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ON GRADIENTS OF FUNCTIONS
DEFINABLE IN O-MINIMAL STRUCTURES

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0. Introduction.

Many results in subanalytic or semialgebraic geometry of $\mathbb{R}^n$ hold true in a more general setting called “the theory of o-minimal structures on the real field” (see [DM]). This theory has presented a strong interest since 1991 when A. Wilkie [W1] proved that a natural extension of the family of semialgebraic sets containing the exponential function ($(\mathbb{R}, \exp)$-definable sets) is an o-minimal structure. A similar extension of subanalytic sets ($(\mathbb{R}_{an}, \exp)$-definable sets) was then treated by L. van den Dries, A. Macintyre, D. Marker in [DMM] (geometric proofs of these facts were found recently by J-M. Lion and J.-P. Rolin [LR1], [LR2]). Another type of o-minimal structure ($(\mathbb{R}^K_{an})$-definable sets) was obtained by C. Miller [Mi], by adding to subanalytic sets all functions $x \to x^r$, $r \in K$, where $K$ is a subfield of $\mathbb{R}$. We give a list and examples of o-minimal structures in section 1. An extension of semialgebraic and subanalytic geometry was also undertaken by M. Shiota [S1], [S2].

Theorem 1 (Section 2), the first main result of this paper, is an o-minimal generalization of the famous Łojasiewicz inequality $\|\nabla f\| \geq |f|^\alpha$ with $\alpha < 1$, where $f$ is an analytic function in a neighborhood of $a \in \mathbb{R}^n$, $f(a) = 0$. We prove that if $f$ is a differentiable function in a

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bounded domain, definable in some o-minimal structure, then there exists a $C^1$ function $\Psi$ in one variable such that $\|\text{grad} \Psi \circ f\| \geq c > 0$. It is rather surprising that the result holds also for infinitely flat functions. Theorem 1 implies that the set of asymptotic critical values of $f$ is finite (Proposition 2). We recall in the beginning of the section the already known o-minimal version of another Lojasiewicz inequality for continuous definable functions on a compact set.

The main result of Section 3 is Theorem 2 which states: if $U$ is an open, bounded subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ is a $C^1$ function definable in some o-minimal structure, then all trajectories of $-\text{grad} f$ (i.e. solutions of the equation $\dot{x} = -\text{grad} f$) have their length bounded by a constant independent of the trajectory. The function $f$ may be unbounded and may not have a continuous extension on $\bar{U}$. We prove also, that for a non negative definable $g$, the flow of $-\text{grad} g$ defines a deformation retraction onto $g^{-1}(0)$. Some applications of this result in the real analytic case can be found in [Si], [Sj]. We finish the paper by a discussion of Thom’s Gradient Conjecture for o-minimal structures.

In Section 1 we gather basic facts on o-minimal structures. To make the paper self-contained and accessible for a wider audience we add a proof of Lemma 2 (on definable functions in one variable). We give also an elementary proof (suggested by C. Miller and J-M. Lion) of the curve selection lemma, the crucial tool in the proof of Theorem 1.

General references of various facts, when not specified, will be as follows: for semialgebraic geometry – [BCR], for subanalytic geometry – [BM] or [L4], for o-minimal structures – [DM].

In this paper we take the gradient with respect to the canonical euclidian metric in $\mathbb{R}^n$.

1. o-minimal structures on the real field.

**Definition 1.** — Let $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$, where each $\mathcal{M}_n$ is a family of subsets of $\mathbb{R}^n$. We say that the collection $\mathcal{M}$ is an o-minimal structure on $(\mathbb{R}, +, \cdot)$ if:

1. each $\mathcal{M}_n$ is closed under finite set-theoretical operations;
2. if $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$;
(3) let $A \in \mathcal{M}_{n+m}$ and $\pi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ be projection on the first $n$ coordinates, then $\pi(A) \in \mathcal{M}_n$;

(4) let $f, g_1, \ldots, g_k \in \mathbb{Q}[X_1, \ldots, X_n]$, then $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\} \in \mathcal{M}_n$;

(5) $\mathcal{M}_1$ consists of all finite unions of open intervals and points.

For a fixed o-minimal structure $\mathcal{M}$ on $(\mathbb{R}, +, \cdot)$ we say that $A$ is an $\mathcal{M}$-set if $A \in \mathcal{M}_n$ for some $n \in \mathbb{N}$. We say that a function $f : A \rightarrow \mathbb{R}^m$, where $A \subset \mathbb{R}^n$, is an $\mathcal{M}$-function if its graph is an $\mathcal{M}$-set.

Axiom (5) will be called the o-minimality of $\mathcal{M}$.

Examples. — We give below a list of o-minimal structures on $(\mathbb{R}, +, \cdot)$ (see also [DM] for detailed definitions and comparisons between the above examples) with examples of functions definable in those o-minimal structures:

(1) Semialgebraic sets (by Tarski-Seidenberg); $f(x, y) = \sqrt{x^4 + y^4}$.

