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CURVATURE FLOWS OF MAXIMAL INTEGRAL TRIANGULATIONS

by Roland BACHER

We describe maximal integral triangulations in terms of a function $\varphi : E^{\rightarrow} \rightarrow \mathbb{Z}$ on the set of oriented edges $E^{\rightarrow}$ of such triangulations.

Such functions φ, called *curvature flows*, satisfy for each interior vertex of the triangulation an identity which is a kind of discrete Gauss-Bonnet formula (Theorem 1.3).

We give then locally necessary and sufficient conditions (Theorem 3.2 for interior vertices and Theorem 2.2 for vertices on straight parts of the boundary) on a function $\psi : E^{\rightarrow} \rightarrow \mathbb{Z}$ to be the curvature flow of a maximal integral triangulation.

1. Maximal integral triangulations.

We denote by $\mathrm{Aff}(\mathbb{R}^2)$ the group of affine transformations of the plane $\mathbb{R}^2$. This group is the semi-direct product of the general linear group $\mathrm{GL}_2(\mathbb{R})$ with the group $\mathbb{R}^2$ of translations. We denote by $\mathrm{Aff}^+(\mathbb{R}^2)$ its subgroup of index 2 consisting of all orientation-preserving elements.

By $\mathrm{Aff}^+(\mathbb{Z}^2)$ we mean the subgroup of $\mathrm{Aff}^+(\mathbb{R}^2)$ whose elements induce bijections on the affine lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ of integral points. This group is isomorphic to the semi-direct product of $\mathrm{SL}_2(\mathbb{Z})$ with the lattice $\mathbb{Z}^2$ and its elements preserve the usual Lebesgue measure (*i.e.* the area) of $\mathbb{R}^2$.

Keywords : Curvature flow – Farey sequence – Farey tree – Maximal integral triangulation.
Math. classification : 05B45 – 11A55 – 11H06 – 52C05.

1.1. *Definitions.* — A *triangulation* of a subset $D \subset \mathbb{R}^2$ is a set $\tau = \{\Delta_i\}_{i\in I}$ of triangles such that $D = \bigcup_{i\in I} \Delta_i$ and the intersection $\Delta_i \cap \Delta_j$ of two distinct triangles $\Delta_i \neq \Delta_j$ is either a common edge, a common vertex or empty.

A triangulation is *locally finite* if every compact set $K \subset D$ of D intersects only a finite number of triangles.

In the sequel we consider only locally finite triangulations of connected subsets $D \subset \mathbb{R}^2$ with boundary ∂D a (generally non-smooth) 1-submanifold of $\mathbb{R}^2$.

A triangle Δ of $\mathbb{R}^2$ is *integral* if all vertices of Δ have integral coordinates.

An *integral triangulation* is a triangulation containing only integral triangles (this implies that the boundary ∂D of D is a union of segments joining points in $\mathbb{Z}^2$).

An integral triangulation is *maximal* if all triangles have area $\frac{1}{2}$.

1.2. *Remark.* — Any integral triangulation can be refined to a maximal integral triangulation. This results from *Picks Formula* (*cf.* for instance [C]) which states that

$$\mathrm{Area}(D) = \sharp\{\mathbb{Z}^2 \cap D^0\} + \tfrac{1}{2}\sharp\{\mathbb{Z}^2 \cap \partial D\} - \chi(D)$$

(where D^0 denotes the interior, ∂D the boundary and $\chi(D)$ the Euler characteristic of D) for any compact subset $D \subset \mathbb{R}^2$ with boundary ∂D a union of segments having endpoints in $\mathbb{Z}^2$.

The affine group $\mathrm{Aff}^+(\mathbb{R}^2)$ acts on triangulations. $\mathrm{Aff}^+(\mathbb{Z}^2)$ acts on integral triangulations and maximal integral triangulations. Two triangulations are *isomorphic* or *equivalent* if they are in the same orbit of $\mathrm{Aff}^+(\mathbb{R}^2)$. For integral triangulations we require equivalent triangulations to be in the same orbit of $\mathrm{Aff}^+(\mathbb{Z}^2)$. Two triangulations are *combinatorially isomorphic*, if their 1-skeletons are isomorphic as embedded planar graphs.

Let Δ and Δ' be two triangles as in Figure 1.1 which have the same area and share a common edge AB.

