JARED WUNSCH

The trace of the generalized harmonic oscillator


<http://www.numdam.org/item?id=AIF_1999__49_1_351_0>


THE TRACE OF THE GENERALIZED HARMONIC OSCILLATOR

by Jared WUNSCH

1. Introduction.

Let \( M \) be a compact manifold with boundary endowed with a scattering metric \( g \) as defined by Melrose [9]. Thus in a neighborhood of \( \partial M \), we can write

\[
g = \frac{dx^2}{x^4} + \frac{h}{x^2}
\]

where \( x \) is a boundary-defining function for \( \partial M \), i.e. is smooth, nonnegative, and vanishes exactly at \( \partial M \) with \( dx \neq 0 \) at \( \partial M \), and where \( h \in C^\infty(Sym^2(T^*M)) \) restricts to be a metric on \( \partial M \). Scattering metrics form a class of complete, asymptotically flat metrics that includes asymptotically Euclidian metrics on \( \mathbb{R}^n \), radially compactified to the \( n \)-ball; this class also includes metrics on \( \mathbb{R}^n \) that are not asymptotically Euclidian but that look like arbitrary; non-round metrics on the sphere at infinity (see [9] for details).

We consider a generalization of the quantum-mechanical harmonic oscillator on the manifold \( M \): let \( x \) be a boundary-defining function for \( \partial M \) with respect to which \( g \) has the form (1.1), e.g. \(|z|^{-1}\) on flat \( \mathbb{R}^n \) (modified to be a smooth function at \( z = 0 \)). For any \( \omega \in \mathbb{R}_+ \), we consider the associated time-dependent Schrödinger equation

\[
\left(D_t + \frac{1}{2} \Delta + \frac{\omega^2}{2x^2} + v\right)\psi = 0
\]

Keywords: Schrödinger equation – Harmonic oscillator – Propagation of singularities – Trace theorem – Scattering metric.
where \( v \) is a formally self-adjoint perturbation term that can include both magnetic and electric potential terms. We will take \( v \) to be an error term in a sense to be made precise later on; potentials of the form \( v \in C^\infty(M) \) are certainly allowed. Note that for such a \( v \),

\[
\frac{1}{2} \Delta + \frac{\omega^2}{2x^2} + v
\]

is semi-bounded, hence the Friedrichs extension gives a self-adjoint operator on \( L^2(M) \) (with respect to the metric \( dg \)). Our class of operators thus includes compactly supported metric and potential perturbations of the standard harmonic oscillator on \( \mathbb{R}^n \).

Perturbations of the free-particle Schrödinger equation on manifolds with scattering metrics were studied in [13] using a calculus of pseudodifferential operators on manifolds with boundary called the quadratic-scattering (or qsc) calculus and denoted \( \Psi_{qsc}(M) \). This calculus is a microlocalization of the Lie algebra of “quadratic-scattering vector fields” on \( M \), given by

\[
\mathcal{V}_{qsc}(M) = x^2 \mathcal{V}_b(M)
\]

where

\[
\mathcal{V}_b(M) = \{ \text{vector fields on } M \text{ tangent to } \partial M \}\.
\]

Near \( \partial M \), \( \mathcal{V}_{qsc}(M) \) is locally spanned over \( C^\infty(M) \) by vector fields of the form \( x^3 \partial_x, x^2 \partial_y \), where \( x, y \) are product-type coordinates on \( M \) near \( \partial M \), i.e. the \( y \)’s are coordinates on \( \partial M \). The Lie algebra \( \mathcal{V}_{qsc}(M) \) can be written as the space of sections of a vector bundle:

\[
\mathcal{V}_{qsc}(M) = C^\infty(M, {}^{qsc}TM);
\]

we call \( {}^{qsc}TM \) the quadratic scattering tangent bundle of \( M \). Let \( {}^{qsc}T^*M \) be the dual bundle (the quadratic scattering cotangent bundle). Let \( {}^{qsc}\overline{T}^*M \) be the unit-ball bundle over \( M \) obtained by radially compactifying the fibers of \( {}^{qsc}T^*M \) (see [9] or [13]). This is a manifold with corners. The principal symbols of operators in the qsc-calculus are conormal distributions on \( {}^{qsc}\overline{T}^*M \) with respect to the boundary (a precise definition of such distributions will be given in § 2). There is an associated wavefront set, \( WF_{qsc} \), which is a closed subset of \( \partial({}^{qsc}\overline{T}^*M) \).

In [13], propagation of \( WF_{qsc} \) was described for perturbations of the free particle Schrödinger equation on \( M \). In this paper, we discuss
the analogous results for the harmonic oscillator, referring to [13] for all technical details. We can conclude from the propagation results that if there are no trapped geodesics on $\hat{M}$, then except at a certain set of times

\begin{equation}
S_\omega = \left\{ \frac{L}{\omega} : \text{there exists a closed geodesic in } \partial M \text{ of length } \pm L \right\} \\
\cup \left\{ \pm \frac{n\pi}{\omega} : \text{there exists a geodesic } n\text{-gon in } M \text{ with vertices in } \partial M \right\} \cup \{0\},
\end{equation}

there is no recurrence of $WF_{qsc}$ for solutions to (1.2). In the above definition of $S_\omega$ we adopt the convention that the sides of a geodesic $n$-gon in $M$ with vertices in $\partial M$ are maximally extended geodesics in $\hat{M}$ (which automatically have infinite length) and geodesics in $\partial M$ of length $\pi$; the latter geodesics appear naturally as limits of geodesics through $\hat{M}$ – cf. Prop. 1 of [10]. Using very general properties of the $qsc$ calculus, in § 5 we use the non-recurrence result to conclude that if $U(t)$ is the solution operator for the Cauchy problem for (1.2) then

\begin{equation}
\text{sing supp } \text{Tr } U(t) \subset S_\omega.
\end{equation}

For example, if we have a compactly-supported potential perturbation of the standard harmonic oscillator on $\mathbb{R}^n$, $S_\omega = 2\pi \mathbb{Z}$: If $M$ is the radial compactification of $\mathbb{R}^n$, $\partial M$ is the unit $(n - 1)$-sphere. Geodesics on $\hat{M}$ connect antipodal points on $\partial M$ and geodesics in $\partial M$ are great circles, hence consecutive vertices of a geodesic $n$-gon are antipodal points and there exist geodesic $n$-gons if $n$ is even; closed geodesics in $\partial M$ also only occur with lengths in $2\pi \mathbb{Z}$. Hence for a potential perturbation of the harmonic oscillator on $\mathbb{R}^n$, the trace of the solution operator can only be singular at multiples of $2\pi$. One can deduce this easily from Mehler's formula in the unperturbed case.

