\). By working near a generic point of \(T\), we may assume that there exist holomorphic coordinates \(z = (z_1, \ldots, z_n)\) on \(M\) in which \(T\) is the zero set of \(z_1\) and, for \(j = 1, 2\), \(f_j(z) = z_1^{m_j} g_j(z)\) where \(g_j\) is a nonvanishing holomorphic function and \(m_j \geq 0\). By replacing \(z_1\) by \(z_1 u\) for some (local) \(m_j^{th}\) root \(u\) of \(g_j\), we may also assume that \(g_2 \equiv 1\). Since \(df_1 \wedge df_2 \equiv 0\), we then get \(f_1 = f_1(z_1)\) and \(g_1 = g_1(z_1)\). Hence the function \(h(z) = f_1'(z_1)/f_2'(z_1)\) depends only on \(z_1\) and is therefore constant on \(T\). Thus the lemma is proved.

The main step in the proof of the theorem stated in the introduction is the following proposition, which is, in a sense, a generalization of a theorem of Ohsawa [O1] and a result in [NR].

**Proposition 2.3.** — Let \((M, g)\) be a connected noncompact Kähler manifold of dimension \(n\) on which there exists a continuous \((n-1)\)-plurisubharmonic exhaustion function \(\varphi\). Suppose there is a compact subset \(K\) of \(M\) such that, on each connected component \(E\) of \(M \setminus K\), there exists a pair of real-valued pluriharmonic functions \(\rho_1\) and \(\rho_2\) with linearly independent differentials \(d\rho_1\) and \(d\rho_2\). Then \(M\) satisfies (BHD).

**Remarks.**

1. The above condition on \(E\) holds if, for example, there exists a
2. If $M$ satisfies the above conditions and has more than one end, then there exists a proper holomorphic mapping of $M$ onto a Riemann surface.

Proof. — Choosing $a \in \mathbb{R}$ sufficiently large, we may assume without loss of generality that 

$$K = \{ x \in M \mid \varphi(x) \leq a \} \neq \emptyset.$$ 

Let $E$ be a connected component of $M \setminus K$ with noncompact closure and let $\rho_1$ and $\rho_2$ be pluriharmonic functions on $E$ as in the statement of the proposition. We may assume that $\rho_1$ and $\rho_2$ extend to pluriharmonic functions on a neighborhood of $\overline{E}$.

There are three possibilities:

(a) The analytic subset $X = \{ x \in E \mid (\partial \rho_1 \wedge \partial \rho_2)_x = 0 \}$ of $E$ is nowhere dense,

(b) $X = E$ and the set $Q$ of points in $E$ which lie in a compact level of the holomorphic mapping 

$$h = \frac{\partial \rho_1}{\partial \rho_2} : E \to \mathbb{P}^1$$

(see Lemma 2.2) is nonempty, or

(c) $X = E$ and $Q = \emptyset$.

We will show that, if (a) or (c) holds, then $M$ admits an exhaustion by $C^\infty$ domains which are strongly hyper-$(n-1)$-convex at each boundary point in $E$; and if (b) holds, then $E$ admits a proper holomorphic mapping onto a Riemann surface. Briefly, in the case (a), we will work on a relatively compact (weakly) hyper-$(n-1)$-convex domain $\Omega$ in $M$. Combining $\varphi$, $\rho_1^2 + \rho_2^2$, and a strictly $(n-1)$-plurisubharmonic function near $X$ (obtained from Theorem 1.2), we will obtain a strictly $(n-1)$-plurisubharmonic function near $E \cap \partial \Omega$. In the case (b), we will show that $Q = E$ and hence, by Stein factorization, one obtains the desired mapping. Finally, in the case (c), we will again work on a hyper-$(n-1)$-convex domain $\Omega$ and we will apply a standard patching argument to strictly $(n-1)$-plurisubharmonic functions on neighborhoods of fibers of $h$ (or of a suitable holomorphic function if $h$ is constant) to obtain a strictly $(n-1)$-plurisubharmonic function as in the case (a).

