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A generalization of Jaeger-Nomura’s Bose Mesner algebra associated to type II matrices


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A GENERALIZATION OF
JAEGER-NOMURA’S BOSE MESNER
ALGEBRA ASSOCIATED TO TYPE II MATRICES

by Makoto MATSUMOTO


Let $X$ be a finite set with cardinality $n$, and $\mathbb{C}[X]$ be the vector space with basis $X$ over the complex number field $\mathbb{C}$. We consider the set of matrices $M_X := \text{Hom}(\mathbb{C}(X), \mathbb{C}(X))$, which is identified with the algebra on $n \times n$ matrices, by identifying $X = \{1, 2, \ldots, n\}$. An $n \times n$ matrix $W \in M_X$ is said to be a (two-weight\(^1\)) spin model, if it satisfies three algebraic conditions called type I, type II, and type III conditions. These names came from the corresponding Reidemeister moves in the Knot theory, see Jones [6] and Bannai-Bannai [1]. Spin models were introduced to describe a link invariant. For a matrix $M \in M_X$, we denote by $M(a, b)$ its $(a, b)$-component for every $a, b \in X$. The usual product of matrices, denoted by $MM'$ or $M \cdot M'$, is defined by $M'M(a, b) = \sum_{x \in X} M'(a, x)M(x, b)$, while the Hadamard product $M' \circ M$ is defined by $(M' \circ M)(a, b) := M'(a, b)M(a, b)$, which is commutative. We denote by $J$ a matrix whose components are all 1, which is the unit for Hadamard product. If $M$ is invertible with respect to the Hadamard product, its inverse is denoted by $M^\circ$. Then, the type II condition is described by

$$M \cdot M^\circ = nI,$$

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\(^1\) Here a spin model means two weight, unless otherwise stated.
where $I$ denote the unit matrix of size $n$. The type I condition is automatically satisfied if $M$ has both type II and type III properties. Type III condition will be explained later.

1.1. A Bose-Mesner algebra associated with a type II matrix.

A matrix $B$ which satisfies the type II condition above is called a type II matrix. For a type II matrix, we introduced [4] after Jaeger [5] and Nomura [7] an algebra $N(W)$ as follows. For each $b, c \in X$, define a vector

$$y_{b,c} := \left( \frac{W(b,x)}{W(c,x)} \right)_{x \in X} \in \mathbb{C}[X].$$

Then we define an algebra (possibly called Jaeger-Nomura algebra of $W$)

$$N(W) := \{ M \in M_X \mid y_{b,c} \text{ is } M\text{'s eigen vector for all } b, c \in X \}.$$

It can be proved that $N(W)$ is a Bose-Mesner (BM) algebra, i.e. a commutative subalgebra of $M_X$ closed under Hadamard product and transpose with identity $I \in M_X$ for usual product and identity $J \in M_X$ for Hadamard product. (Thus, it corresponds to an association scheme.)

2. Spin models of index two.

Jaeger-Nomura [5] introduced the notion of index of a spin model. I don’t recall the definition, but according to their result we may define a spin model $W$ of index two to be a spin model with the following form:

$$W = \begin{pmatrix}
A & A & B & -B \\
A & A & -B & B \\
-B & B & C & C \\
B & -B & C & C
\end{pmatrix},
$$

which was fully used in [5] to construct a spin model with chiral link invariant, i.e. one which distinguishes some link from its mirror image.

**Proposition 1** ([5], Proposition 8). — Let $A, C$ be a symmetric matrix of the same size. Then, the matrix $W$ in (1) is a spin model if and only if the following conditions are satisfied:

(A) $A, C$ are spin models and $B$ is a type II matrix.
Let \( n \) be the size of \( W \), and \( D \) be one of \( \pm\sqrt{n} \). Then the following four equalities hold:

\[
\sum_y A(a, y) \frac{B(y, b)}{B(y, c)} = DC(b, c)^{-1} B(a, b) / B(a, c)
\]

\[
\sum_y C(a, y) \frac{B(b, y)}{B(c, y)} = DA(b, c)^{-1} B(b, a) / B(c, a)
\]

\[
\sum_y A(a, y)^{-1} B(y, b) B(y, c) = -DC(b, c)^{-1} B(a, b)^{-1} B(a, c)^{-1}
\]

\[
\sum_y C(a, y)^{-1} B(b, y) B(c, y) = -DA(b, c)^{-1} B(b, a)^{-1} B(c, a)^{-1}
\]

The concrete result of this article is just the following.