(2) Global subanalytic sets (by Gabrielov);

$$f(x, y) = \frac{y}{\sin x}, x \in (0, \pi).$$

(3) $(\mathbb{R}, \exp)$-definable sets (by Wilkie);

$$f(x, y) = x^2 \exp \left(-\frac{y^2}{x^4 + y^2}\right) \ln x.$$

(4) $(\mathbb{R}_{an}, \exp)$-definable sets (by van den Dries, Macintyre, Marker);

$$f(x, y) = x^{\sqrt{2}} \ln(\sin y), x > 0, y \in (0, \pi).$$

(5) $(\mathbb{R}_{an})$-definable sets (by Miller);

$$f(x, y) = x^{\sqrt{2}} \exp \left(\frac{x}{y}\right), 0 < x < y < 1.$$

Recently another example of an o-minimal structure was found by van den Dries and Speissegger [DS] which is larger than $\mathbb{R}_{an}$ but polynomially bounded (i.e. any definable function in one variable is bounded by a polynomial at infinity). Finally we mention a result of Wilkie [W2] in which he gives a general method for construction of o-minimal structures; this method can be applied to Pfaffian functions.

In the rest of this paper $\mathcal{M}$ will denote some fixed, but arbitrary, o-minimal structure on $(\mathbb{R}, +, \cdot)$. We will give now several elementary properties of $\mathcal{M}$-sets and $\mathcal{M}$-functions.

Remark 1. — Let $E$ be an $\mathcal{M}$-set in $\mathbb{R}^{n+1}$. Axioms (1)--(4) imply
that the sets
\[ \{ x \in \mathbb{R}^n : \exists x_{n+1} \ (x, x_{n+1}) \in E \} \quad \text{and} \quad \{ x \in \mathbb{R}^n : \forall x_{n+1} \ (x, x_{n+1}) \in E \} \]
are \( \mathcal{M} \)-sets. Actually the first set is the image of \( E \) by projection, the second is the complement of the image of the complement of \( E \) by projection.

**Remark 2.** — The sum, product, inverse, composition of \( \mathcal{M} \)-functions is again an \( \mathcal{M} \)-function. Also the image and inverse image of an \( \mathcal{M} \)-set by an \( \mathcal{M} \)-function are again \( \mathcal{M} \)-sets. Proofs of these facts are quite standard applications of Remark 1 and axioms (1)-(4) and actually the same as in the semialgebraic case (see e.g. [BCR]).

**Lemma 1.** — Let \( f : A \rightarrow \mathbb{R} \) be an \( \mathcal{M} \)-function such that \( f(x) \geq 0 \) for all \( x \in A \). Let \( G : A \rightarrow \mathbb{R}^m \) be an \( \mathcal{M} \)-mapping and define a function
\[ \varphi : G(A) \rightarrow \mathbb{R} \]
by
\[ \varphi(y) = \inf_{x \in G^{-1}(y)} f(x). \]
Then \( \varphi \) is an \( \mathcal{M} \)-function.

**Proof.** — Write a formula for the graph of the function \( \varphi \) and apply Remark 1.

**Corollary 1.** — Let \( A \) be an \( \mathcal{M} \)-set in \( \mathbb{R}^n \). Then the distance function \( d_A : \mathbb{R}^n \rightarrow \mathbb{R} \) is an \( \mathcal{M} \)-function, where \( d_A(x) = \inf_{y \in A} |x - y| \).

**Corollary 2.** — Let \( A \) be an \( \mathcal{M} \)-set in \( \mathbb{R}^n \). Then \( \overline{A} \) and \( \text{Int } A \) are \( \mathcal{M} \)-sets.

**Proof.** — Actually by Corollary 1 we know that \( d_A \) is an \( \mathcal{M} \)-function, hence \( \overline{A} = d_A(0)^{-1} \) is an \( \mathcal{M} \)-set. To prove that the interior of \( A \) is an \( \mathcal{M} \)-set we use the fact that by axiom (1) the complement of an \( \mathcal{M} \)-set is an \( \mathcal{M} \)-set.

**Lemma 2 (Monotonicity Theorem).** — Let \( f : (a, b) \rightarrow \mathbb{R} \) be an \( \mathcal{M} \)-function. Then there exist real numbers \( a = a_0 < a_1 < \ldots < a_k = b \) such that \( f \) is continuously differentiable on each interval \((a_i, a_{i+1})\). Moreover \( f' \) is an \( \mathcal{M} \)-function and the function \( f \) is strictly monotone or constant on every interval \((a_i, a_{i+1})\).

