ANNALES DE L'INSTITUT FOURIER

MICHAEL TARSI

The graph polynomial and the number of proper vertex coloring

Annales de l'institut Fourier, tome 49, nº 3 (1999), p. 1089-1093

<http://www.numdam.org/item?id=AIF_1999__49_3_1089_0>

Ann. Inst. Fourier, Grenoble
49, 3 (1999), 1089–1093

THE GRAPH POLYNOMIAL AND THE NUMBER OF PROPER VERTEX COLORINGS

by Michael TARSI

---

1. Introduction.

Let $G = (V, E)$ be a graph, $V = \{1, 2, \ldots, n\}$, let

$$P_G = \prod_{ij \in E, i<j} (x_j - x_i)$$

be its graph polynomial, and let $\overline{P^k}_G$ be the remainder of this polynomial modulo the ideal generated by the polynomials $x_i^k - 1$, $1 \leq i \leq n$. Put $Z_k^n = \{0, 1, \ldots, k-1\}^n$. For a polynomial

$$P = P(x_1, \ldots, x_n) = \sum_{v \in Z_k^n} a_v \prod_{i=1}^{n} x_i^{v_i}.$$

Define $||P||_2^2 = \sum_{v \in Z_k^n} |a_v|^2$.

In a recent joint work with Noga Alon, [2] we proved the folowing result, in which $C_k(G)$ is the set of all proper colorings c of a graph G by the k colors $\{0, \ldots, k-1\}$.

THEOREM 1.1. — *In the above notation*

$$(1) \qquad ||\overline{P^k}_G||_2^2 = \frac{4^{|E|}}{k^n} \sum_{c \in C_k(G)} \prod_{ij \in E, i<j} \sin^2 \left[\frac{\pi(c(i) - c(j))}{k}\right].$$

Keywords: Graph coloring – Graph polynomial.
Math. classification: 05C35.

Theorem 1.1 clearly provides a lower bound for the number of proper k-colorings of a graph G in terms of $||\overline{P^k}_G||_2^2$. For the special case $k = 3$ this is an equality, showing that the precise number $|C_3(G)|$ of proper 3-colorings satisfies

$$||\overline{P^3}_G||_2^2 = 3^{|E|-n}|C_3(G)|. \tag{2}$$

The following notation is used in the sequel: A *Partial orientation* of a graph is obtained when some edges are assigned with an orientation while the other edges remain undirected.

Given a partial orientation and a vertex x, the number of oriented edges with their tails, (heads) at x is denoted by $d^+(x)$, $(d^-(x))$.

A $k, 1-$flow in a graph $G = (V, E)$ is a partial orientation of G, where for every vertex $x \in V$, $d^+(x) - d^-(x) \equiv 0 \pmod k$.

$F_k^1(G)$ stands for the set of all $k, 1-$flows of G.

The *support* $\sigma(f)$ of a flow $f \in F_k^1(G)$ is the set of edges of G which are oriented in f.

The main result of this paper is the following equality, which gives another combinatorial interpretation to the left hand side of 1:

THEOREM 1.2.

$$||\overline{P^k}_G||_2^2 = (-1)^{|E|} \sum_{f \in F_k^1(G)} (-2)^{|E|-|\sigma(f)|}. \tag{3}$$

Note that $F_k^1(G)$ corresponds to the subset of $F_k(G)$ (the set of all Z_k-flows in G) where the permitted flow values are $1, -1$ and 0. Since $Z_3 = \{0, 1, -1\}$, in the special case of $k = 3$ we obtain

$$||\overline{P^3}_G||_2^2 = (-1)^{|E|} \sum_{f \in F_3(G)} (-2)^{|E|-|\sigma(f)|} \tag{4}$$

and by 2

$$3^{|E|-n}|C_3(G)| = (-1)^{|E|} \sum_{f \in F_3(G)} (-2)^{|E|-|\sigma(f)|}. \tag{5}$$

Considering the parity of the right hand side, all summands are clearly even, except those corresponding to flows f for which $\sigma(f) = E$. These flows are the $3-$ *Nowhere Zero Flows* $(3 - NZF)$ of G. This implies:

THEOREM 1.3. — *The set of* 3*−Nowhere zero flows* ($NZF_3(G)$) *and the set of proper 3-colorings of a graph G are of the same parity, or more precisely*

$$|C_3(G)| \equiv |NZF_3(G)| (\bmod 4). \tag{6}$$

Remark. — Clearly $|C_3(G)|$ is always divisible by 6 (permutations of the 3 colors) and flows come in pairs (obtained by reversing the entire orientation). It is a common convention, however, to refer to permutations of the colors as the same coloring (e.g. phrases like "uniquely k-colorable" etc.). Accordingly, when talking of an "odd" $|C_3(G)|$ we actually refer to 2 modulo 4 and the same holds to the parity of $NZF_3(G)$.

One can easily observe that in 4-regular graphs, Eulerian orientations are the only 3−Nowhere zero flows. The following is then a direct consequence of Theorem 1.3:

THEOREM 1.4. — *The set of proper 3-colorings of a 4-regular graph G and the set of Eulerian orientations of G have the same parity.*

Fleischner and Stiebitz [3] proved that in every *'Cycle and triangles'* graph there is an odd number of Eulerian orientations. They used this result and the *'Graph polynomial method'* presented in [1], to prove that such graphs are 3-colorable. Recently Sachs [4] proved, by means of direct induction, that the number of 3-colorings of every *Cycle and triangles* graph is odd. Theorem 1.4 shows that Sachs result can be derived from the Fleischner and Stiebitz Lemma. This observation provides an affirmative answer to a question posted by Jack Edmonds.