Suppose first that $E$ satisfies the condition (a) and let $Y$ be the union of all of the compact irreducible components of $X$ of dimension $n-1$. The
maximum principle (Proposition 1.5) implies that \( \varphi(Y) \) is a countable set, so we may choose \( b \in (a, \infty) \setminus \varphi(Y) \) so large that \( K \) is contained in some connected component \( \Omega \) of \( \{ x \in M \mid \varphi(x) < b \} \). By Theorem 1.2 (Demailly, Greene-Wu, Ohsawa), there exists a positive \( C^{\infty} \) function \( \psi \) on \( E \setminus Y \) which is strictly \((n-1)\)-plurisubharmonic on a neighborhood of \( X \setminus Y \) in \( E \). Therefore, since \( \rho_1^2 + \rho_2^2 \) is strictly \((n-1)\)-plurisubharmonic on \( E \setminus X \) and since \( \varphi^{-1}(b) \cap Y = \emptyset \), the function \( \gamma \equiv r \cdot (\rho_1^2 + \rho_2^2) + \psi \), where \( r \) is a sufficiently large positive constant, is strictly \((n-1)\)-plurisubharmonic on a relatively compact neighborhood \( U \) of \( \varphi^{-1}(b) \cap E \) in \( E \setminus Y \). Applying Wu’s approximation theorem [W] (Proposition 1.1) to the function \( \gamma - \log(b - \varphi) \) on \( \Omega \cap U \), we get a \( C^{\infty} \) strictly \((n-1)\)-plurisubharmonic function \( \lambda \) on \( U \cap \Omega \) which approaches infinity at \( \partial \Omega \). Finally, let \( c > 0 \) and let \( \chi : \mathbb{R} \to \mathbb{R} \) be a \( C^{\infty} \) nondecreasing convex function such that \( \chi(t) = 0 \) for \( t \leq c \), \( \chi'(t) > 0 \) for \( t > c \), and \( \chi(t) \to \infty \) as \( t \to \infty \). If \( c \) is sufficiently large, then the function \( \chi(\lambda) \) extends to a \( C^{\infty} \) nonnegative \((n-1)\)-plurisubharmonic function on \( \Omega \cup (M \setminus E) \) which vanishes on \( M \setminus E \), which is strictly \((n-1)\)-plurisubharmonic near \( E \cap \partial \Omega \), and which exhausts \( \Omega \cap \overline{E} \).

Assuming now that (b) holds, we show that \( Q = E \). We first observe that \( Q \) is open for point-set topological reasons. For if \( x_0 \in Q \) and \( L_0 \) is the (compact) level of \( h \) through \( x_0 \), then there is a relatively compact neighborhood \( U \) of \( L_0 \) in \( E \setminus (h^{-1}(h(x_0)) \setminus L_0) \). Since \( \partial U \) is compact, there is a neighborhood \( V \) of \( h(x_0) \) in \( \mathbb{P}^1 \) such that \( h^{-1}(V) \cap \partial U = \emptyset \). Hence the level through each point in the neighborhood \( h^{-1}(V) \cap U \) does not meet \( \partial U \) and must, therefore, be a compact subset of \( U \).

Next, let \( b > a \) be so large that there is a connected component \( \Omega \) of the set \( \{ x \in M \mid \varphi(x) < b \} \) such that \( Q \cap \Omega \neq \emptyset \), \( K \subset \Omega \), and \( \Omega \cap E \) is connected. If \( x_0 \in Q \cap \Omega \), then the irreducible component \( A \) of \( h^{-1}(h(x_0)) \) containing \( x_0 \) is a compact subset of \( \Omega \cap E \). For if \( \{ x_\nu \} \) is a sequence in \( Q \cap \Omega \setminus \{ x_0 \} \) converging to \( x_0 \) and, for each \( \nu \), \( L_\nu \) is the (compact) level of \( h \) through \( x_\nu \), then, by the Remmert-Stein-Thullen theorem (see [Gu]), \( A \) lies in the closure of \( \bigcup L_\nu \). On the other hand, \( \varphi|_{L_\nu} \) is constant for each \( \nu \) and \( \varphi(x_\nu) \to \varphi(x_0) \) where \( a < \varphi(x_0) < b \). Therefore, since \( \varphi = a \) or \( b \) at each boundary point of \( \Omega \cap E \), \( \bigcup L_\nu \) must lie in some compact subset of \( \Omega \cap E \) and the claim follows.

Since \( h \) extends to a holomorphic mapping on a neighborhood of \( \overline{E} \) \((\rho_1 \text{ and } \rho_2 \text{ extend by the choice of } a)\), the set of critical values of \( \text{h|}_{(\Omega \cap E)} \) is finite and the inverse image of this finite set is an analytic subset \( B \) of \( \Omega \cap E \). The above discussion implies that, if \( L_0 \) is the level of \( h \) through
a point $x_0 \in \overline{Q} \cap \Omega$ and $L_0 \cap \Omega$ is smooth, then $L_0 \subset Q \cap \Omega$. Therefore $Q \cap \Omega \setminus B$ is a closed subset of the connected set $\Omega \cap E \setminus B$ and hence, since $Q$ is open, we must have equality. In particular, $Q \cap \Omega$ is dense in $\Omega \cap E$. Applying the above again, we see that if $L_0$ is any level of $h$ which meets $\Omega$, then every irreducible component of $L_0$ that meets $\Omega$ must be a compact subset of $\Omega \cap E$. It follows that $L_0$ is compact. Thus $\Omega \cap E \subset Q$ and, since the choice of $b$ was arbitrary, we get $Q = E$. Therefore every level of $h$ is compact and, by Stein factorization [St], we obtain a proper holomorphic mapping of $E$ onto a Riemann surface.