**Proof (Due essentially to van den Dries [vD]).** — We may assume that the set \( f((a, b)) \) is infinite. First we prove that \( D(f) \), the set of discontinuity points of \( f \), is finite.
Writing the definition of continuity of a function at a point and using Remark 1 we deduce that $D(f)$ is an $M$-set in $\mathbb{R}$, hence by o-minimality, it is enough to prove that $f$ is continuous at some point of $(a, b)$. Since the set $f((a, b))$ is an infinite $M$-set it contains an open interval. Thus by induction we can construct a descending sequence of intervals $[\alpha_n, \beta_n] \subset (a, b)$ such that $\alpha_n < \alpha_{n+1}, \beta_{n+1} < \beta_n, \beta_n - \alpha_n < 1/n$ and $f([\alpha_n, \beta_n])$ is contained in an interval of length smaller than $1/n$. Clearly $f$ is continuous at the point $\cap_{n \in \mathbb{N}} [\alpha_n, \beta_n]$. So we have proved that the complement of $D(f)$ is dense in $(a, b)$, hence $D(f)$ is finite.

We can assume now that $f$ is continuous on $(a, b)$. To prove differentiability observe first that by o-minimality we have:

**Observation.** — For each $x \in (a, b)$ and each $c \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that $f(t) \geq f(x) + c(t - x)$ for all $t \in (x, x + \varepsilon)$ or $f(t) \leq f(x) + c(t - x)$ for all $t \in (x, x + \varepsilon)$.

Let us write $f'_-(x) = \lim_{t \to x} \frac{1}{t - x} (f(x + t) - f(x))$ for $x \in (a, b)$ and $f'_+(x) = \lim_{t \to x} \frac{1}{t - x} (f(x + t) - f(x))$ for $x \in [a, b)$. Note that $f'_+$ and $f'_-$ are $M$-functions, by Remark 1. From the above observation it is not difficult to obtain the following consequences:

i) for each $x \in (a, b)$ the values of $f'_-(x)$ and $f'_+(x)$ are well defined (possibly equal to $+\infty$ or $-\infty$),

ii) for each $x \in (a, b)$ there exists $y$ arbitrary close to $x$, $y > x$ such that $f'_+(y) \leq f'_+(x), f'_-(y) \leq f'_+(x)$ or $f'_+(y) \geq f'_+(x), f'_-(y) \geq f'_+(x)$.

Clearly the sets

$$\{x \in (a, b); f'_+(x) = +\infty\}, \{x \in (a, b); f'_+(x) = -\infty\}$$

are $M$-sets, hence are finite unions of open intervals and points. By ii) these sets are finite. So we can assume that $f'_+$ and $f'_-$ take values in $\mathbb{R}$. Since $f'_+$ and $f'_-$ are $M$-functions we may also assume that these functions are continuous on $(a, b)$. It follows easily now from ii) that $f'_+ = f'_-$ on $(a, b)$, but this means that $f$ is $C^1$ on $(a, b)$.

We proved also that $f'$ is an $M$-function, hence the claim on monotonicity follows from the fact that $\{f' = 0\}$ is an $M$-set and so is a finite union of points and open intervals.

Writing the definition of partial derivatives and using Remark 1 we obtain:
Lemma 3. — Let \( f : U \to \mathbb{R}^k \) be a differentiable \( \mathcal{M} \)-function, where \( U \) is open in \( \mathbb{R}^n \). Then \( \partial f / \partial x_j, j = 1, \ldots, n \) are \( \mathcal{M} \)-functions, and hence \( \text{grad } f \) is an \( \mathcal{M} \)-mapping.

Proposition 1 (Curve Selection Lemma). — Let \( A \) be an \( \mathcal{M} \)-set in \( \mathbb{R}^n \) and suppose that \( a \in A \setminus \{a\} \). Then there exists an \( \mathcal{M} \)-function \( \gamma : [0, \varepsilon) \to \mathbb{R}^n \) which is \( C^1 \) on \( [0, \varepsilon) \) and such that

\[
a = \gamma(0) \quad \text{and} \quad \gamma((0, \varepsilon)) \subset A \setminus \{a\}.
\]

Proof. — The key point is to construct a “definable” selection operator \( e \), which assigns to each nonempty set \( A \in \mathcal{M}_n \) an element \( e(A) \in A \). Let \( n = 1 \). Then \( e(A) \) is the smallest element of \( A \) if \( A \) has one. Otherwise, let \( a := \inf A \) and let \( b \in \mathbb{R} \cup \{+\infty\} \) be maximal such that \( (a, b) \subseteq A \). If \( a, b \in \mathbb{R} \), then \( e(A) := (a + b)/2 \). If \( a \in \mathbb{R} \) and \( b = +\infty \), then \( e(A) := a + 1 \). If \( a = -\infty \) and \( b \in \mathbb{R} \), then \( e(A) := b - 1 \). If \( a = -\infty \) and \( b = +\infty \) (i.e., \( A = \mathbb{R} \)), then \( e(A) := 0 \). Assume \( e(A) \) has been defined for all nonempty \( A \in \mathcal{M}_n \). Let \( B \in \mathcal{M}_{n+1} \) be nonempty, and let \( A \) be its image in \( \mathbb{R}^n \) under the projection map \( (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n) \). Put \( a := e(A) \). Then

\[
e(B) := (a, e(B_a)) \text{ where } B_a := \{r \in \mathbb{R} : (a, r) \in B\}.
\]