2. Proving the main result.

As observed in [1], each term in the expansion of P_G to the sum of $2^{|E|}$ monomials, corresponds to an orientation of G. Similar terms, that is monomials with the same degree for every variable, correspond to orientations which share the same outdegree sequence $\{d^+(x)|x \in V\}$.

When computing $\overline{P^k}_G$, two monomials are similar if the corresponding degrees are congruent modulo k. The outdegree sequences corresponding to such two monomials are clearly, term by term, congruent modulo k. We refer to such orientations as $k-$equivalent.

Let ω_1 and ω_2 be two orientations of a graph $G = (V, E)$, and let $\omega_1 \triangle \omega_2$ denote the set of edges whose orientation is reversed when going from ω_1 to ω_2. For ω_1 and ω_2 to be k-equivalent, $\omega_1 \triangle \omega_2$ should satisfy (as a subgraph of any of the two orientations) $d^+(x) \equiv d^-(x) \pmod{k}$ for every $x \in V$. In other words, $\omega_1 \triangle \omega_2$ should be the support of a $k, 1-$flow in G.

Furthermore, selecting either x or y as the head of an edge (x, y), corresponds to the selection of either x or $-y$ from the term $(x - y)$, in the expansion of P_G (and of $\overline{P^k}_G$ as well). Two $k-$equivalent orientations ω_1 and ω_2 then yield similar monomials, of the same sign, if $|\omega_1 \triangle \omega_2|$ is even, or of different signs, if it is odd.

Let us agree that an edge (x_i, x_j) with $i < j$ is *oriented backwards* if its head is at x_i, and define parity$(\omega) = 0$ if the number of backwards oriented edges is even and parity$(\omega) = 1$ if this number is odd.

For $v \in Z_k^n$ let S_v be the $k-$equivalence class of orientations of G with $d^+(x_i) \equiv v_i \pmod{k}$, $1 \le i \le n$, and let S_v^0, respectively S_v^1 denote the subsets of even, respectively odd orientations in that class.

For any $\omega \in S_v$, let $(F_k^1)^0(\omega)$, respectively $(F_k^1)^1(\omega)$ be the sets of $k, 1-$flows, the supports of which are even, respectively odd oriented subgraphs of ω.

Following the discussion above we obtain

$$a_v = |S_v^0| - |S_v^1| \tag{7}$$

and for $i = 0, 1$ ($\overline{0} = 1$, $\overline{1} = 0$) and any $\omega \in S_v^i$:

$$|(F_k^1)^i(\omega)| = |S_v^0| \quad \text{and} \quad |(F_k^1)^{\overline{i}}(\omega)| = |S_v^1|. \tag{8}$$

Consider now the quantity

$$\sum_{\omega \in S_v} (|(F_k^1)^0(\omega)| - |(F_k^1)^1(\omega)|).$$

By 8 it can be rewritten as:

$$\sum_{\omega \in S_v^0} |S_v^0| - |S_v^1| + \sum_{\omega \in S_v^1} |S_v^1| - |S_v^0|.$$

By 7, this equals:

$$\sum_{\omega \in S_v^0} a_v - \sum_{\omega \in S_v^1} a_v = (|S_v^0| - |S_v^1|) a_v = a_v^2.$$

Summing up the above over all equivalent classes we obtain

$$(9) \qquad ||P_G^k||_2^2 = \sum_{\omega \in D(G)} (|(F_k^1)^0(\omega)| - |(F_k^1)^1(\omega)|)$$

where $D(G)$ is the set of all orientations of G.

Let us now change the order of summation in the right hand side of 9 by taking $k, 1-$flows, instead of orientation, as the leading index:

$$(10) \qquad ||P_G^k||_2^2 = \sum_{f \in F_k^1(G)} (-1)^{|\sigma(f)|} |D(f)|.$$

Here $D(f) = \{\omega | f \in F_k^1(\omega)\}$ is the set of all expansions of f to (complete) orientations (of all the edges) of G.

An expansion of a partial orientation is constructed by selecting an orientation for each undirected edge and hence $|D(f)| = 2^{|E|-|\sigma(f)|}$ which completes the proof of Theorem 1.2.

BIBLIOGRAPHY

[1] N. ALON and M. TARSI, Colorings and orientations of graphs, Combinatorica, 12 (1992), 125–134.

[2] N. ALON and M. TARSI, A note on graph colorings and graph polynomials, J. Comb. Theory, B 70 (1997), 197–201.

[3] H. FLEISCHNER and M. STIEBITZ, A solution to a Colouring problem of P. Erdös, Discrete Math., 101 (1992), 39–48.

[4] H. SACHS, Elementary proof of the cycle-plus-triangles theorem, in: Combinatorics, Paul Erdös is Eighty, Janos Bolyai Math. Soc., Budapest, 1993, Vol 1, 347–359.

Michael TARSI,
Tel Aviv University
Raymond and Beverly Sackler
Faculty of Exact Sciences
School of Mathematical Sciences
Tel Aviv (Israel).
tarsi@math.tau.ac.il