The trace theorem (1.6) closely resembles a result of Chazarain [1] and Duistermaat-Guillemin [6] which says that on a compact Riemannian manifold without boundary,

\[
\text{sing supp } \text{Tr } e^{it\sqrt{\Delta}} \subset \{\pm \text{lengths of closed geodesics} \} \cup \{0\};
\]

related results of Colin de Verdière using heat kernels can be found in [3] and [4]. Chazarain [2] has also proved a semi-classical trace theorem for the time-dependent Schrödinger equation, in which the lengths of closed bicharacteristics of the total symbol appear. By contrast, the trace theorem
of this paper is a non-semi-classical result, and over $S^*\hat{M}$, the relevant bicharacteristic flow is that of the symbol $\frac{1}{2} |\xi|^2$ rather than the full symbol as in [2]. Results on singularities of perturbations of the harmonic oscillator have been obtained by Zelditch [15], Weinstein [12], Fujiwara [7], Yajima [14], Kapitanski-Rodnianski-Yajima [8], and Treves [11]. Periodic recurrence of singularities for perturbations of the harmonic oscillator on $\mathbb{R}^n$ was demonstrated by Zelditch [15] and Weinstein [12], and the trace theorem (1.6) was proven by Zelditch for perturbations of the harmonic oscillator in $\mathbb{R}^n$ by potentials in $\mathcal{B}(\mathbb{R}^n)$.

The author is grateful to Richard Melrose, who supervised the Ph.D. thesis of which this work formed a part. The comments of an anonymous referee were also helpful, as was Hubert Goldschmidt’s help in reducing the level of illiteracy of the French abstract. The work was supported by a fellowship from the Fanny and John Hertz Foundation.

2. The quadratic-scattering calculus.

In this section, we briefly review the properties of the algebra $\Psi_{qsc}(M)$, which was constructed in [13], and is closely related to the “scattering algebra” of Melrose [9].

Let $\mathcal{V}_{qsc}(M)$ and $\mathcal{V}_{b}(M)$ be defined by (1.3) and (1.4), and let $\text{Diff}_{qsc}(M)$ and $\text{Diff}_{b}(M)$ be the order-filtered algebras of smooth linear combinations of products of elements of $\mathcal{V}_{qsc}(M)$ and $\mathcal{V}_{b}(M)$ respectively. There exists a bi-filtered star-algebra $\Psi_{qsc}(M)$, the “quadratic-scattering calculus” of pseudodifferential operators on $M$ such that

- $\text{Diff}_{qsc}^{m}(M) \subset \Psi_{qsc}^{m,0}(M)$.
- $\Psi_{qsc}^{m,\ell}(M) = x^\ell \Psi_{qsc}^{m,0}(M) = \Psi_{qsc}^{m,0}(M)x^\ell$.
- $\Psi_{qsc}^{m,\ell}(M) \subset \Psi_{qsc}^{m',\ell'}(M)$ if $m \leq m'$ and $\ell' - m' \leq \ell - m$.
- $\bigcap_{m,\ell} \Psi_{qsc}^{m,\ell}(M) \equiv \Psi_{\infty,\infty}^{-\infty}(M)$ consists of operators whose Schwartz kernels are smooth functions on $M \times M$, vanishing to infinite order at $\partial(M \times M)$.
- Elements of $\Psi_{qsc}^{0,0}(M)$ are bounded operators on $L^2(M)$.
- Given a sequence $A_j \in \Psi_{qsc}^{m-j,\ell+j}(M)$ for $j = 0, 1, 2, \ldots$, there exists an “asymptotic sum” $A \in \Psi_{qsc}^{m,\ell}(M)$, uniquely determined modulo $\Psi_{qsc}^{-\infty,\infty}(M)$, such that $A - \sum_{j=0}^{N-1} A_j \in \Psi_{qsc}^{m-N,\ell+N}(M)$. 
Let \( C_{\text{qsc}}M = \partial(\text{qsc}\,\overline{T}^*M) \). Let \( \sigma \) be a boundary defining function for the boundary face \( \text{qsc}\,S^*M \) of \( \text{qsc}\,\overline{T}^*M \) created by the fiber compactification. Let \( x \) be the lift of a boundary defining function on \( M \) to \( \text{qsc}\,\overline{T}^*M \) — thus \( x \) defines the boundary face \( \text{qsc}\,\overline{T}^*_\partial M \). Let \( \hat{C}^\infty(M) \) denote smooth functions on \( M \) vanishing to infinite order at \( \partial M \) and \( C^{\infty}(M) \) the dual space to \( \hat{C}^\infty(M) \)-valued densities. Following Melrose [9], we define conormal distributions on \( \text{qsc}\,\overline{T}^*M \) with respect to \( C_{\text{qsc}}M \) as follows:

\[
\mathcal{A}^{p,q}(\text{qsc}\,\overline{T}^*M) = \{ u \in C^{\infty}(\text{qsc}\,\overline{T}^*M) : \text{Diff}^k_{\text{qsc}}(\text{qsc}\,\overline{T}^*M) u \subset \sigma^0 x^q L^{\infty}(\text{qsc}\,\overline{T}^*M) \text{ for all } k \};
\]

here \( \text{Diff}^k_{\text{qsc}} \) is defined on the manifold with corners \( \text{qsc}\,\overline{T}^* \) exactly as it was defined on manifolds with boundary: as the span of products of vector fields tangent to (all faces of) the boundary. Let

\[
\mathcal{A}^{m,\ell}(C_{\text{qsc}}M) = \mathcal{A}^{m,\ell}(\text{qsc}\,\overline{T}^*M) / \mathcal{A}^{m-1,\ell+2}(\text{qsc}\,\overline{T}^*M).
\]

There exists a symbol map

\[
\hat{\gamma}_{\text{qsc}} : \hat{\gamma}^{m,\ell}(C_{\text{qsc}}M) \rightarrow \mathcal{A}^{m,\ell}(C_{\text{qsc}}M)
\]

such that

* There is a short exact sequence

\[
0 \rightarrow \mathcal{A}^{m-1,\ell+1}(M) \rightarrow \mathcal{A}^{m,\ell}(M) \rightarrow \mathcal{A}^{[-m,\ell-m]}(C_{\text{qsc}}M) \rightarrow 0.
\]

* The symbol map is multiplicative.

* The Poisson bracket extends continuously from the usual bracket defined on the interior of \( \text{qsc}\,\overline{T}^*M \) to \( \mathcal{A}^{[-m,\ell-m]}(C_{\text{qsc}}M) \), and

\[
\hat{j}_{\text{qsc},m_1+m_2-1,\ell_1+\ell_2}([P,Q]) = \frac{1}{i} \{ \hat{j}_{\text{qsc},m_1,\ell_1}(P), \hat{j}_{\text{qsc},m_2,\ell_2}(Q) \}.
\]

Furthermore, if \( a \in \mathcal{A}^{m,\ell}(\text{qsc}\,\overline{T}^*M) \), \( \{ a, b \} = H_a(b) \) where \( H_a \) is the extension of the usual Hamilton vector field on the interior of \( \text{qsc}\,\overline{T}^*M \) to an element of \( \sigma^{-m+1}x^{\ell+2}\mathcal{N}_b(\text{qsc}\,\overline{T}^*M) \). (We refer to the flow along \( H_a \) or \( \sigma^{-m-1}x^{-\ell-2}H_a \) as bicharacteristic flow.)

* There exists a (non-unique) “quantization map”

\[
\text{Op} : \mathcal{A}^{m,\ell-m}(\text{qsc}\,\overline{T}^*M) \rightarrow \mathcal{A}^{m,\ell}(M)
\]

such that

\[
\hat{j}_{\text{qsc},m,\ell}(\text{Op}(a)) = [a] \in \mathcal{A}^{[-m,\ell-m]}(C_{\text{qsc}}M).
\]
DEFINITION 2.1. — An operator $P \in \Psi^{m,\ell}_{\mathrm{qsc}}(M)$ is said to be elliptic at a point $p \in C_{\mathrm{qsc}} M$ if $j_{\mathrm{qsc}} m, \ell$ is locally invertible near $p$. The set of points at which $P$ is elliptic is denoted $\text{ell } P$. If $P$ is elliptic everywhere, it is simply said to be elliptic.