This selection operator \( e \) has several applications, and Curve Selection is only one of them: let \( A \in \mathcal{M}_n \) and \( a \in A \setminus \{a\} \). By \( \mathcal{O} \)-minimality the set \( \{a - x : x \in A\} \in \mathcal{M}_1 \) contains an interval \( (0, \varepsilon), \varepsilon > 0 \). For \( 0 < t < \varepsilon \), let

\[
\gamma(t) := e(\{x \in A : |a - x| = t\}).
\]

It is routine to check that \( \gamma : (0, \varepsilon) \to A \) belongs to \( \mathcal{M} \). By the monotonicity theorem \( \gamma \) is \( C^1 \) after suitable shrinking of \( \varepsilon \). After composition on the right with a sufficiently flat (at 0) function in \( \mathcal{M} \) (e.g. the inverse of the biggest component of \( \gamma \)) we can further arrange that \( \gamma \) extends to a \( C^1 \)-function on \( [0, \varepsilon) \).

2. Lojasiewicz inequalities for \( \mathcal{O} \)-minimal structures.

We begin this section recalling an already well-known generalization of the Lojasiewicz inequality for continuous \( \mathcal{M} \)-functions on a compact set. This result was observed by T. Loi [Lo] for (\( \mathbb{R} \), exp)-definable sets (actually his version is more precise than the theorem stated below); M. Shiota [S1], [S2] and L. van den Dries and C. Miller [DM] also noticed this fact.

Theorem 0. — Let \( K \) be a compact subset of \( \mathbb{R}^n \) and let \( f, g : K \to \mathbb{R} \) be two continuous \( \mathcal{M} \)-functions. If \( f^{-1}(0) \subset g^{-1}(0) \), then there
exists a strictly increasing positive $\mathcal{M}$-function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ of class $C^1$, such that for any $x \in K$ we have

$$|f(x)| \geq \sigma(g(x)).$$

The idea of the proof goes back to the original argument of Lojasiewicz (see [L2], [KLZ]). Let $\Sigma \subset \mathbb{R}^2$ be the image of $K$ by the mapping $K \ni u \mapsto (g(u), f(u)) = (x, y)$. Clearly $\Sigma$ is an $\mathcal{M}$-set; moreover it is compact and $\Sigma \cap \{y = 0\} = \{(0,0)\}$. It is not difficult to find (by Lemma 2) a strictly increasing positive $\mathcal{M}$-function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ of class $C^1$, such that $\Sigma \subset \{y \geq \sigma(x), x \geq 0\}$. It is proved in [DM] that for each $k \in \mathbb{N}$ one can find $\sigma$ of class $C^k$.

We state now the main result of this section. Recall that $\mathcal{M}$ is any fixed o-minimal structure on $(\mathbb{R}, +, \cdot)$.

**THEOREM 1.** — Let $f : U \to \mathbb{R}$ be a differentiable $\mathcal{M}$-function, where $U$ is an open and bounded subset of $\mathbb{R}^n$. Suppose that $f(x) > 0$ for all $x \in U$. Then there exists $c > 0$, $\rho > 0$ and a strictly increasing positive $\mathcal{M}$-function $\Psi : \mathbb{R}_+ \to \mathbb{R}$ of class $C^1$, such that

$$\|\text{grad} (\Psi \circ f)(x)\| \geq c,$$

for each $x \in U$, $f(x) \in (0, \rho)$.

The proof is given in the end of the section. We shall see now that in the subanalytic case our Theorem 1 is equivalent to the classical Lojasiewicz inequality for gradients of analytic functions (see [L1], [L2], [BM]). We state this result in the form generalized in [KP]:

**THEOREM (LI).** — Let $f : \Omega \to \mathbb{R}$ be a subanalytic function which is differentiable in $\Omega \setminus f^{-1}(0)$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$. Then there exist $C > 0$, $\rho > 0$ and $0 \leq \alpha < 1$ such that:

$$\|\text{grad} f(x)\| \geq C|f(x)|^\alpha,$$

for each $x \in \Omega$ such that $|f(x)| \in (0, \rho)$. If in addition $\lim_{x \to a} f(x) = 0$ for some $a \in \overline{\Omega}$ (which holds in the classical case, where $f$ is analytic and $a \in \Omega$, $f(a) = 0$), then the above inequality holds for each $x \in \Omega \setminus f^{-1}(0)$ close to $a$.