Finally, assuming that $X = E$ and $Q = \emptyset$ (i.e. that $E$ satisfies the condition (c)), we apply a modification of a construction due to Ohsawa [O1] for the case of a weakly 1-complete surface. We also assume for now that the mapping $h : E \to \mathbb{P}^1$ is nonconstant.

We first show that the union $C$ of the collection $C$ of all compact irreducible components of fibers of $h$ is a nowhere dense analytic subset of $E$. For if $K_0$ is a compact subset of $E$ and $L_0$ is a level of $h$, then any compact irreducible component $C_0$ of $L_0$ which meets $K_0$ must lie in the compact subset $\varphi^{-1}(\varphi(K_0))$. Moreover, since $C_0 \neq L_0$, $C_0$ must meet some irreducible component $H_0$ of the analytic set $H \equiv \{x \in E \mid (h_x)_x = 0\}$ and, since $h$ is locally constant on $H$, we have $H_0 \subset L_0$. Only finitely many irreducible components of $H$ meet $\varphi^{-1}(\varphi(K_0))$, so the collection of all levels $L_0$ with such an irreducible component $C_0$ is finite. It follows that $C$ is locally finite in $E$ and hence that $C$ is an analytic set.

The set $\varphi(C)$ is discrete, so we may choose a number $b \in (a, \infty) \setminus \varphi(C)$ so large that there is a connected component $\Omega$ of $\{x \in M \mid \varphi(x) < b\}$ such that $K \subset \Omega$ and $\Omega \cap E$ is connected. We may also choose a relatively compact neighborhood $W$ of $\varphi^{-1}(b) \cap E$ in $E \setminus C$. For each point $x \in \varphi^{-1}(b) \cap E$, the analytic set $h^{-1}(h(x)) \setminus C$ has no compact irreducible components and hence, by Theorem 1.2, there is a $C^\infty$ strictly $(n-1)$-plurisubharmonic function $\gamma_1$ on a neighborhood $V_1$ of $h^{-1}(h(x)) \setminus C$ in $E$. Moreover, we have $h^{-1}(D'_1) \cap W \subset V_1$ for any sufficiently small neighborhood $D'_1$ of $h(x)$ in $\mathbb{P}^1$. There is also a nonnegative $C^\infty$ function $\lambda_1$ with $\text{supp} \lambda_1 \subset D'_1$ and $\lambda_1 \equiv 1$ on some neighborhood $D_1$ of $h(x)$. We may, therefore, choose $C^\infty$ strictly $(n-1)$-plurisubharmonic functions $\gamma_1, \ldots, \gamma_m$ on open sets $V_1, \ldots, V_m$ in $E$, respectively; open sets $D_1, \ldots, D_m$ and $D'_1, \ldots, D'_m$ in $\mathbb{P}^1$; and nonnegative $C^\infty$ functions $\lambda_1, \ldots, \lambda_m$ on $\mathbb{P}^1$ such that, for each $j = 1, \ldots, m$, we have $\lambda_j \equiv 1$ on $D_j$, $\text{supp} \lambda_j \subset D'_j$, $h^{-1}(D'_j) \cap W \subset V_j$, and $\varphi^{-1}(b) \cap E \subset h^{-1}(D_1) \cup \ldots \cup h^{-1}(D_m)$. Moreover,
by Lemma 2.2, \( h \) is locally constant on the analytic set \( S \equiv \{ x \in E \mid (\partial \rho_2)_x = 0 \} \), so \( h(S \cap \phi^{-1}(b)) \) is a finite set and we may assume that, for each \( j \), \( \lambda_j(h) \) is constant near each point of \( W \cap S \).