DEFINITION 2.2. — Let $P \in \Psi^{m,\ell}_{\mathrm{qsc}}(M)$. A point $p \in C_{\mathrm{qsc}} M$ is in the complement of $WF'_\mathrm{qsc} P$ (the operator wavefront set or microsupport of $P$) if there exists $Q \in \Psi^{-m,\ell}_{\mathrm{qsc}}(M)$ such that $Q$ is elliptic at $p$ and $PQ \in \Psi^{-\infty,\infty}_{\mathrm{qsc}}(M)$.

We can now define the qsc wavefront set of $u \in C^{-\infty}(M)$ as the subset $WF'_\mathrm{qsc} u$ of $C_{\mathrm{qsc}} M$ such that $p \notin WF'_\mathrm{qsc} u$ if and only if there exists $A \in \Psi^0_0(M)$ with $p \in \text{ell } A$ such that $Au \in \dot{C}^{\infty}(M)$.

The qsc wavefront set and microsupport enjoy the following properties:

- If $A, B \in C_{\mathrm{qsc}}(M)$, then $WF'_\mathrm{qsc} AB \subset WF'_\mathrm{qsc} A \cap WF'_\mathrm{qsc} B$ and $WF'_\mathrm{qsc} A^* = WF'_\mathrm{qsc} A$.

- Microlocal parametrices exist at elliptic points: if $P \in \Psi^{m,\ell}_{\mathrm{qsc}}(M)$ is elliptic at $p \in C_{\mathrm{qsc}} M$ then there exists $Q \in \Psi^{-m,\ell}_{\mathrm{qsc}}(M)$ such that $p \notin WF'_\mathrm{qsc} (PQ - I)$ and $p \notin WF'_\mathrm{qsc} (QP - I)$.

- Microlocality: let $P \in \Psi_{\mathrm{qsc}}(M)$ and $u \in C^{-\infty}(M)$. Then
  \[WF_{\mathrm{qsc}} Pu \subset WF'_\mathrm{qsc} P \cap WF_{\mathrm{qsc}} u.\]

- Microlocal elliptic regularity: Let $P \in \Psi_{\mathrm{qsc}}(M)$ and $u \in C^{-\infty}(M)$. Then
  \[WF_{\mathrm{qsc}}(u) \subset WF_{\mathrm{qsc}}(Pu) \cup (\text{ell } P)^c.\]

- We can (and do) choose the map $Op$ in such a way that
  \[WF'_{\mathrm{qsc}} Op(a) \subset \text{ess supp } a\]

  (ess sup $a$ is the set of points in $C_{\mathrm{qsc}} M$ near which $a$ does not vanish to infinite order).

We will also require a notion of qsc wavefront set that is uniform in a parameter.
DEFINITION 2.3. — Let \( u \in C(\mathbb{R}; C^{-\infty}(M)) \). For \( S \subset \mathbb{R} \) compact, we say that \( p \notin WF_{qsc}^S(u) \) if there exists a smooth family \( A(t) \in \Psi_{qsc}^{0,0}(M) \) such that \( A(t) \) is elliptic at \( p \) for all \( t \in S \) and \( Au \in C(S; \hat{C}^\infty(M)) \).

Associated to \( \Psi_{qsc}(M) \) is a family of Sobolev spaces

\[
H_{qsc}^{m,\ell}(M) = \{ u \in C^{-\infty}(M) : \Psi_{qsc}^{m,-\ell}(M)u \subset L^2(M) \}
\]

such that

- If \( A \in \Psi_{qsc}^{m',\ell'}(M) \) then
  \[
  A : H_{qsc}^{m,\ell}(M) \to H_{qsc}^{m'-m,\ell+\ell'}(M)
  \]
  is continuous for any \( m, \ell \).

- For any \( \ell \in \mathbb{R} \),
  \[
  \bigcap_m H_{qsc}^{m,\ell}(M) = \hat{C}^\infty(M) \quad \text{and} \quad \bigcup_m H_{qsc}^{m,\ell}(M) = C^{-\infty}(M).
  \]

- If \( a_n \) is a bounded sequence in \( \mathcal{A}^{-m,\ell-m}(M) \) and \( a_n \to a \) in some \( \mathcal{A}^{p,q}(M) \), then \( \text{Op}(a_n) \to \text{Op}(a) \) in the strong operator topology on
  \[
  B(H_{qsc}^{M,L}(M), H_{qsc}^{M-m,M+\ell}(M))
  \]
  for all \( M, L \).

3. The propagation of \( WF_{qsc} \).

For details of all computations in this section, see [13], especially §11.

We consider the symbol and corresponding bicharacteristic flow for the operator

\[
\mathcal{H} = \frac{1}{2} \Delta + \frac{\omega^2}{2x^2} + v
\]

where

\[
v \in \text{Diff}^{1,1}_{qsc}(M)
\]

is formally self-adjoint and \( x \) is a boundary-defining function with respect to which \( g \) takes the form (1.1).
Let the canonical one-form on $\mathcal{Q}^*M$ be
\[ \lambda \frac{dx}{x^3} + \mu \frac{dy}{x^2}. \]

The joint symbol of $\mathcal{H}$ is represented in $\mathcal{A}^{[-2,-2]}(C_{\text{qsc}}M)$ by a conormal distribution of the form
\[ j_{\text{qsc},2,0}(\mathcal{H}) = \frac{1}{2x^2} (\lambda^2 + |\mu|^2 + \omega^2 + 2r(\lambda, \mu)); \]
\[ r(\lambda, \mu) = \lambda^2 x C^\infty(x,y) + \lambda \mu C^\infty(x,y) + \mu^2 C^\infty(x,y) \]
where $|\mu|$ denotes the norm of $\mu$ with respect to the metric $\bar{h} = h|_{\partial M}$. Note that (3.1) shows that $\mathcal{H}$ is an elliptic element of $\mathcal{A}^{[-2,-2]}(M)$; the perturbation $v$ does not enter into the expression (3.1) as it has lower order than $\frac{1}{2} \Delta + \frac{\omega^2}{2x^2}$ in both indices. The Hamilton vector field of $\mathcal{H}$ is
\[ X = \tilde{X} + P \]
where
\[ (3.2) \quad \tilde{X} = \lambda x \partial_x + (\lambda^2 - |\mu|^2 + \omega^2) \partial_\lambda + (\mu, \partial_y) + 2\lambda \mu \cdot \partial_\mu - \frac{1}{2} \partial_y |\mu|^2 \cdot \partial_\mu \]
is the Hamilton vector field for the symbol $\frac{1}{2x^2} (\lambda^2 + |\mu|^2 + \omega^2)$, and
\[ (3.3) \quad P = p_1 x^2 \partial_x + p_2 x \partial_y + q_1 x \partial_\lambda + q_2 x \partial_\mu \]
is the Hamilton vector field for the "error term" $\frac{1}{2} x^{-1} r(\lambda, \mu)$. Here we adopt the convention that
\[ \langle a, b \rangle = \sum a_i b_j \tilde{h}^{ij}(y) \quad \text{and} \quad a \cdot b = \sum a_i b_i. \]

The vector field $P$ is identically zero if $h$ is a function of $y$ only, and always vanishes at $x = 0$.