To see that in the subanalytic case (LI) $\Rightarrow$ Theorem 1 it is enough to put $\Psi(t) = t^{1-\alpha}$. To prove the converse in the subanalytic case, recall first that every subanalytic function in one variable is actually
semianalytic (see [L2], [KLZ]). Hence \( \Psi \) has the Puiseux expansion of the form \( \Psi(t) = \sum_{\nu=0}^{\infty} a_{\nu} t^{\nu/k} \). Thus, for \( t \) small enough we have \( |\Psi'(t)| \leq D t^{k-1} \) for some \( D > 0 \). The last inequality and Theorem 1 yield
\[
\| \text{grad} f(x) \| = \frac{\| \text{grad} (\Psi \circ f)(x) \|}{|\Psi'(f(x))|} \geq \frac{c}{D} |f(x)|^{1-\frac{1}{k}}.
\]

Remark. — The above argument and Theorem 1 imply that (LI) holds in any polynomially bounded o-minimal structure on \((\mathbb{R}, +, \cdot)\).

We discuss now a consequence of Theorem 1. Let \( f : U \to \mathbb{R} \) be a differentiable function, where \( U \) is an open subset of \( \mathbb{R}^n \). We shall say that \( \lambda \in \mathbb{R} \cup \{ -\infty, +\infty \} \) is an asymptotic critical value of \( f \) if there exists a sequence \( x_n \in U \) such that
\[
f(x_n) \to \lambda \quad \text{and} \quad \text{grad} f(x_n) \to 0.
\]
Clearly any "true" critical value of \( f \) (i.e., \( \lambda = f(x) \) and \( \text{grad} f(x) = 0 \), for some \( x \in U \)) is also an asymptotic critical value. Notice that this notion depends heavily on the domain \( U \), in particular on whether \( U \) is bounded or not.

Suppose now that \( U \) is bounded and that our \( f \) is an \( M \)-function, where \( M \) is an o-minimal structure on \((\mathbb{R}, +, \cdot)\). Let \( \lambda \) be an asymptotic critical value of \( f \). It follows immediately from Theorem 1 that \( f \) has no asymptotic critical values in \((\lambda - \rho, \lambda) \cup (\lambda, \lambda + \rho)\) for some \( \rho > 0 \). But on the other hand the set of all asymptotic critical values of \( f \) is an \( M \)-subset of \( \mathbb{R} \), so it must be finite. Thus we have proved:

PROP 2. — If \( U \) is bounded and \( f \) is an \( M \)-function, then the set of all asymptotic critical values of \( f \) is finite.

It is easily seen that \(-\infty\) and \(+\infty\) cannot be an asymptotic critical value of an \( M \)-function defined in a bounded set. As the following example shows the assumption of boundness on \( U \) is necessary.

Example. — The function \( f(x, y) = \frac{x}{y} \) on \( U = \{ y > 0 \} \subset \mathbb{R}^2 \), being semialgebraic, belongs to any o-minimal structure on \((\mathbb{R}, +, \cdot)\). But clearly any \( \lambda \in \mathbb{R} \) is an asymptotic critical value of \( f \).

Proof of Theorem 1. — It follows from Lemma 3 that \( U \ni x \mapsto \| \text{grad} f(x) \| \) is an \( M \)-function. We may suppose that \( f^{-1}(t) \neq \emptyset \) for any
small enough $t > 0$, since otherwise, by o-minimality, the theorem is trivial. Hence the function

$$\varphi(t) = \inf\{\|\text{grad } f(x)\| : x \in f^{-1}(t)\}$$

is well-defined in some interval $(0, \varepsilon)$. By Lemma 1, $\varphi$ is an $\mathcal{M}$-function.

**CLAIM.** There exists $\varepsilon' > 0$ such that $\varphi(t) > 0$ for any $t \in (0, \varepsilon')$.

Assume that this is not the case and put

$$\Sigma = \{x \in U : \|\text{grad } f(x)\| < (f(x))^2\}.$$

Clearly $\Sigma$ is an $\mathcal{M}$-set. Let $f|\Sigma$ denote the graph of $f$ restricted to $\Sigma$. If the claim doesn’t hold, then there exists a sequence of positive numbers $t_n \to 0$ such that $\varphi(t_n) = 0$ for all $n \in \mathbb{N}$. Let $x_n \in \Sigma$ be a sequence such that $f(x_n) = t_n$, in other words $(x_n, t_n) \in f|\Sigma$. Let $b$ be an accumulation point of $\{x_n\}$, then $(b, 0)$ belongs to the closure of the set $(f|\Sigma \setminus \{(b, 0)\})$. By the curve selection lemma (Proposition 1) we have an $\mathcal{M}$-function (arc) $\gamma : (-\delta, \delta) \to \mathbb{R}^n \times \mathbb{R}$ of class $C^1$, such that $\gamma(0) = (b, 0)$, and $\gamma(0, \delta) \subset f|\Sigma$. Write $\gamma(s) = (\gamma(s), f \circ \gamma(s))$, where $\gamma(s) \in \Sigma \subset \mathbb{R}^n$. Let $h(s) = f \circ \gamma(s)$ for $s \in (0, \delta)$, then clearly $\lim_{s \to 0} h(s) = 0 = \lim_{s \to 0} h'(s)$, since $\gamma(s) \in \Sigma$. Of course $h$ and $h'$ are $\mathcal{M}$-functions, so by Lemma 2 we may suppose that $h$ and $h'$ are monotone; actually they must be strictly increasing. Thus we have