We now show that, for a sufficiently large positive constant \( s \), the \( C^\infty \) function

\[
\gamma \equiv s \cdot (\rho_2)^2 + \sum_{j=1}^{m} \lambda_j(h) \gamma_j
\]

is strictly \((n-1)\)-plurisubharmonic on some neighborhood of \( \varphi^{-1}(b) \cap E \). It is easy to see that \( \gamma \) is strictly \((n-1)\)-plurisubharmonic on a neighborhood of \( S \cap \varphi^{-1}(b) \cap E \), since, near each point of this set, each of the nonnegative functions \( \lambda_1(h), \ldots, \lambda_m(h) \) is constant and at least one of the functions is positive. Given a point \( x \) near \( \varphi^{-1}(b) \cap E \) and a tangent vector \( v \in T_x T_{x,0} M \), we have

\[
\mathcal{L}(\gamma)(v, v) = 2s|\partial \rho_2(v)|^2 + \sum_{j=1}^{m} \gamma_j(x) \mathcal{L}(\lambda_j)(h_*v, h_*v)
\]

\[
+ \sum_{j=1}^{m} 2 \text{Re} \left[ \frac{\partial \lambda_j(h_*v)}{\partial \gamma_j(v)} \right] + \sum_{j=1}^{m} \lambda_j(h(x)) \mathcal{L}(\gamma_j)(v, v).
\]

If \( x \) is not near \( S \), then, since \( (\partial \rho_2)_x \neq 0 \) and \( \partial \rho_1 \wedge \partial \rho_2 \equiv 0 \), we may choose holomorphic coordinates \( (z_1, \ldots, z_n) \) near \( x \) in which \( \rho_2 = 2 \text{Re} z_1 \), \( \rho_1 = \rho_1(z_1) \), and

\[
h = \frac{\partial \rho_1}{\partial \rho_2} = \frac{\partial \rho_1}{\partial z_1} = \rho_1(z_1) \in \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}.
\]

Thus \( \partial h = \frac{\partial h}{\partial z_1} dz_1 \) and hence \( |\partial \rho_2(v)|^2 = |dz_1(v)|^2 \geq q_0|\partial h(v)|^2 \) for some positive constant \( q_0 \). Since \( \lambda_1, \ldots, \lambda_m \geq 0 \), since \( \max(\lambda_1(h), \ldots, \lambda_m(h)) > 0 \) near \( \varphi^{-1}(b) \cap E \), and since, for each \( j \), \( \gamma_j \) is strictly \((n-1)\)-plurisubharmonic on the neighborhood \( V_j \) of \( \text{supp} \lambda_j(h) \cap \varphi^{-1}(b) \), there exist positive constants \( q_1 \) and \( q_2 \) (independent of \( s \)) such that, for every \( \epsilon > 0 \), for every point \( x \in \varphi^{-1}(b) \cap E \), and for every collection of orthonormal tangent vectors \( e_1, \ldots, e_{n-1} \in T_{x,0} M \), we have

\[
\sum_{i=1}^{n-1} \mathcal{L}(\gamma)(e_i, e_i) \geq (2sq_0 - q_1 \cdot (1 + (2\epsilon)^{-1})) \cdot \sum_{i=1}^{n-1} |\partial h(e_i)|^2 = \frac{1}{2} q_1 + q_2.
\]

Thus if we choose \( \epsilon < 2q_2/q_1 \) and \( s > (2q_0)^{-1} q_1 \cdot (1 + (2\epsilon)^{-1}) \), then \( \gamma \) will be strictly \((n-1)\)-plurisubharmonic near points in \( \varphi^{-1}(b) \cap E \) which lie outside an (arbitrarily small) neighborhood of \( S \) as well as those which lie near \( S \).
Proceeding now as in the case (a), one gets a $C^\infty$ nonnegative $(n-1)$-plurisubharmonic function on $\Omega \cup (M \setminus E)$ which vanishes on $M \setminus E$, which exhausts $\overline{E} \cap \Omega$, and which is strictly $(n-1)$-plurisubharmonic near $E \cap \partial \Omega$.

If $h$ is constant, then $\partial (c_1 \rho_1 + c_2 \rho_2) \equiv 0$ for some pair of constants $c_1, c_2 \in \mathbb{C}$ which are not both zero. The function $f = \bar{c}_1 \rho_1 + \bar{c}_2 \rho_2$ on $E$ is holomorphic and nonconstant; because the functions $1, \rho_1, \text{and } \rho_2$ are linearly independent. One may now proceed as above by using $f$ in place of the mapping $h$.