Under the flow along $\tilde{X}$,
\[ \frac{d}{dt} (\lambda + i|\mu|) = (\lambda + i|\mu|)^2 + \omega^2, \]
hence
\[ (3.4) \quad \lambda + i|\mu| = \omega \frac{\sin \omega(t - t_0) + iR \cos \omega(t - t_0)}{\cos \omega(t - t_0) - iR \sin \omega(t - t_0)} \]
for some $R \in [0, 1]$. For $R > 0$, this gives a periodic orbit with period $\pi/\omega$.

On $\{\mu \neq 0\}$ (i.e. $R > 0$), we set $\tilde{\mu} = \mu/|\mu|$, and introduce the rescaled time parameter $s = \int |\mu| \, dt$ to rewrite the flow along $\widetilde{X}$ as

$$
\frac{dy_i}{ds} = \tilde{h}^{ij} \tilde{\mu}_j,
$$

$$
\frac{d\tilde{\mu}_i}{ds} = -\frac{1}{2} \tilde{\mu}_j \tilde{\mu}_k \partial_{\mu} \tilde{h}^{jk},
$$

$$
\frac{d\lambda}{ds} = \frac{\lambda^2 - |\mu|^2}{|\mu|} + \omega^2,
$$

$$
\frac{d|\mu|}{ds} = 2\lambda,
$$

$$
\frac{dx}{ds} = \frac{\lambda x}{|\mu|}.
$$

As the set $\mu = 0$ plays an important role in the geometry of $\widetilde{X}$, we give it a name:

**Definition 3.1.** — Let $N \subset q^*TM$ be the set given in our coordinates by $\{x = \mu = 0\}$. Let $N_{\pm} \subset N$ be the subsets on which $\pm \lambda \geq 0$. Let $N_{\pm}^c = N_{\pm} \cap q^*S^*M$ (i.e. $N^c$ is the intersection of $N$ with the corner). We refer to $N$ as the “normal set,” with $N_+$ being the “incoming normal set” and $N_-$ the “outgoing normal set.”

![Figure 1](image)

*Figure 1. Integral curves of $\widetilde{X}$, projected onto the $(\lambda, \mu)$-plane and radially compactified. The vertical line is the solution $\mu = 0$."

While $(\lambda, |\mu|)$ are undergoing a flow described by (3.4) (see Fig. 1), then provided $R \neq 0$, (3.5) shows that $(y, \tilde{\mu})$ are undergoing unit speed geodesic flow in $\partial M$ with rescaled time parameter $s$. For $R = 0$, $\mu$ is
identically zero, \( y \) is constant, and \( \lambda \) blows up at \( t - t_0 = \pm \pi/2\omega \), i.e. the flow crosses \( \mathcal{N} \) from \( \mathcal{N}_c^+ \) to \( \mathcal{N}_c^- \) in time \( \pi/\omega \). More generally the integral curve starting at \( \mu = 0, \lambda = \lambda_0 \), reaches the corner at time \( t = \omega^{-1} \arctan(\omega/\lambda_0) \).

Note that all terms in \( X \) are homogeneous of degree 1 in \( (\lambda, \mu) \) except the term \( \omega^2 \partial \lambda \), which is homogeneous of degree \(-1\). If we let \( \sigma \) be the defining function for \( q^{sc}S^*M \) in \( q^{sc}\overline{T}^*M \) given by

\[
\sigma = \left( \lambda^2 + |\mu|^2 \right)^{-1/2}
\]

and set

\[
\tilde{\lambda} = \sigma \lambda, \quad \tilde{\mu} = \sigma \mu
\]

then the vector field \( \sigma X \) is tangent to the boundary of \( q^{sc}\overline{T}^*M \), and we have

\[
\sigma X = \tilde{\lambda} x \partial_x - |\tilde{\mu}|^2 \partial_{\tilde{\lambda}} + \langle \tilde{\mu}, \partial_y \rangle + (\tilde{\lambda} \mu - \frac{1}{2} \partial_y |\tilde{\mu}|^2) \partial_{\tilde{\mu}} - \tilde{\lambda} \sigma \partial_\sigma + O(\sigma^2) + O(x)
\]

where \( O(\sigma^2) \) and \( O(x) \) denote error terms of the form \( \sigma^2 Y_1 \) and \( x Y_2 \), with \( Y_i \) tangent to \( \partial(q^{sc}\overline{T}^*M) \); the \( O(\sigma^2) \) term is just \( \sigma \omega^2 \partial \lambda \), while the \( O(x) \) term is what has above been denoted \( P \).

The vector field \( X \) differs from the free-particle Hamilton vector-field \( X_{fp} \) described in [13] only in the term \( \omega^2 \partial \lambda \), hence since this term is \( O(\sigma) \), we have

\[
\sigma X|_{q^{sc}S^*M} = \sigma X_{fp}|_{q^{sc}S^*M} \tag{3.9}
\]

**Definition 3.2.** — A maximally extended integral curve of \( \sigma X \) on \( q^{sc}S^*M \) is said to be non-trapped forward/backward if

\[
\lim_{t \to \pm \infty} x(t) = 0.
\]

A point in \( q^{sc}S^*M \setminus \mathcal{N}_c \) is said to be non-trapped forward/backward if the integral curve through it is non-trapped. A point in \( \mathcal{N}_c \) is said to be non-trapped forward/backward if it is not in the closure of any

(1) Unfortunately, this vector field is called \( X \) as well in [13].
forward-/backward-trapped integral curves. Let $T_\pm$ denote the set of forward-/backward-trapped points in $^{qsc}S^*M$.

The only zeros of $\sigma X$ on $^{qsc}S^*M$ are on the manifolds $N_-^c$ (attracting) and $N_+^c$ (repelling), so we can define

$$N_{\pm\infty} : ^{qsc}S^*M \setminus (N_\pm \cup T_\pm) \longrightarrow N_\mp^c$$

by

$$p \mapsto \lim_{t \to \pm\infty} \exp(t\sigma X)[p].$$

We extend this definition of $N_{\pm\infty}$ to $^{qsc}\overline{T}^*M \setminus (N_\pm \cup T_\pm)$ by homogeneity. We further define

$$Y_{\pm\infty} : ^{qsc}\overline{T}^*M \setminus (N_\pm \cup T_\pm) \longrightarrow \partial M$$

to be the projection of $N_{\pm\infty}$ to $\partial M$.

**THEOREM 3.3.** — $N_{\pm\infty}$ and $Y_{\pm\infty}$ are smooth maps. If we let $C_\pm^c$ be the submanifold of $^{qsc}S^*M$ given by

$$C_\pm^c = \{ x^2 + |\bar{\mu}|^2 = \epsilon, \bar{\lambda} \gtrless 0 \}$$

then for $\epsilon$ sufficiently small, $C_\pm^c$ is a fibration over $\partial M$, and every integral curve of $\sigma X$ which is not trapped forward/backward passes through $C_\mp^c$. The sets $T_\pm \setminus N_\mp^c$ are closed subsets of $^{qsc}S^*M \setminus N_\pm^c$.

By (3.9), this theorem follows from Theorem 11.6 of [13].