$$0 < h'(s) \leq A(h(s))^2, \quad \text{for } s \in (0, \delta),$$

where $A$ is a bound for $\|\gamma'(s)\|$. But by the Mean Value Theorem we have $h(s) \leq s h'(s)$, because $h'$ is increasing. Finally, we get $0 < h'(s) \leq A s^2 (h'(s))^2$ for any $s \in (0, \delta)$, which is impossible since $\lim_{s \to 0} h'(s) = 0$.

So we have proved that $\varphi(t) > 0$ for all $t \in (0, \varepsilon)$, provided that $\varepsilon > 0$ is small enough. We define now:

$$\Delta = \{x \in U \setminus f^{-1}(0) : f(x) < \varepsilon, \|\text{grad } f(x)\| \leq 2 \varphi(f(x))\}.$$

Observe that $\Delta$ is also an $\mathcal{M}$-set and moreover $\Delta \cap f^{-1}(t) \neq \emptyset$ for every $t \in (0, \varepsilon)$. Hence as before there exists $d \in \overline{U}$ such that $(d, 0) \in f|\Delta \setminus \{(d, 0)\}$. Applying again the curve selection lemma to $f|\Delta$ at the point $(d, 0)$ we obtain an $\mathcal{M}$-function (arc) $\eta : (-\delta, \delta) \to \mathbb{R}^n$ of class $C^1$, such that $\eta(0) = (d, 0)$, and $\eta(0, \delta) \subset f|\Delta$. Write as before $\eta(s) = (\eta(s), f \circ \eta(s))$, where $\eta(s) \in \Delta \subset \mathbb{R}^n$. Let $g(s) = f \circ \eta(s)$ for $s \in (0, \delta)$, then clearly $\lim_{s \to 0} g(s) = 0$ and $g(s) > 0$ for each $s \in (0, \delta)$. It follows from Lemma 2 that for $\delta' > 0$ small enough the function $g : (0, \delta') \to \mathbb{R}$ is a diffeomorphism onto $(0, \rho)$, for some $\rho > 0$. We put

$$\Psi(t) = g^{-1}(t) \quad \text{for } t \in (0, \rho).$$
We shall check now the inequality claimed in Theorem 1. Let $B$ be some bound for $\|\eta'(s)\|$ in $(0, \delta')$. Take any $x \in U$ such that $t = f(x) \in (0, \rho)$, and write $s = \Psi(t) = g^{-1}(t)$. Then we have

$$\|\nabla \Psi \circ f(x)\| = \Psi'(f(x))\|\nabla f(x)\| \geq \frac{1}{2B} (f \circ \eta)'(s) = \frac{1}{2B} = c,$$

since $\|\nabla f(\eta(s))\| \|\eta'(s)\| \geq \langle \nabla f(\eta(s)), \eta'(s) \rangle = (f \circ \eta)'(s)$ and $B \geq \|\eta'(s)\|$. Theorem 1 follows.

3. Trajectories of gradients of $M$-functions.

Let $f : U \rightarrow \mathbb{R}$ be a $C^1$ function, where $U$ is an open subset of $\mathbb{R}^n$. We shall consider a vector field,

$$U \ni x \mapsto -\nabla f(x) \in \mathbb{R}^n.$$

Let $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$. We shall say that $\gamma : (\alpha, \beta) \rightarrow U$ is a trajectory of the vector field $-\nabla f$ if it is a maximal differentiable curve verifying $\gamma'(t) = -\nabla f(\gamma(s))$. Actually we shall consider $\gamma$ as an equivalence class of all curves obtained from $\gamma$ by a strictly increasing $C^1$ reparametrization. Observe that if $\psi$ is an increasing $C^1$ diffeomorphism between two intervals in $\mathbb{R}$, then the trajectories of $-\nabla \psi \circ f$ and those of $-\nabla f$ are the same.

Let $a, b \in \gamma$. We denote by $|\gamma(a, b)|$ the length of $\gamma$ between $a$ and $b$.