Working with affine coordinates we get for the distinct vertices $C \neq C'$ of Δ and Δ'

$$C' = -C + (A+B) + \lambda(B-A) \qquad = -C + (A+B) + (-\lambda)(A-B),$$

$$C = -C' + (A+B) + (-\lambda)(A-B) = -C' + (A+B) + \lambda(B-A),$$

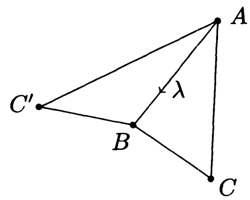

Figure 1.1. Two adjacent triangles of the same area.

for some real number λ, which is unique and depends only on the isomorphism class of the triangulation $\Delta \cup \Delta'$.

If both triangles Δ and Δ' are integral triangles of area $\frac{1}{2}$, then it is easy to check that the number λ relating them is integral.

We associate the number λ to the oriented edge starting at A and ending at B. The oriented edge starting at B and ending at A corresponds hence to $-\lambda$. The number λ shows how to construct the triangle at the right side of an oriented edge from the triangle at the left side. Such an edge, which is always common to exactly two triangles of the triangulation is called an *interior* edge.

Given a triangulation τ consisting of triangles with the same area, we get in this way a real function φ on the set $E_i^{\rightarrow}$ of oriented interior edges which satisfies $\varphi(a^{\rightarrow}) = -\varphi(a^{\leftarrow})$ with $a^{\rightarrow}$ and $a^{\leftarrow}$ denoting the two opposite orientations of an edge a.

We call this function φ the *curvature flow* of the triangulation τ. It describes the triangulation τ up to equivalence.

We denote by $\omega(a^{\rightarrow})$ the endpoint of an oriented edge $a^{\rightarrow}$. A *flow* is a function $f : E^{\rightarrow} \longrightarrow \mathbb{R}$ on the set of oriented edges of a graph such that $f(a^{\rightarrow}) = -f(a^{\leftarrow})$ and

$$\sum_{a^{\rightarrow},\, \omega(a^{\rightarrow})=v} f(a^{\rightarrow}) = 0$$

for any vertex v. The following result shows that the curvature flow φ of a maximal integral triangulation τ is almost a flow.

1.3. Theorem. — *Let τ be a maximal integral triangulation of some subset $D \subset \mathbb{R}^2$ and let $\varphi : E_i^{\rightarrow} \longrightarrow \mathbb{Z}$ be its curvature flow. Let v be an* *interior vertex with degree* $\deg(v)$ *of the triangulation* τ*. We have then*

$$2(6-\deg(v)) = \sum_{\substack{a^{\rightarrow}\in E^{\rightarrow}\\ \omega(a^{\rightarrow})=v}} \varphi(a^{\rightarrow}).$$

This result will be an easy corollary of Theorem 3.2 which describes all possible configurations around a vertex in a maximal integral triangulation.

1.4. *Remarks.*

(i) Theorem 1.3 does not hold for non-integral triangles. The following figure shows five triangles of area $\frac{1}{2}$ surrounding a central vertex at the origin:

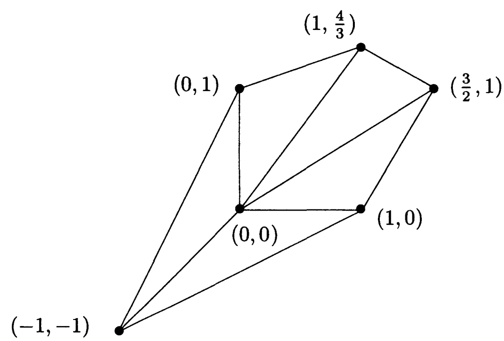

Figure 1.2. A triangulation with non-integral triangles.

The curvature flows of the five edges directed toward the vertex $(0,0)$ are $1/2$, $-1/3$, $-1/2$, $2/3$, 2 (read counterclockwise, starting at the horizontal edge). Since

$$\frac{1}{2}-\frac{1}{3}-\frac{1}{2}+\frac{2}{3}+2=\frac{7}{3}\neq 2(6-5)=2,$$

the curvature flow of this triangulation does not satisfy the equation of Theorem 1.3.

(ii) An easy corollary of Theorem 1.3 is a special case of the following well-known result: A compact orientable surface Σ without boundary which can be endowed with an affine structure is necessarily a torus.

If the affine structure on Σ is integral, then we can always get a maximal integral triangulation τ on Σ. Realizing Σ by gluing isometrical regular Euclidean triangles accordingly to τ, one sees that the total curvature of Σ is necessarily zero.