We can thus define the scattering relation:

**DEFINITION 3.4.** — Let $S \subset N_-^c \setminus T_-$. The scattering relation on $S$ is

$$\text{Scat}(S) = N_{-\infty} \left( N_{+\infty}^{-1}(S) \right) \subset N_+^c.$$
Example 3.5. — If $M$ is the radial compactification of $\mathbb{R}^n$ with an asymptotically Euclidean metric, we can identify the manifolds $N^\pm_\varphi$ with $S^{n-1} = \partial M$. Then for $\theta \in S^{n-1}$, $\text{Scat} \theta$ consists of all $\theta' \in S^{n-1}$ such that there exists a geodesic $\gamma$ in (uncompactified) $\mathbb{R}^n$ with $\lim_{t \to -\infty} \gamma'(t) = -\theta'$ and $\lim_{t \to +\infty} \gamma'(t) = \theta$. In other words, $\text{Scat}$ consists of all directions in $\mathbb{R}^n$ that can scatter to the direction $\theta$. In the Euclidean case, $\text{Scat}$ is the antipodal map on $S^{n-1}$.

We now state theorems on propagation of $WF_{\text{qsc}}$ that will suffice to obtain results on $\text{sing supp} \ Tr U(t)$. (Slightly more sophisticated theorems, corresponding to Theorems 12.1–12.5 of [13], in fact hold here as well.)

**Theorem 3.6** (propagation over the boundary). — Let $p \in (^{\text{qsc}}T^\star_{\partial M} M)^\circ$ and assume

$$\exp(TX)[p] \in (^{\text{qsc}}T^\star_{\partial M} M)^\circ.$$ 

Then $p \notin WF_{\text{qsc}} \psi(0)$ if and only if there exists $\delta > 0$ such that $\exp(TX)[p] \notin WF_{\text{qsc}}^{[T-\delta, T+\delta]} \psi$.

**Theorem 3.7** (propagation into the interior). — Let $p \in qsc S^*M \setminus N^\subset_\varphi$ be non-backward-trapped and let $T \in (0, \pi/\omega)$. If

$$\exp(-TX) [N_{-\infty}(p)] \notin WF_{\text{qsc}} \psi(0)$$

then there exists $\delta > 0$ such that $p \notin WF_{\text{qsc}}^{[T-\delta, T+\delta]} \psi$.

**Theorem 3.8** (scattering across the interior). — Let $q \in N^\subset_\varphi$ be non-backward-trapped. If

$$\exp(-T_0 X) [\text{Scat}(q)] \cap WF_{\text{qsc}} \psi(0) = \emptyset$$

for some $T_0 \in (0, \pi/\omega)$, then for every $T \in (T_0, T_0 + \pi/\omega)$, there exists $\delta > 0$ such that $\exp((T - T_0) X)[q] \notin WF_{\text{qsc}}^{[T-\delta, T+\delta]} \psi$.

**Theorem 3.9** (global propagation into the boundary). — Let $q \in N^\subset_\varphi$ be non-backward-trapped. If

$$\overline{N_{+\infty}^{-1}(q)} \cap WF_{\text{qsc}} \psi(0) = \emptyset$$
(closure taken in $\text{qsc} S^* M$), then for $T \in (0, \pi/\omega)$, there exists $\delta > 0$ such that
\[
\exp(TX)[q] \notin \text{WF}_{\text{qsc}}^{[T-\delta, T+\delta]} \psi.
\]

The proofs are by the same positive-commutator arguments used in [13] (which were in turn adapted from Craig-Kappeler-Strauss [5]), although the symbol constructions need to be slightly modified from those in [13] because the maps $Y_{\pm \infty}$ are not exactly constant along the flow of $X$; we discuss these issues in an appendix.


We assume throughout this section that there are no trapped geodesics in $\hat{M}$.

This section is devoted to proving

**Theorem 4.1.** — Let $S_\omega$ be defined by (1.5). For $T \not\in S_\omega$ and for any $p \in C_{\text{qsc}} M$, there exists an open neighborhood $O$ of $p$ and $\epsilon > 0$ such that
if $\text{WF}_{\text{qsc}} \psi(0) \subset O$ then $\text{WF}_{\text{qsc}}^{[T-\epsilon, T+\epsilon]} \psi \cap O = \emptyset$.

In order to deduce this theorem from Theorems 3.6–3.9, we first define a relation on $C_{\text{qsc}} M$ which describes from what points singularities may reach a point $p \in C_{\text{qsc}} M$:

**Definition 4.2.** — Let $p, q \in C_{\text{qsc}} M$. We write $p \sim q$ if there exists a continuous path $\gamma$ from $p$ to $q$ in $C_{\text{qsc}} M$ that is a concatenation of maximally extended integral curves of $\sigma X$ such that
\[
\sum \text{(lengths of integral curves in $\text{qsc} \bar{T}^*_{\partial M} M$)} = t,
\]
where we define the length of an integral curve in $\text{qsc} \bar{T}^*_{\partial M} M$ to be its length as an integral curve of $X$ (and hence a finite number).

Then for $S \subset C_{\text{qsc}} M$, let
\[
G_t(S) = \{ p \in C_{\text{qsc}} M : p \sim q \text{ for some } q \in S \}.
\]

If $p \sim q$ and $q \sim r$, then $q \sim r$, hence
\[
G_{s+t}(S) = G_s \circ G_t(S).
\]
We also have

\[(4.3) \quad \mathcal{G}_t(S \cup T) = \mathcal{G}_t(S) \cup \mathcal{G}_t(T).\]

The relation \(p \sim q\) is closed in the following sense:

**Lemma 4.3.** — Let \(R \subset C_{\text{qsc}} M \times C_{\text{qsc}} M \times \mathbb{R}\) be defined by

\[(p, q, t) \in R \iff p \sim q.\]

Then \(R\) is a closed subset of \(C_{\text{qsc}} M \times C_{\text{qsc}} M \times \mathbb{R}\).

**Proof.** — Suppose \(p_i \to p\), \(q_i \to q\), and \(t_i \to t\) as \(i \to \infty\), and that \((p_i, q_i, t_i) \in R\). We will show that \((p, q, t) \in R\).

For simplicity, we reformulate (4.1) as follows: let \(k\) be a Riemannian metric on the manifold \((q\text{sc} \overline{\partial}_M M)^0\) such that the norm of \(X\) with respect to \(k\) is one. (As \(X = O(\sigma^{-1})\), \(k\) vanishes at \(q\text{sc} \partial_M M\).) Let \(\theta = k(\cdot, X) \in \Omega^1((q\text{sc} \overline{\partial}_M M)^0)\); extend \(\theta\) to be zero on the interior of the boundary face \(q\text{sc} \partial^* M\). Then the condition (4.1) is equivalent to

\[(4.4) \quad \int_\gamma \theta = t.\]

Now by hypothesis there exists a sequence \(\gamma_i\) of paths as in Definition 4.2 such that \(\gamma_i(0) = p_i\), \(\gamma_i(1) = q_i\), and \(\int_{\gamma_i} \theta = t_i\) for all \(i\). As the \(\gamma_i\) are all integral curves of \(\sigma X\), we apply Ascoli-Arzelà to obtain a path \(\gamma\) between \(p\) and \(q\), made up of integral curves of \(\sigma X\) with \(\int_\gamma \theta = t\).

**Definition 4.4.** — Let

\[\mathcal{G}_t^{-1} S = \{p : \mathcal{G}_t(p) \subset S\}.