Lojasiewicz derived (see [L1], [L3]) from (LI) that all trajectories of $-\nabla f$ are of finite length, when $f$ is analytic in a neighborhood of a compact $\overline{U}$. We have:

**Theorem 2.** — Let $f : U \rightarrow \mathbb{R}$ be a function of class $C^1$, where $U$ is an open and bounded subset of $\mathbb{R}^n$. Suppose that $f$ is an $M$-function, for some $\sigma$-minimal structure $M$.

a) Then there exists $A > 0$ such that all trajectories of $-\nabla f$ have length bounded by $A$.

b) More precisely, there exists $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a continuous strictly increasing $M$-function, with $\lim_{t \rightarrow 0} \sigma(t) = 0$, such that if $\gamma$ is a trajectory of $-\nabla f$ and $a, b \in \gamma$, then

$$|\gamma(a, b)| \leq \sigma(|f(b) - f(a)|).$$
Proof of theorem 2. — Taking, if necessary the composition \( \psi \circ f \), where \( \psi(t) = \frac{t}{\sqrt{1 + t^2}} \), we may suppose that \( f \) is bounded; more exactly that the image of \( f \) lies in \((-1,1)\). We consider again the \( \mathcal{M} \)-function \( \varphi : (-1,1) \to \mathbb{R} \) defined by
\[
\varphi(t) = \inf \{ \| \nabla f(x) \| : x \in f^{-1}(t) \},
\]
when \( f^{-1}(t) \neq \emptyset \), and \( \varphi(t) = 1 \) when \( f^{-1}(t) = \emptyset \). Let \( \Sigma \) be the set of all asymptotic critical values of \( f \). Observe that \( \lambda \in \Sigma \) if \( \varphi(\lambda) = 0 \), or \( \lim_{t \to \lambda} \varphi(t) = 0 \), or \( \lim_{t \to \lambda} \varphi(t) = 0 \).

Let \( I \subset (-1,1) \) be an open interval. Assume that \( \varphi \) is bounded from below in \( I \) by some \( c > 0 \). Let \( \gamma \) be a trajectory of \( -\nabla f \) and \( a, b \in \gamma \). Suppose that the part of \( \gamma \) lying between \( a \) and \( b \) is contained in \( f^{-1}(I) \). We parametrise \( \gamma \) by arc-length (i.e \( \| \gamma'(s) \| = 1 \)), so by the Mean Value Theorem we have that \( |f \circ \gamma(\beta) - f \circ \gamma(\alpha)| \geq c|\beta - \alpha| \), in other words
\[
|\gamma(a, b)| \leq \frac{1}{c}|f(b) - f(b)|.
\]
This observation explains the idea of the proof. By a partition \( -1 = t_0 < t_1 < \ldots < t_k = 1 \) we shall decompose \((-1,1)\) in such a way that \( \varphi \) is strictly monotone on \((t_i, t_{i+1})\). Moreover we shall distinguish between two disjoint types of intervals, namely

1. there exists \( c_i > 0 \) such that \( \varphi(t) \geq c_i \) on \((t_i, t_{i+1})\) (we write \( i \in I_1 \) in this case), or

2. one of \( t_i, t_{i+1} \) is an asymptotic critical value of \( f \), hence by Theorem 1, there exist \( c_i > 0 \) and \( \Psi_i : (t_i, t_{i+1}) \to \mathbb{R} \) a strictly increasing, bounded \( C^1 \) function such that
\[
\| \nabla (\Psi_i \circ f)(x) \| \geq c_i
\]
for all \( x \in f^{-1}(t_i, t_{i+1}) \) (we write \( i \in I_2 \) in this case).

Take now any trajectory \( \gamma \) of \( -\nabla f \), and let \( \gamma_i = \gamma \cap f^{-1}(t_i, t_{i+1}) \). We denote by \( |\gamma| \) (resp. \( |\gamma_i| \)) the length of \( \gamma \) (resp. \( \gamma_i \)). Clearly \( |\gamma_i| \leq \frac{1}{c_i} |t_i - t_{i+1}| \) if \( i \in I_1 \). Extending by continuity, we may suppose that each \( \Psi_i \) is defined also at \( t_i \) and \( t_{i+1} \). Hence for \( i \in I_2 \) we have \( |\gamma_i| \leq \frac{1}{c_i} |\Psi_i(t_i) - \Psi_i(t_{i+1})| \), since the trajectories of \( -\nabla (\Psi_i \circ f) \) and \( -\nabla f \) are the same in \( f^{-1}(t_i, t_{i+1}) \). Finally, we can write
\[
|\gamma| = \sum_{i=0}^{k-1} |\gamma_i| \leq \sum_{i \in I_1} \frac{1}{c_i} |t_i - t_{i+1}| + \sum_{i \in I_2} \frac{1}{c_i} |\Psi_i(t_i) - \Psi_i(t_{i+1})| = A,
\]
which proves part a) of Theorem 2.