Thus for each of the (finitely many) connected components $E$ of $M \setminus K$ which have noncompact closure, either $E$ admits a proper holomorphic mapping onto a Riemann surface (and hence a $C^\infty$ plurisubharmonic exhaustion function), or there exists an arbitrarily large relatively compact domain $\Omega$ in $M$ and a $C^\infty$ nonnegative $(n-1)$-plurisubharmonic function on $\Omega \cup (M \setminus E)$ which vanishes on $M \setminus E$, which exhausts $\overline{E} \cap \Omega$, and which is strictly $(n-1)$-plurisubharmonic near $E \cap \partial \Omega$. If all of these connected components of $M \setminus K$ have the former property, then $M$ admits a $C^\infty$ plurisubharmonic exhaustion function and a complete Kähler metric and Lemma 2.1 implies that there is a proper holomorphic mapping of $M$ onto a Riemann surface. If at least one of these connected components has the latter property, then $H_c^1(M, \mathcal{O}) = 0$. For if $\alpha$ is a $C^\infty$ compactly supported form of type $(0,1)$ and $\bar{\partial} \alpha = 0$, then we may choose a $C^\infty$ relatively compact domain $\Omega$ which contains the support of $\alpha$ and which admits a $C^\infty$ $(n-1)$-plurisubharmonic defining function which is strictly $(n-1)$-plurisubharmonic near at least one connected component of $\partial \Omega$. The theorem of Grauert and Riemenschneider [GR] and Siu [S] (Theorem 1.6) then implies that $H_c^1(\Omega, \mathcal{O}) = 0$ and hence that there is a $C^\infty$ compactly supported function $\beta$ on $\Omega$, and therefore on $M$, such that $\bar{\partial} \beta = \alpha$. Thus $M$ has the property (BHD).

Q. E. D.

To apply Proposition 2.3, it will be convenient to have the following lemma, which is essentially a special case of [NR], Theorem 2.6.

**Lemma 2.4.** — Let $(M, g)$ be a connected noncompact complete hyperbolic Kähler manifold, let $M_0$ be a $C^\infty$ relatively compact domain in $M$, and let $E$ be a connected component of $M \setminus \overline{M_0}$ with noncompact closure. Then there exists a pluriharmonic function $\tau$ on $M$ such that $0 \leq \tau \leq 1$, $\tau$ has finite energy, and, for every regular sequence $\{x_\nu\}$ in $M$ approaching $\infty$ (see Section 1), $\tau(x_\nu) \to 1$ if $x_\nu \in E$ for $\nu \gg 0$ and $\tau(x_\nu) \to 0$ if $x_\nu \in M \setminus E$ for $\nu \gg 0$. 

The construction of $\tau$ is contained within the proof of [NR], Theorem 2.6. That $\tau \leq 1$ is not proved explicitly, but one can easily verify this property by forming an exhaustion of $M$ by $C^\infty$ relatively compact domains and writing the Green's function on $M$ as the limit of the corresponding sequence of Green's functions.

We are now ready to prove the main result.

**THEOREM 2.5.** Let $(M, g)$ be a connected Kähler manifold of dimension $n$ which has exactly one end and which admits a continuous $(n-1)$-plurisubharmonic exhaustion function $\varphi$. Assume that at least one of the following conditions is satisfied:

(i) $\varphi$ plurisubharmonic,

(ii) $(M, g)$ is complete and hyperbolic, or

(iii) $\varphi$ is of class $C^\infty$.

Then $M$ satisfies (BHD).

**Remark.** An example of Cousin [Co] shows that one cannot remove the requirement that $M$ have only one end (see, for example, [NR], Example 3.9).

**Proof.** Assuming that there exists a $C^\infty$ compactly supported form $\alpha$ of type $(0,1)$ such that $\bar{\partial}\alpha = 0$ and $[\alpha] \neq 0$ in $H^1_{M, \mathcal{O}}$, we will show that there is a proper holomorphic mapping of $M$ onto a Riemann surface.