We now prove that \(\mathcal{G}_t\) is, in an appropriate sense, a continuous set map.

**Lemma 4.5.** — If \(K \subset \mathbb{R}\) is compact then

\[\bigcup_{t \in K} \mathcal{G}_t\]

takes closed sets to closed sets, and

\[\bigcap_{t \in K} \mathcal{G}_t^{-1}\]

takes open sets to open sets.
Proof. — Let $\pi_L$ and $\pi_R$ denote the projections of $C_{qsc}M \times C_{qsc}M \times \mathbb{R}$ onto the “left” and “right” factors of $C_{qsc}$ and let $\pi_t$ denote projection to $\mathbb{R}$. Then we can write

$$\bigcup_{t \in K} G_t(S) = \pi_L(\pi_R^{-1}S \cap \pi_t^{-1}K \cap R)$$

and

$$\bigcap_{t \in K} G_t^{-1}(S) = [\pi_R(\pi_L^{-1}(S^c) \cap \pi_t^{-1}K \cap R)]^c$$

hence the result follows from Lemma 4.3. □

Theorems 3.6–3.9 can now be conveniently recast as

MAIN PROPAGATION THEOREM. — If $S \subset C_{qsc}M$ and

$$G_t(S) \cap WF_{qsc} \psi(0) = \emptyset$$

then there exists $\epsilon > 0$ such that

$$S \cap WF_{qsc}^{[T-\epsilon,T+\epsilon]} \psi = \emptyset.$$

Proof. — By (4.3), it suffices to prove the result for $S = \{p\}$, a single point in $C_{qsc}M$. By (4.2), it suffices to prove the result for small $t$; we take $t < \pi/\omega$ for simplicity. If

$$p \in (^{qsc}T_{\partial M}^*M)^* \setminus \mathcal{N},$$

then for any $t$, as discussed in §3, $G_t(p)$ is a single point in $(^{qsc}T_{\partial M}^*M)^*$, and the result follows from Theorem 3.6.

Let $\arctan_+$ denote the branch of arctan taking values in $[0,\pi)$. If $p \in \mathcal{N}^c$, then for $t \in (0, \omega^{-1}\arctan_+(\lambda(p)/\omega))$, $G_t(p)$ is again a point in $\mathcal{N}^c$, and again the theorem follows from Theorem 3.6. At $t = \omega^{-1}\arctan_+(\lambda(p)/\omega)$, $\exp(-tX)[p] \in \mathcal{N}^c$, and

$$G_t(p) = \overline{N_{-\infty}^{-1}(\exp(-tX)[p])} \subset ^{qsc}S^*M,$$

hence Theorem 3.9 takes care of this case. For

$$\omega^{-1}\arctan_+(\lambda(p)/\omega) < t < \pi/\omega,$$

we once again have $G_t(p) \subset (^{qsc}T_{\partial M}^*M)^*$, and Theorem 3.8 finishes the proof.

If, on the other hand, $p \in ^{qsc}S^*M$, $G_t(p) \subset (^{qsc}T_{\partial M}^*M)^*$ for $t \in (0, \pi/\omega)$: $G_t(p)$ is a single point if $p \notin \mathcal{N}_+^c$, or a whole set, given by the scattering relation, if $p \in \mathcal{N}_+^c$. The theorem then follows from Theorem 3.7 in the former case, and Theorem 3.8 in the latter. □
The relation $G_t$ is non-recurrent except at certain times:

**Lemma 4.6.** — For any $T \notin S_\omega$ and $p \in Cqsc M$, there exists an open neighborhood $O$ of $p$ and $\epsilon > 0$ such that

\[ G_t(O) \cap O = \emptyset \quad \text{for all } t \in [T - \epsilon, T + \epsilon]. \]

**Proof.** — By compactness of $\partial M$, $S_\omega$ is closed. Hence if $T \notin S_\omega$, there exists $\epsilon > 0$ such that

\[ K = [T - \epsilon, T + \epsilon] \subset \mathbb{R} \setminus S_\omega. \]

By Lemma 4.5, $\bigcup_{t \in K} G_t(p)$ is closed. If this set does not contain $p$ then we can choose an open set $U$ containing $\bigcup_{t \in K} G_t(p)$ but such that $p \notin U$. By Lemma 4.5, we can then set

\[ O = \bigcap_{t \in K} G_t^{-1}(U) \setminus U. \]

Thus it will suffice to prove that for $t \notin S_\omega$, $p \notin G_t(p)$.

First we take the case $p \in qsc S^* M \setminus N^c$. Then for $t \in (0, \pi/\omega)$,

\[ G_t(p) = \exp(-tX)[N_{-\infty}(p)] \subset (qsc \tilde{T}_{\partial M}^* M)^c, \]

and this set certainly doesn’t contain $p$. Let $I$ be the involution of $N^c$ swapping $N^c_\pm$ and $N^c_\mp$. Then

\[ G_{\pi/\omega}(p) = N_{-\infty}^{-1} \circ I \circ N_{-\infty}(p), \]

and this set doesn’t contain $p$ unless $Y_{\pm}(p) = Y_{-\infty}(p)$, i.e. unless $p$ lies on a geodesic 1-gon with vertex in $\partial M$. For $t \in (\pi/\omega, 2\pi/\omega)$,

\[ G_t(p) = \exp(-(t - \pi/\omega)X)[Scat \circ I \circ N_{-\infty}(p)], \]

again a subset of $(qsc \tilde{T}_{\partial M}^* M)^c$. The set

\[ G_{2\pi/\omega}(p) = N_{-\infty}^{-1} \circ I \circ Scat \circ I \circ N_{-\infty}(p), \]

and this set certainly does contain $p$. Continuing in this manner, we find that if $t = n\pi/\omega + r$ with $r \in (0, \pi/\omega)$ then

\[ G_t(p) = \exp(-rX)(Scat \circ I)^n N_{-\infty}(p) \subset (qsc \tilde{T}_{\partial M}^* M)^c, \]
while
\[ G_{n\pi/\omega}(p) = \frac{1}{N_{-\infty}} \circ T \circ (\text{Scat} \circ T)^n \circ N_{-\infty}(p), \]
hence \( p \in G_t(\omega) \) iff there exists a geodesic \( n \)-gon passing through \( p \) with vertices in \( \partial M \) (this is always the case for \( n \) even, as we are allowed to repeat edges).

\[ \begin{array}{c}
Y_\infty(p) \quad \partial M \\
M \\
p \quad \circ \\
Y_{-\infty}(p)
\end{array} \]

Figure 2. A point \( p \) on a geodesic triangle with vertices in \( \partial M \).