We are now going to construct the function $\sigma$ of part b). For $i \in I_2$ we put

$$\sigma_i(r) = \frac{1}{c_i} \sup\{|\Psi_i(p) - \Psi_i(q)| : p, q \in (t_i, t_{i+1}), r = p - q\},$$

and $\sigma_i(r) = \frac{r}{c_i}$ for $i \in I_1$. Extend each $\sigma_i$ to a continuous strictly increasing $\mathcal{M}$-function on $\mathbb{R}$. It is easily seen that $\sigma = \sup \sigma_i$ satisfies b) of Theorem 2.

We finish this section by a short discussion of some consequences of Theorem 2, which extend and generalize those known in the real analytic (compact) setting.

Observe that if $\gamma : (\alpha, \beta) \to U$ is a trajectory then $x_0 = \lim_{s \to \beta} \gamma(s)$ exists, and in general $x_0$ belongs to $\overline{U}$. Notice that if $x_0 \in U$, then $x_0$ is a critical point of $f$. Let us take $E$ a closed $\mathcal{M}$-subset in an open set $U$; by 4.22 of [DM], $E$ is the zero set of an $\mathcal{M}$-function $f : U \to \mathbb{R}$ of class $C^2$. Let $g = f^2$. We want to show that the flow of $-\text{grad} \, g$ defines a strong deformation retraction of a neighborhood of $E$ onto $E$. This is actually a new result even in the subanalytic case since the retraction is global and $E$ is not necessarily compact. By Proposition 2, taking a neighborhood of $E$, we may suppose that $0$ is the only asymptotic critical value of $g$ in $U$. Clearly the set

$$V = \{x \in U : \text{dist}(x, \partial U) < \sigma(g(x))\}$$

is an $\mathcal{M}$-set, it is an open neighborhood of $E$. For each $x \in V$ we denote by $\gamma_x : (\alpha_x, \beta_x) \to U$ the trajectory passing through $x$. It is clearly unique if $g(x) \neq 0$ and constant (hence unique) if $g(x) = 0$. Put $R(x) = \lim_{s \to \beta_x} \gamma_x(s)$, and observe that $R(x) \in E$. We have:

**Proposition 3.** — There exists an open neighborhood $V_1$ of $E$ such that $R : V_1 \to E$ is a strong deformation retraction.

**Proof.** — First we shall prove that $R$ is continuous. Take $x_0 \in V$ and $\Omega_0$ a neighborhood of $R(x_0)$. Let $x_1 \notin E$ be close to $R(x_0)$ so that there is (by Theorem 2 b)) a neighborhood $\Omega_1$ of $x_1$ with the following property: any trajectory passing through $\Omega_1$ has its limit in $\Omega_0$. By continuity of the flow of $-\text{grad} \, g$ there exists a neighborhood $G$ of $x_0$ such that any trajectory passing by $G$ must cross $\Omega_1$. So we have $R(G) \subset \Omega_0$, which proves the continuity of $R$. 
Let $\gamma$ be the trajectory passing through $x$. Let $\gamma_x$ be the part of $\gamma$ between $x$ and the limit $R(x)$. Assume that $\gamma_x : [0, \beta_x] \to U$ is parametrized by arc-length; moreover that $\gamma_x(0) = x$, and $\gamma_x(\beta_x) = R(x)$. Clearly $\beta_x$ is the length of $\gamma_x$. Notice that the argument in the proof of continuity of $R$ yields that the function $V \ni x \to \beta_x$ is continuous. Let $V_1$ be the set of all $x \in V$ such that $\gamma_x$ lies in $V$. We define a homotopy $F : [0,1] \times V_1 \to V_1$ as follows: $F_t(x) = \gamma_x(t\beta_x)$.

In general the retraction $R$ is not an $M$-mapping. Take $g(x,y) = (x^2 - y^3)^2$; it was observed by Hu [Hu] that the retraction $R$ is not hoelderian (at $(0,0)$) in this case, hence it cannot be subanalytic. Observe also that, in general, the set $V_1$ is not an $M$-set. It would be interesting to prove that actually $R$ belongs to some larger o-minimal structure. Even a weaker problem is open (also in the subanalytic case):

**Conjecture (F).** — Let $\gamma$ be a trajectory of $-\nabla f$, where $f$ is an $M$-function of class $C^1$, and let $H$ be any $M$-subset. Then $\gamma \cap H$ has a finite number of connected components.

This is connected with the Gradient Conjecture of R. Thom, proved recently in [KM]. R. Thom asked whether for an analytic function $f$ every trajectory $\gamma$ of $-\nabla f$ has a tangent at the limit point (i.e. whether $\lim_{s \to \beta_x} \frac{\gamma(s) - R(x)}{|\gamma'(s)|}$ exists). We can of course ask the same question for a trajectory of the gradient of any $M$-function of class $C^1$.

It is easily seen that (F) implies that $\lim_{s \to \beta_x} \frac{\gamma'(s)}{|\gamma'(s)|}$ exists, thus that the tangent to $\gamma$ at the limit point exists.
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