We first assume that (i) or (ii) holds. Fix a $C^\infty$ relatively compact domain $M_0$ in $M$ such that supp $\alpha \subset M_0$ and $M \setminus \overline{M_0}$ is connected. If $\varphi$ is plurisubharmonic (i.e. (i) holds), let $a > \sup_{M_0} \varphi$ and let $\Omega$ be the connected component of $\{x \in M | \varphi(x) < a\}$ containing $\overline{M_0}$. Then, by a theorem of Nakano [N] and Demailly [D1], the (weakly 1-complete) domain $\Omega$ admits a complete Kähler metric $g'$. Moreover, $(\Omega, g')$ is hyperbolic (in fact, the Green's function vanishes at the boundary) and if $a$ is sufficiently large, then $\Omega \setminus \overline{M_0}$ is connected. It suffices to show that $\Omega$ admits a proper holomorphic mapping onto a Riemann surface (for a suitable choice of $a$). For we may then form an exhaustion of $M$ by such domains. Applying a theorem of Narasimhan [Na], Corollary 1 to the Cartan-Remmert reductions, we get the required mapping on $M$. If $\varphi$ is not plurisubharmonic, we set $(\Omega, g') = (M, g)$.
Thus, in either case, \((\Omega, g')\) admits a continuous \((n-1)\)-plurisubharmonic exhaustion function and a positive symmetric Green’s function \(G(x, y)\), and the Kähler metric \(g'\) is complete. Therefore, since \(\bar{\partial}^*\alpha\) is a \(C^\infty\) function with compact support on \(\Omega\), the function \(\beta \equiv -2G(\bar{\partial}^*\alpha)\) is a \(C^\infty\) bounded function with finite energy and \(-\frac{1}{2} \Delta \beta = \bar{\partial}^*\bar{\partial}\beta = \bar{\partial}^*\alpha\). Moreover, \(\beta(x, \nu) \to 0\) for any regular sequence \(\{x, \nu\}\) in \(\Omega\) approaching \(\infty\) (see Section 1). The form \(\gamma \equiv \alpha - \bar{\partial}\beta\) is a harmonic form in \(L^2_{0,1}(\Omega, g')\); the harmonic projection of \(\alpha\). By the Gaffney theorem [G], \(\gamma\) is closed (and coclosed). In particular, \(\gamma\) is a holomorphic 1-form on \(\Omega\) (since \(\partial\gamma = 0\)) and \(\beta\) is pluriharmonic on the connected set \(E \equiv \Omega \setminus M_0\). Since \([\alpha] \neq 0\) in \(H^1_c(M, \mathcal{O})\) and \(\text{supp} \alpha \subset M_0 \subset \subset \Omega\), we have \([\alpha] \neq 0\) in \(H^1_\mathcal{O}(\Omega, \mathcal{O})\).

It follows that \(\beta\) is not constant on \(E\). For, if \(\beta\) were constant on \(E\), then, since \(\beta\) vanishes at infinity along any regular sequence, we would have \(\beta \equiv 0\) on \(\Omega \setminus M_0\). Hence \(\gamma\) would vanish on \(E\) and, therefore, on \(\Omega\), since \(\gamma\) is harmonic. But we would then get \(\alpha = \bar{\partial}\beta\), where \(\beta\) is a \(C^\infty\) function with compact support in \(\Omega\); which contradicts the nonvanishing of \([\alpha]\). Thus \(\beta \mid E\) is not constant (and hence is nowhere locally constant).

If \(\gamma = 0\), then \(\beta \mid E\) is a nonconstant holomorphic function on the end \(E\). Proposition 2.3 then implies that \(\Omega\) satisfies (BHD) and hence, since \(H^1_\mathcal{O}(\Omega, \mathcal{O}) \neq 0\), there is a proper holomorphic mapping onto a Riemann surface. Thus we may assume that \(\gamma\) is not everywhere zero. Let \(\rho = \text{Re} \beta \mid E\) and \(\rho' = \text{Im} \beta \mid E\). If the functions \(1, \rho,\) and \(\rho'\) are linearly independent on \(E\), then, again, Proposition 2.3 gives one the required mapping. If \(1, \rho,\) and \(\rho'\) are linearly dependent on \(E\), then, since \(\rho\) and \(\rho'\) vanish at infinity along any regular sequence, the functions \(\rho\) and \(\rho'\) must be linearly dependent. Therefore, after multiplying \(\alpha\) by a suitable nonzero complex constant, we may assume that \(\beta \mid E = \rho\) (i.e. \(\beta \mid E\) is real-valued) and hence that \(\gamma \mid E = -\bar{\partial}\rho\). To obtain a second pluriharmonic function (i.e. to apply Proposition 2.3) we will use Lemma 2.4 to obtain a pluriharmonic function on a covering space of \(\Omega\) with two ends and then push down to \(E\).

We first observe that, for any point \(x_0 \in E\), the mapping \(\pi_1(E, x_0) \rightarrow \pi_1(\Omega, x_0)\) is not surjective. For if the mapping is surjective, then, since the \(C^\infty\) closed real 1-form \(\theta = -\gamma - \bar{\gamma}\) on \(\Omega\) is equal to \(d\rho\) on \(E\), it follows that the function \(\rho_0\) given by

\[
\rho_0(x) = \rho(x_0) + \int_{x_0}^x \theta \quad \forall x \in \Omega
\]

is a well-defined \(C^\infty\) function with \(d\rho_0 = \theta = d\rho\) on \(E\) and \(\rho_0(x_0) = \rho(x_0)\). Hence \(\rho_0 = \rho\) on \(E\), so \(\beta - \rho_0\) is a \(C^\infty\) function on \(\Omega\) which vanishes on
\( E = \Omega \setminus \overline{M}_0 \) and which satisfies
\[
\overline{\partial}(\beta - \rho_0) = \overline{\partial}\beta - \overline{\partial}\rho_0 = \alpha - \gamma + \gamma = \alpha.
\]
But this contradicts the assumption that \([\alpha] \neq 0\) in \(H^1_c(\Omega, \mathcal{O})\), so the mapping cannot be surjective.