**Theorem 2.** — Let \( A, B, C, D \) be type II matrices. Then the conditions (2) and (3) are equivalent, and the conditions (4) and (5) are equivalent.

The first equivalence is a direct consequence of [4], Theorem 1, as shown below, but the second equivalence is more delicate and I prove it by introducing something like a duality on something like BM algebra.

**Proof of (2) \( \Leftrightarrow \) (3).** — Let us define \( y = b, c \) as in §1.1. Then (2) for all \( a \in X \) is equivalent to

\[
Ay_{b, c} = DC(b, c)y_{b, c},
\]

and thus \( A \in N(B) \). Also this shows that \( \Phi_W(A) = D \cdot C^0 \), where \( \Phi_W \) is defined as follows.

**Theorem 3 ([4], Theorem 1).** — Let \( W \) be a type II matrix of size \( n \). Let \( M \in N(W) \). Write \( \Phi_W(M)(a, b) \) for the eigenvalue of \( M \) with respect to \( y_{a,b} \). Thus, \( \Phi_W(M) \in MX \). Then, \( \Phi_W \) induces a dual pair

\[
\Phi_W : N(W) \to N(^tW),
\]

i.e. a linear isomorphism satisfying \( \Phi_W(M'M) = \Phi_W(M') \circ \Phi_W(M) \), \( \Phi_W(M' \circ M) = \frac{1}{n} \Phi_W(M')\Phi_W(M) \), and \( \Phi_{W'}(\Phi_W(M)) = n^tM \).

By this, the condition (2) is equivalently transformed to \( \Phi_W(A^{-1}) = D^{-1}C \) (since inverse is mapped to Hadamard inverse), then by taking \( \Phi_{W'} \) we have \( \Phi_{W'}(\Phi_W(A^{-1})) = D^{-1}\Phi_{W'}(C) \). The left hand side is \( n^t(A^{-1}) \),
which is $A^\circ$, since by type II condition on $A$ we have $^tA^\circ = n \cdot A^{-1}$. Thus it is equivalent to $\Phi_{tW}(C) = DA^\circ$, i.e. (3).

For the rest of Theorem 2, we use a similar generalization according to a categorical concept.

3. A category $N$.

We shall define a category $N$ (possibly called Jaeger-Nomura category), which is a generalization of Jaeger-Nomura algebra $N(W)$. $N$ has a beautiful duality $\Phi$, which is not a functor unfortunately.

3.1. An abstract nonsense.

Although it is a bit confusing, by $y$ we denote a function $X \times X \to C[X] - \{0\}$, or equivalently $n \times n$ nonzero vectors in $C[X]$ indexed by $X \times X$, i.e.

$$y = (y_{a,b} \in C[X], \neq 0 \mid a, b \in X).$$

(I.e. $y$ is not a single vector but an ordered set of vectors, from now on.) Let $y' := (y'_{a,b} \mid a, b \in X)$ be another set of vectors. We define the set of morphisms from $y$ to $y'$ by

$$\text{Hom}_N(y, y') := \{M \in M_X \mid My_{a,b} = c_{a,b}y'_{a,b} \text{ for some } c_{a,b} \in C$$

for all $a, b \in X\},$$

which is a subvector space of $M_X = \text{Hom}(C[X], C[X])$. It is clear that a morphism from $y$ to $y'$ can be composed with one from $y'$ to $y''$ to obtain one from $y$ to $y''$ (i.e. we defined a category $C$ with object $y$'s and morphisms $\text{Hom}_N(y, y')$). We define a linear homomorphism

$$\Phi : \text{Hom}_N(y, y') \to M_X$$

by $\Phi(M) := (c(a, b))_{a,b}$. It follows that

$$\Phi(M'M) = \Phi(M') \circ \Phi(M)$$

for $M' \in \text{Hom}_N(y', y'')$. Thus, $\Phi$ is a functor from $C$ to the category consisting of one object $*$, with homomorphisms $M_X$, with composition by Hadamard product and identity $J$. We denote this category by $M_X^\circ$.

If there is no fear of confusion, we omit the suffix of $\Phi$. 
3.2. An object associated with a pair of type II matrices.