(iii) Theorem 1.3 reminds the Gauss-Bonnet theorem (*cf.* for instance [DC])

$$2\pi\chi(\Sigma) = \int_\Sigma \kappa \, \mathrm{d}\mu$$

where κ denotes the curvature of a compact oriented surface Σ without boundary endowed with a Riemannian metric and associated area measure $\mathrm{d}\mu$.

2. Halfstars and halfstar-sequences.

2.1. *Definitions.* — A *k-halfstar* is (up to action of $\mathrm{Aff}^+(\mathbb{Z}^2)$) a maximal integral triangulation of "half" a neighbourhood of $(0,0)$ which is contained in the the halfplane $H = \{(x,y),\ y \geq 0\}$ and which consists of k triangles all containing $(0,0)$ among their vertices. We require that the points $(1,0)$ and $(-1,0)$ are among the vertices of the halfstar. The segment $[(-1,0),(1,0)]$ is the *base* of the halfstar (see Figure 2.1 for an example).

The curvature flow of a k-halfstar τ is only defined on the $(k-1)$ interior (oriented) edges. Its values (for instance read clockwise around the origin) yield a sequence $\Lambda = (\lambda_1, \ldots, \lambda_{k-1})$ which we call the *k-halfstar-sequence* of τ.

Halfstars are determined (up to action of $\mathrm{Aff}^+(\mathbb{Z}^2)$) by their halfstar sequence. The Figure 2.1 shows an example.

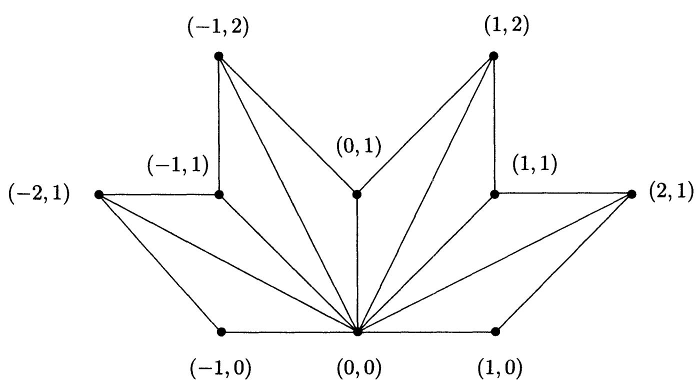

Figure 2.1. The 8-halfstar $(0,-2,0,-3,0,-2,0)$.

The following theorem yields an algorithm which checks whether a given sequence $(\lambda_1, \ldots, \lambda_{k-1})$ is a k-halfstar-sequence.

2.2. THEOREM. — *A sequence* $\Lambda = (\lambda_1, \ldots, \lambda_{k-1})$ *is a* k*-halfstar-sequence if and only if either*

(i) $k = 2$ *and* $\Lambda = (1)$ *or*

(ii) *there exists* $1 \leq i \leq k-1$ *such that* $\lambda_i = 0$ *and the sequence*

$$(\lambda_1, \ldots, \lambda_{i-2}, \lambda_{i-1} + 1, \lambda_{i+1} + 1, \lambda_{i+2}, \ldots, \lambda_{k-1})$$

is a $(k-1)$*-halfstar-sequence (of course, if* $i = 1$*, respectively* $i = k-1$*, the term* $\lambda_0 + 1$*, respectively* $\lambda_k + 1$*, does not occur in the above sequence).*

Moreover, condition (ii) *is equivalent to condition*

(ii′) *there exists an integer* $1 \leq i \leq k-1$ *with* $\lambda_i = 0$ *and for any such integer* i *the sequence*

$$(\lambda_1, \ldots, \lambda_{i-2}, \lambda_{i-1} + 1, \lambda_{i+1} + 1, \lambda_{i+2}, \ldots, \lambda_{k-1})$$

is a $(k-1)$*-halfstar-sequence.*

2.3. DEFINITION. — *A generalized Farey-sequence is a sequence*

$$\frac{p_0}{q_0} = \frac{0}{1}, \ \frac{p_1}{q_1}, \ \ldots \ , \ \frac{p_{k-1}}{q_{k-1}}, \ \frac{p_k}{q_k} = \frac{1}{1}$$

of rational numbers $p_i/q_i \in [0,1]$ *such that*

$$p_i q_{i+1} - p_{i+1} q_i = -1 \quad \text{for all} \quad 0 \leq i < k.$$

In particular, generalized Farey sequences are strictly increasing.