Now we take the case \( p \in (q_{\text{s}}T_{\partial M}^*)^o \setminus \mathcal{N} \). The flow of \( X \) in \((q_{\text{s}}T_{\partial M}^*)^o \setminus \mathcal{N}\) is, as discussed in §3, given by unit speed geodesic flow in \( \partial M \) with time parameter \( s = \int |\mu| \, dt \), while \((\lambda, |\mu|)\) undergo the motion (3.4). The only fixed-point of the \((\lambda, |\mu|)\) flow is given by \( \lambda = 0, |\mu| = \omega \); all other orbits are periodic with period \( \pi/\omega \). Hence if \( (\lambda(p), |\mu(p)|) \neq (0, \omega) \) and \( t \notin (\pi/\omega)\mathbb{Z} \) then \( p \notin G_t(p) \), since the \((\lambda, |\mu|)\) coordinates distinguish between these two points. If, on the one hand, \( t = n\pi/\omega \), we have by (3.4)

\[ (4.5) \quad s = \int_0^{n\pi/\omega} |\mu| \, dt \\
= \Im \int_0^{n\pi/\omega} \frac{\sin \omega(t - t_0) + iR \cos \omega(t - t_0)}{\cos \omega(t - t_0) - iR \sin \omega(t - t_0)} \, dt \\
= n\omega \Im \int_{-\pi/2\omega}^{\pi/2\omega} \frac{\tan \omega t - iR}{1 + iR \tan \omega t} \, dt \\
= n\pi \]
(recall that \( R = 0 \) only on \( \mathcal{N} \)). Thus by (3.5), for \((\lambda, |\mu|) \neq (0, \omega), p = G_{n\pi/\omega}(p)\) only if there is a closed geodesic of length \( n\pi \) in \( \partial M \). On the other hand, if \((\lambda(p), |\mu(p)|) = (0, \omega), (\lambda, |\mu|)\) remains constant along the flow, so \( p = G_t(p) \) only if there is a closed geodesic in \( \partial M \) of length \( \omega t \). This proves the result for \( p \in (q_{\text{s}}T_{\partial M}^*)^o \setminus \mathcal{N} \).
The proof for $p \in \mathcal{N}$ (including $\mathcal{N}^c$) proceeds like the proof for $p \in qsc S^*M \setminus \mathcal{N}_-$; certainly if $t \notin (\pi/\omega)\mathbb{Z}$, $p \notin G_t(p)$, as $\lambda$ is constant on $G_t(p)$ at fixed $t$, and equals $\lambda(p)$ only for $t \in (\pi/\omega)\mathbb{Z}$. The same geometrical discussion used in the proof for points in $(qsc S^*M)^0$ also shows that $p \notin G_{\pi/\omega}(p)$ unless there is a geodesic $n$-gon with vertices in $\partial M$, with one vertex at $y(p)$.

**Proof of Theorem 4.1.** — The theorem follows directly from the Main Propagation Theorem and Lemma 4.6.

From Theorem 4.1, we deduce the following, which is the key result for our trace theorem.

**Corollary 4.7.** — Given $T \notin \mathcal{S}_\omega$, there exists $\epsilon > 0$, $k \in \mathbb{Z}^+$, and $A_i \in \Psi^{0,0}_{qsc}(M)$, $i = 1, \ldots, k$ such that

$$A_i U_{\omega}(t) A_i \in C^\infty([T - \epsilon, T + \epsilon]; \Psi^{-\infty,\infty}_{qsc}(M))$$

and

$$I = \sum_{i=1}^{k} A_i^2 + R$$

($I$ denotes the identity operator) with $R \in \Psi^{-\infty,\infty}_{qsc}(M)$.

**Proof.** — By Theorem 4.1, we can find a partition of unity $(b_{1,i})^2$, subordinate to a cover $\mathcal{O}_i$ of $C_{qsc} M$, such that $WF_{qsc}\psi(0) \subset \mathcal{O}_i$ implies that $WF_{qsc}^{[T-\epsilon,T+\epsilon]} \psi \cap \mathcal{O}_i = \emptyset$. Extend the $b_{1,i}$ to be smooth functions on $qsc \mathcal{T}^* M$ with ess supp $b_{1,i} \subset \mathcal{O}_i$. Set $B_{1,i} = Op(b_{1,i})$. Then

$$\sum_i B_{1,i}^2 - I = C_1 \in \Psi^{-1,1}_{qsc}(M).$$

Let $c_1$ denote a representative of the symbol of $C_1$ in $A^{1,2}_{qsc}(\mathcal{T}^* M)$. Setting $b_{2,i} = -\frac{1}{2} c_1 b_{1,i}$ and $B_{2,i} = Op(b_{2,i})$, we have

$$\sum_i (B_{1,i} + B_{2,i})^2 - I = C_2 \in \Psi^{-2,2}_{qsc}(M).$$

Now let $c_2$ represent the symbol of $C_2$, set $b_{3,i} = -\frac{1}{2} c_2 b_{1,i}$ and $B_{3,i} = Op(b_{3,i})$, and continue in this manner, defining $B_{j,i}$ inductively. Then use asymptotic summation to obtain

$$A_i \sim \sum_j B_{j,i},$$

with $WF_{qsc} A_i \subset \mathcal{O}_i$ and $I = \sum A_i^2 + R$ with $R \in \Psi^{-\infty,\infty}_{qsc}(M)$.
By our construction of $O_j$, for all $i = 1, \ldots, k$ we have
\[ WF_{\text{qsc}}' A_i \cap WF_{\text{qsc}} [T-\epsilon, T+\epsilon] U(t) A_i \psi(0) = \emptyset \]
for $t \in [T - \epsilon, T + \epsilon]$, hence by microlocality,
\[ WF_{\text{qsc}} [T-\epsilon, T+\epsilon] A_i U(t) A_i \psi(0) = \emptyset \]
for any $\psi(0) \in C^{-\infty}(M)$, i.e. $A_i U(t) A_i \in C ([T - \epsilon, T + \epsilon]; \Psi^{-\infty, \infty}_{\text{qsc}}(M))$.

Smoothness in $t$ follows similarly, as
\[ D^k_i A_i U(t) A_i = A_i (-\mathcal{H})^k U(t) A_i, \]
and since $\mathcal{H} \in \Psi^{\infty}_{\text{qsc}}(M)$,
\[ WF_{\text{qsc}} [T-\epsilon, T+\epsilon] (-\mathcal{H})^k U(t) A_i \psi(0) \subset WF_{\text{qsc}} [T-\epsilon, T+\epsilon] U(t) A_i \psi(0). \]

5. The trace.

We begin the study of $\text{Tr} U(t)$ by showing that it exists as a distribution:

**Proposition 5.1.** — For $\phi \in \mathcal{S}(\mathbb{R})$,
\[ \int \phi(t) U(t) \, dt \in \Psi^{-\infty, \infty}_{\text{qsc}}(M) \]
and
\[ \phi \mapsto \text{Tr} \int \phi(t) U(t) \, dt \]
is a tempered distribution on $\mathbb{R}$.

**Proof.** — The structure of the argument is standard – see, for example, part II of [2]. We reproduce it only owing to the slight novelty of the Sobolev spaces involved.

Choose $\kappa \in \mathbb{R}$ below the spectrum of $\mathcal{H}$. Then by ellipticity of $\mathcal{H}$,
\[ (\kappa + \mathcal{H})^{-k} : H^{0,0}_{\text{qsc}}(M) \rightarrow H^{2k,0}_{\text{qsc}}(M). \]
Since
\[ U(t) = (\kappa + \mathcal{H})^k(\kappa + \mathcal{H})^{-k}U(t) = (\kappa - D_t)^k(\kappa + \mathcal{H})^{-k}U(t), \]
we can write
\[ (5.1) \int \phi(t)U(t)\, dt = \int (\kappa - D_t)^k\phi(t)(\kappa + \mathcal{H})^{-k}U(t)\, dt. \]

\( U(t) \) is unitary on \( H^0_{\text{qsc}}(M) \), so
\[ (\kappa + \mathcal{H})^{-k}U(t) : H^0_{\text{qsc}}(M) \to H^{2k,0}_{\text{qsc}}(M) \]
is bounded uniformly in \( t \). Since \( \bigcap_k H^{2k,0}_{\text{qsc}}(M) = \mathcal{C}^\infty(M) \), (5.1) shows that
\[ \int \phi(t)U(t)\, dt : \mathcal{C}^{-\infty}(M) \to \mathcal{C}^\infty(M), \]
i.e.
\[ \int \phi(t)U(t)\, dt \in \Psi^{-\infty,\infty}_{\text{qsc}}(M). \]

Furthermore, if we take \( k \) large enough so that \( (\kappa + \mathcal{H})^{-k}U(t) \) is trace-class, we see that \( \phi \mapsto \text{Tr} \int \phi(t)U(t)\, dt \) is a tempered distribution of order at most \( k \).