Let $K, L$ be two type-II matrices on the same set $X$. Define a column vector by
$$y_{a,b}^{K,L} := (K(a,x)L(b,x))_{x \in X}.$$  
This gives a set of ordered $n \times n$ vectors indexed by $X \times X$, i.e. an object of $C$. We denoted it by $y^{K,L}$.

Remark. — In this terminology, $y_{b,c}$ defined in §1.1 is $y^{W,W^o}_{b,c}$. Moreover,
$$N(W) = \text{Hom}_N(y^{W,W^o}, y^{W,W^o}) = \text{End}_N(y^{W,W^o}),$$
which becomes an $C$-algebra by the general fact on additive category. Non trivial part is that it is closed also under Hadamard product.

3.3. A quasi duality.

The following is the main result of this article.

**Theorem 4.** — Let $K, L, K', L'$ be arbitrary type II matrices of same size. We write $\Phi(K,L;K',L')$ for $\Phi_{y^{K,L},y^{K',L'}}$.

1. $\Phi(K,L;K',L')$ defines a linear map
$$\text{Hom}_N(y^{K,L}, y^{K',L'}) \rightarrow \text{Hom}_N(y^{L',L^o}, y^{K'^o,K}).$$

2. $\Phi_{(L',L^o,K'^o,K)}(\Phi_{(K,L;K',L')}(M)) = nM$
holds. Thus, $\Phi$ is a linear bijection.

**Proof.** — All we need is to prove
$$\Phi(M)y^{L',L^o}_{(g,h)} = nM(g,h)y^{K'^o,K}_{(g,h)},$$
or equivalently
$$\sum_b \Phi(M)(a,b)L'(b,g)L(b,h)^{-1} = nM(g,h)K'^{-1}(a,g)K(a,h).$$

The definition of $\Phi(M)$ is given by
$$\sum_z M(c,z)K(a,z)L(b,z) = \Phi(M)(a,b)K'(a,c)L'(b,c)$$
holds for all \(a, b, c \in X\). We specialize \(c\) to \(g\), then we have
\[
\Phi(M)(a, b)L'(b, g) = K'(a, g)^{-1} \sum_z M(c, z)K(a, z)L(b, z).
\]
Plug this into the left hand side of (6) to have
\[
\sum_b \Phi(M)(a, b)L'(b, g)L(b, h)^{-1}
= K'(a, g)^{-1} \sum_b \sum_z M(c, z)K(a, z)L(b, z)L(b, h)^{-1}
= K'(a, g)^{-1} \sum_z M(c, z)K(a, z) \sum_b L(b, z)L(b, h)^{-1}
= nM(g, h)K'^{-1}(a, g)K(a, h) \quad (\text{since } L \text{ is type II}),
\]
as desired. \(\square\)

From now on, we shall denote
\[
(K, L) := y^{K, L},
N(K, L; K', L') := \text{Hom}_N(y^{K, L}, y^{K', L'}),
\]
and consider only the objects of the form \((K, L)\) in the category \(C\). Let \(N\) be the full subcategory of \(C\) induced by these objects. In other words, we consider pairs of type II matrices \((K, L)\), and a morphism from \((K, L)\) to \((K', L')\), denoted by \(N(K, L; K', L') := \text{Hom}_N(y^{K, L}, y^{K', L'})\).

**Remark.** \(\Phi\) is a generalization of duality between \(N(W)\) and \(N(\mathfrak{f}W)\). We have \(N(W) = N(W, W^\circ; W, W^\circ)\) by definition. If we specialize to \(K = K'\) and \(L = L'\), we have
\[
N(K, L; K, L) \xrightarrow{\Phi} N(\mathfrak{f}L, \mathfrak{f}L^\circ, \mathfrak{f}K^\circ, \mathfrak{f}K).
\]
This shows that \(\Phi\) is not a functor. By the way, the left hand side is a subalgebra of \(M_X\) with respect to the ordinary product. From 2 of Theorem 4, it follows that the right hand side is closed under the Hadamard product, and these two algebras are isomorphic as an abstract algebra. Imposing the right hand side to be an algebra with usual product, we may require \((\mathfrak{f}L, \mathfrak{f}L^\circ) = (\mathfrak{f}K^\circ, \mathfrak{f}K)\), i.e. \(L = K^\circ\). Then the right hand side is
\[
N(\mathfrak{f}K^\circ, \mathfrak{f}K; \mathfrak{f}K^\circ, \mathfrak{f}K),
\]
which is closed also by the usual product, thus we recover the Jaeger-Nomura algebra. Note that the transpose induces a linear bijection
\[
(7) \quad N(K, L; K', L') \xrightarrow{\text{t}} N(L, K; L', K').
\]
By composing with this transpose we have