Fix a point \(x_0 \in E\), let \(\Gamma\) be the image of \(\pi_1(E, x_0)\) in \(\pi_1(\Omega, x_0)\), and let \(\pi : \tilde{\Omega} \to \Omega\) be a connected covering space with \(\pi_* (\pi_1(\tilde{\Omega}, x_1)) = \Gamma\) for some point \(x_1 \in \pi^{-1}(x_0)\). Since \(\Gamma\) is a proper subgroup and \(E\) is a \(C^\infty\) domain, \(\pi\) maps a neighborhood of the closure \(\overline{E}_1\) of the connected component \(E_1\) of \(\tilde{E} = \pi^{-1}(E)\) containing \(x_1\) isomorphically onto a neighborhood of \(\overline{E}\) and \(\tilde{\Omega} \setminus \overline{E}_1\) is not relatively compact in \(\tilde{\Omega}\) (i.e. \(\Omega\) has at least two ends). In particular, \(E_1\) is a hyperbolic end of \(\tilde{\Omega}\) with respect to the complete \(\text{Kähler}\) metric \(\pi^* g'\). Therefore, by Lemma 2.4, there exists a pluriharmonic function \(\tau\) on \(\tilde{\Omega}\) such that \(0 \leq \tau \leq 1\), \(\tau\) has finite energy, and, for every regular sequence \(\{x_\nu\}\) in \(\tilde{\Omega}\) approaching \(\infty\),
\[
\lim_{\nu \to \infty} \tau(x_\nu) = \begin{cases} 1 & \text{if } x_\nu \in E_1 \text{ for } \nu \gg 0 \\ 0 & \text{if } x_\nu \in \tilde{\Omega} \setminus E_1 \text{ for } \nu \gg 0. \end{cases}
\]
Since \(\pi\) maps \(E_1\) isomorphically onto \(E\), the restriction \(\tau|_{E_1}\) determines a pluriharmonic function on \(E\). If the differential of this function and the differential of the pluriharmonic function \(\rho\) are linearly independent on \(E\), then, by Proposition 2.3, \(\Omega\) satisfies (BHD). Therefore, since \(\rho\) is nonconstant, it now suffices to assume that there exist real constants \(r\) and \(s\) such that \(\rho \circ \pi = r\tau + s\) on \(E_1\) and to obtain a contradiction.

The first observation is that if \(\tilde{\gamma} = \pi^* \gamma\), then \(\tilde{\gamma}\) is a closed form of type \((0,1)\) on \(\tilde{\Omega}\) which is equal to the form \(-\overline{\partial}(r\tau + s)\) on the nonempty open set \(E_1\) and hence on the entire set \(\tilde{\Omega}\). Therefore, on the nonempty open set \(\tilde{E} \setminus \overline{E}_1\), we have
\[
-\overline{\partial}(\rho \circ \pi) = \tilde{\gamma} = -\overline{\partial}(r\tau + s).
\]
Hence the restriction of the function \((\rho \circ \pi) - (r\tau + s)\) to \(\tilde{E} \setminus \overline{E}_1\) is real-valued and holomorphic and is therefore locally constant. Thus if \(E_2\) is a connected component of \(\tilde{E}\) which is not equal to \(E_1\), then, for some real constant \(s'\), we have \(\rho \circ \pi = r\tau + s'\) on \(E_2\). Now since \(\pi(E_1) = \pi(E_2) = E\), we may choose a regular sequence \(\{x_\nu\}\) in \(E\) and sequences \(\{y_\nu\}\) and \(\{z_\nu\}\) in \(E_1\) and \(E_2\), respectively, such that \(\pi(y_\nu) = \pi(z_\nu) = x_\nu\) for each \(\nu\). The sequences \(\{y_\nu\}\) and \(\{z_\nu\}\) are then regular sequences in \(\tilde{\Omega}\), because the lifting of the function \(v = -G(x_0, \cdot)\) to \(\tilde{\Omega}\) is a negative subharmonic function and \(v(y_\nu), v(z_\nu) \to 0\). Therefore \(\tau(y_\nu) \to 1\) and \(\tau(z_\nu) \to 0\). Since \(\rho\) vanishes at infinity along any regular sequence, we get
\[
0 = \lim \rho(x_\nu) = \lim (r\tau(y_\nu) + s) = r + s
\]
and
\[ 0 = \lim \rho(x_\nu) = \lim (r\tau(z_\nu) + s') = s'. \]

Therefore, for each point \( x \in E \) and each pair of points \( y \in E_1 \cap \pi^{-1}(x) \) and \( z \in E_2 \cap \pi^{-1}(x) \), we have \( \tau(y) - 1 = r^{-1}\rho(x) = \tau(z) \). Since \( 0 < \tau < 1 \) on \( \overline{\Omega} \), this is not possible. Thus the proof is complete for the cases (i) and (ii).