We are now in a position to prove our main theorem:

**Theorem 5.2.** — If there are no trapped geodesics in \( \tilde{M} \) then
\[ \text{sing supp Tr} U(t) \subset S_\omega. \]

**Proof.** — Let \( \phi \in C^\infty(\mathbb{R}) \) be 0 for \( x > 2 \) and 1 for \( x < 1 \). Set
\[ W_n = \text{Op}[(1 - \phi(nx))(1 - \phi(n\sigma))] \in \Psi^0_{\text{qsc}}(M); \]
then \( W_n \to I \) strongly on \( L^2(M) \). We regularize \( \text{Tr} U(t) \) by examining instead \( \text{Tr} U(t)W_n \); this is a smooth function on \( \mathbb{R} \) since \( D_r^p \text{Tr} U(t)W_n = \text{Tr}(-\mathcal{H})^pU(t)W_n \).

Given \( T \notin S_\omega \), we choose \( A_i, i = 1, \ldots, k \) as in Corollary 4.7, and write
\[ \text{Tr} U(t)W_n = \text{Tr} IU(t)W_n = \sum_{i=1}^k \text{Tr} A_i^2U(t)W_n + \text{Tr} R U(t)W_n. \]
$A_i U(t) W_n$ is trace-class, so we may now rewrite

$$\text{Tr} U(t) W_n = \sum_{i=1}^{k} \text{Tr} A_i U(t) W_n A_i + \text{Tr} R U(t) W_n.$$ 

As $n \to \infty$, $D_p^k R U(t) W_n$ converges to $D_p^k R U(t)$ in the norm topology on operators $H^{m,\ell}_{q}(M) \to H^{m',\ell'}_{q}(M)$ for any $m, \ell, m', \ell'$, and any $p \in \mathbb{Z}_+$; thus $\text{Tr} R U(t) W_n$ approaches a smooth function as $n \to \infty$. Thus, if we can also show that

1) $\lim_{n \to \infty} \text{Tr} U(t) W_n = \text{Tr} U(t)$, and

2) $\lim_{n \to \infty} \text{Tr} A_i U(t) W_n A_i = \text{Tr} A_i U(t) A_i$ for all $i = 1, \ldots, k$,

in the sense of distributions, we will have $\text{Tr} U(t) \in C^\infty([T - \epsilon, T + \epsilon])$ for some $\epsilon > 0$, and we will be done.

Both 1) and 2) follow from the following identity, which holds, in the distributional sense, for any $A \in \Psi_{q}(M)$ (and any $p, q$):

$$\lim_{n \to \infty} \text{Tr} A U(t) W_n A = \text{Tr} A U(t) A.$$

To prove this, let $\phi \in \mathcal{S}(\mathbb{R})$ be a test function, let $\kappa$ lie below the spectrum of $\mathcal{H}$, and write

$$\lim_{n \to \infty} \int \phi(t) \text{Tr} A U(t) W_n A dt \quad \text{.}$$

$$= \lim_{n \to \infty} \text{Tr} \int \phi(t) U(t) W_n A^2 dt$$

$$= \lim_{n \to \infty} \text{Tr} \int \phi(t)(\kappa - D_t)^m(\kappa + \mathcal{H})^{-m} U(t) W_n A^2 dt$$

$$= \lim_{n \to \infty} \int [(\kappa - D_t)^m \phi(t)] \text{Tr} [(\kappa + \mathcal{H})^{-m} U(t) W_n A^2] dt$$

$$= \lim_{n \to \infty} \int [(\kappa - D_t)^m \phi(t)] \text{Tr} [A(\kappa + \mathcal{H})^{-m} U(t) W_n A] dt$$

$$= \int \phi(t) \text{Tr} A U(t) A dt \quad ;$$

here we take $m$ large enough that $(\kappa + \mathcal{H})^{-m} U(t)$ is trace-class; the penultimate equality follows from the norm convergence

$$A(\kappa + \mathcal{H})^{-m} U(t) W_n A \to A(\kappa + \mathcal{H})^{-m} U(t) A$$

as operators $H^{0}_{q}(M) \to H^{2m-2p,2q-\epsilon}_{q}(M)$ for all $p, q$ and all $\epsilon > 0$. \[\Box\]
Appendix: the propagation theorems.

As noted above, the only obstacle to proving Theorems 3.6–3.9 in exactly the same manner as Theorems 12.1–12.5 of [13] is the fact that \((Y_{\pm\infty})_* X \neq 0\) in the harmonic oscillator case; we merely have

\[(Y_{\pm\infty})_* X = O(\sigma).\]

This makes no difference in proving Theorems 3.6 or 3.8, but we must modify the constructions of the symbols \(a_\pm\) and \(\tilde{a}_\pm\) used to prove the other three theorems.

We modify the symbols \(a_+^{m,\ell}\) and \(\tilde{a}_+^{m,\ell}\) defined in §13 of [13] by replacing the factor \(\psi_{-\infty} = \phi(d(Y_{-\infty}(p), y_0))\) (\(\phi\) is a cutoff function) by

\[
\tilde{\psi}_{-\infty} = \phi(d(Y_{-\infty}(p), y_0)^2 - \epsilon\sigma).
\]

Since \(X\sigma = -\tilde{\lambda} + O(\sigma^2) + O(x) = -1 + O(\sigma^2) + O(x) + O(|\tilde{\mu}|^2)\) and since \((Y_{-\infty})_* X = O(\sigma)\),

\[
-X(\tilde{\psi}_{-\infty})
= -\phi'(d(Y_{-\infty}(p), y_0)^2 - \epsilon\sigma) \left[ O(\sigma) + \epsilon + O(\sigma^2) + O(x) + O(|\tilde{\mu}|^2) \right].
\]

The quantity in square brackets is strictly positive for \(x, \sigma, \tilde{\mu}\) sufficiently small, and the constructions of \(a_+\) and \(\tilde{a}_+\) in [13] go through as before, with \(\tilde{\psi}_{-\infty}\) replacing \(\psi_{-\infty}\), and \(b_+\) constructed so as to ensure that \(\sigma\) is small on \(\text{supp } a_+\).

Similarly, in the construction of \(a_-\) and \(\tilde{a}_-\), we replace \(\psi_{+\infty}(q) = \phi(d(Y_{+\infty}(q)), y_0)\) with

\[
\tilde{\psi}_{+\infty}(q) = \phi(d(Y_{+\infty}(q), y_0)^2 + \epsilon\sigma).
\]

BIBLIOGRAPHY