\[ N(K, K^o; K, K^o) \xrightarrow{\Phi} N(tK^o, t^oK; t^oK^o, tK) \xrightarrow{t} N(t^oK, t^oK^o; tK, tK^o), \]

i.e. the duality

\[ N(K) \xrightarrow{\Phi} N(tK) \]

defined in [4], §3.1 (note that \( N(tK) \) is \( N'(K) \) there).

4. Application of the category \( N \).

4.1. Proof of (4) \( \Leftrightarrow \) (5).

(4) is equivalent to

\[ A^o \in N(tB, t^oB; tB^o, t^oB^o) \text{ and } \Phi(A^o) = -DC. \]

We calculate

\[ \Phi(A^o \cdot A) = \Phi(A^o) \circ \Phi(A). \]

The left hand side is \( \Phi(nJ) \) by type II property of \( A \), then by definition of \( \Phi \), it is \( nJ \). The right hand side is, by condition (4), \( -DC \circ \Phi(A) \). By moving \( DC \) to the left hand side, we have

\[ -DC^o = \Phi(A). \]

Applying \( \Phi \), we have

\[ \Phi(C^o) = -D^{-1}\Phi(\Phi(A)) = DA, \]

which is nothing but the condition (5).

4.2. Four-weight spin model.

Four-weight spin model was introduced in [1] as a generalization of two-weight model, which was fully used in determination of the link invariant in [5]. We can paraphrase their definition in terms on \( N \). For the original definition, see [1].

**PROPOSITION 5.** — A pair of \( n \times n \) matrix \( (W_1, W_2) \) is a four weight spin model if and only if the following conditions hold:

1. \( W_1, W_2 \) are type II matrices.
2. $W_1 \in N( t W_2^\circ, t W_1; t W_2, t W_1 )$ and $\Phi( t W_2^\circ, t W_1; t W_2, t W_1 ) = D$ $t W_2^\circ$ (which is automatically in $N( W_1, W_1^\circ; W_2, W_2^\circ )$).

3. $W_1 \in N( W_2, W_1^\circ; W_2, W_1^\circ )$ and $\Phi( W_2, W_1^\circ; W_2, W_1^\circ ) = D W_2^\circ$ (which is automatically in $N( t W_1^\circ, t W_1; t W_2^\circ, t W_2 )$).

**Proof.** — In the terminology in [1], $Y^{i,j} = y^{W_i, W_j}$. Their condition $\text{III}_1$ is equivalent to 2, and their condition $\text{III}_9$ is equivalent to 3. □

**Remark.** — It is shown that $\text{III}_1$ to $\text{III}_8$ are equivalent, and $\text{III}_9$ to $\text{III}_{16}$ are equivalent in [1], where these 16 conditions naturally arise from Reidemeister move of type III with various orientations. These equivalences are easily proved by $N$ and $\Phi$. For example, note that $\text{III}_1$ is equivalent to $W_4 = t W_2^\circ = \frac{1}{D} \Phi(W_1)$, and by taking $\Phi$ again, we have the condition $\text{III}_2$. Then, $t W_3 = W_1^\circ$ and $\Phi(A^\circ) = n \cdot (\Phi(A^\circ))$ implies $\text{III}_3$, etc.

**Remark.** — Huang-Guo [2] proved that $t W_1^\circ \cdot W_1^\circ \in N( t W_2 )$. This can be interpreted in $N$ by

$W_1^\circ \in N( t W_2, t W_2^\circ; t W_1, t W_1^\circ )$ (III$_8$)

and

$t W_1^\circ \in N( t W_1, t W_1^\circ; t W_2, t W_2^\circ )$ (III$_2$)

and thus

$t W_1^\circ \cdot W_1^\circ \in N( t W_2, t W_2^\circ; t W_2, t W_2^\circ ) = N( t W_2 )$.

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**BIBLIOGRAPHY**


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