Finally, suppose \( \varphi \) is of class \( C^\infty \) (i.e. the condition (iii) holds). As in the case (i), we fix \( M_0, a, \) and \( \Omega \). Here, we choose \( a \) to be a regular value of \( \varphi \) and we let \( g' \) be the restriction of \( g \) to \( \Omega \). It suffices to show that there exists an arbitrarily large choice of \( a \) for which \( \Omega \) admits a proper holomorphic mapping \( \Psi \) onto a Riemann surface. For if \( \Psi \) is such a mapping, then \( \varphi \) is constant on each level of \( \Psi \). Hence, near a generic point of \( \Omega \), there exist holomorphic coordinates \( (z_1, \ldots, z_n) \) in which \( \varphi \) is a function of \( z_1 \) and we get
\[
\mathcal{L}(\varphi) = \frac{\partial^2 \varphi}{\partial z_1 \partial \bar{z}_1} dz_1 d\bar{z}_1 = (2g^{11})^{-1} \Delta \varphi dz_1 d\bar{z}_1 \geq 0.
\]
Thus it will follow that \( \varphi \) is plurisubharmonic on each of the domains \( \Omega \). Letting \( a \to \infty \), it will then follow immediately that \( \varphi \) is plurisubharmonic on \( M \) as in the case (i) and the proof in the case (iii) will be complete.

As in the proof of Theorem 1.6 (Grauert-Riemenschneider and Siu), since the boundary of \( \Omega \) is regular, we have \( \gamma = \alpha - \partial \beta \) on \( \Omega \) where \( \gamma \) is harmonic and \( \beta \) vanishes on \( \partial \Omega \). The metric \( g' \) is not complete, but the proof of Theorem 1.6 shows that \( \gamma \) is closed. In particular, \( \beta \) is pluriharmonic on the connected set \( E = \Omega \setminus \overline{M_0} \). Moreover, since \( [\alpha] \neq 0 \), \( \beta \) is nonconstant on \( E \).

If \( \partial \Omega \) is not connected, then we may form a \( C^\infty \) function \( \tau \) on \( \overline{\Omega} \) which is harmonic on \( \Omega \) and locally constant, but not constant, on \( \partial \Omega \). By Theorem 1.6 (a), \( \tau \) is then pluriharmonic on \( \Omega \) and, since \( \beta \) vanishes on \( \partial \Omega \), \( d\tau \) and \( d\beta \) are linearly independent on \( E \). Therefore, by Proposition 2.3, \( \Omega \) admits a proper holomorphic mapping onto a Riemann surface. Thus we may assume that \( \partial \Omega \) is connected.

If \( a' \) is a regular value in the interval \( (\sup_{M_0} \varphi, a) \) which is sufficiently close to \( a \) and \( \Omega' \) is the connected component of \( \{ x \in M \mid \varphi(x) < a' \} \) containing \( \overline{M_0} \), then \( \Omega' \subset \subset \Omega \) and the set \( E' = \Omega' \setminus \overline{M_0} \) is connected. As above, we may write \( \gamma' = \alpha - \partial \beta' \) on \( \Omega' \) where \( \gamma' \) harmonic and \( \beta' \) vanishes on \( \partial \Omega' \). If \( d\beta \) and \( d\beta' \) are linearly independent on \( E' \), then Proposition 2.3 implies that \( \Omega' \) satisfies (BHD). If \( d\beta \) and \( d\beta' \) are linearly dependent, then \( \beta \) is constant on \( \partial \Omega' \). Since \( \beta \) is nowhere locally constant in \( \Omega \setminus \overline{\Omega'} \) and \( \beta \equiv 0 \) on \( \partial \Omega \), the maximum principle implies then that \( \beta \) is equal to a nonzero
constant on $\partial \Omega'$. Hence the restriction of the real part or the imaginary part of $\beta$ to $\Omega \setminus \overline{\Omega'}$ is a positive or negative pluriharmonic function which vanishes at points in $\partial \Omega$. It follows that $\Omega$ admits a plurisubharmonic exhaustion function and hence, a proper holomorphic mapping onto a Riemann surface. Therefore, by the above remarks, $M$ satisfies (BHD).

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