The *Farey-tree* FT is the infinite planar 3-regular tree in the oriented plane $\mathbb{R}^2$ with one semi-infinite edge streching down from the "root" $(0, +\infty) \in \overline{\mathbb{R}^2}$, splitting into two downward edges which both in turn split into two such edges and so on (see Figure 2.2). The connected components of $\mathbb{R}^2 \setminus FT$ are then labeled by rational numbers as follows: Label the two uppermost components $0/1$ and $1/1$. Every other component C has then exactly two adjacent components C', C'' which stretch out higher than C. Label C by $(a+c)/(b+d)$ where a/b and c/d are the labels of C' and C''.

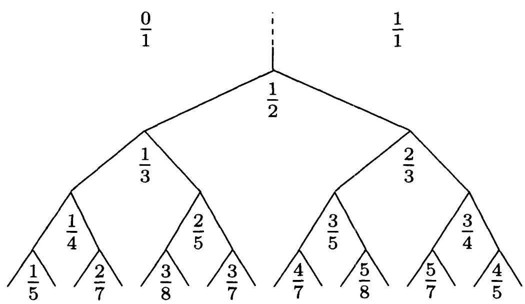

Figure 2.2. A part of the Farey tree.

This labelling induces a bijection of all connected components in $\mathbb{R}^2 \setminus FT$ and the set of rational numbers in $[0,1]$ and the combinatorics of the Farey tree encodes all the arithmetic properties of continued fractions. In particular, finite connected subtrees stretching out to the root $(0,+\infty)$ of FT and which have only interior vertices of degree 3 (we call such trees *finite* 3*-regular rooted trees*) are easily proved to be in bijection with Farey sequences (*cf.* Figure 2.2 which shows a small part of the Farey tree.

One has the following result which we state without (the straightforward) proof.

2.4. THEOREM. — *Given a generalized Farey sequence*

$$\frac{0}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \cdots < \frac{p_k}{q_k} = \frac{1}{1}$$

there exists a unique locally finite k-halfstar with base $[(0,-1),(0,1)]$, *center* $(0,0)$ *and vertices*

$$(0,0) \cup \{(p_0,q_0) = (0,1), (p_1,q_1), \ldots, (p_k,q_k) = (1,1), (0,-1)\}.$$

Moreover, this application induces a bijection between generalized Farey sequences (or finite 3-regular rooted subtrees of FT) and equivalence classes of (locally finite) halfstars.

2.5. COROLLARY. — *There exist exactly*

$$\binom{2(k-2)}{k-2}\frac{1}{k-1}$$

distinct equivalence classes of k-halfstars.

Proofs.

Given a k-halfstar τ and a triangle $\Delta \in \tau$ with vertices $(0,0), A, B$, we get a $(k+1)$-halfstar τ' by replacing the triangle Δ with the two triangles Δ' and Δ'' having respectively vertices $(0,0), A, (A+B)$ and $(0,0), (A+B), B$. We say that τ' is obtained by an *elementary subdivision* of τ.

2.6. LEMMA. — (i) *Every $(k+1)$-halfstar with $k \geq 2$ contains an interior edge with zero curvature flow.*

(ii) *An interior edge of zero curvature flow in a $(k+1)$-halfstar separates two triangles Δ' and Δ'' obtained by elementary subdivision of some k-halfstar $\tilde{\tau}$ with respect to some triangle $\Delta \in \tilde{\tau}$.*

Proof of Lemma 2.6. — Let τ be a $(k+1)$-halfstar. Choose a vertex $(x,y) \in \mathbb{Z} \times \mathbb{N}$ of maximal Euclidean norm among all vertices of τ. The edge e between $(0,0)$ and (x,y) is then necessarily an interior edge. Denote by λ its curvature flow (directed from (x,y) to $(0,0)$) and let Δ' and Δ'' be the two triangles of τ which are at the left and at the right of the oriented edge e. The triangle Δ' has vertices $(0,0)$, (a,b) and (x,y) and the triangle Δ'' has vertices $(0,0)$, (x,y) and (α,β). We have then

$$(\alpha,\beta) = -(a,b) + (1-\lambda)(x,y)$$

with $\lambda \in \mathbb{Z}$. Since τ is a halfstar, the curvature flow λ must be ≤ 0. The triangle inequality and the maximality of the norm of (x,y) imply $\lambda = 0$. This proves (i).

A more geometric proof of (i) is given by remarking that the quadrangle with vertices $(0,0), (\alpha,\beta), (x,y)$ and (a,b) is strictly convex and has area 1. The area of the integral triangle with vertices $(0,0), (\alpha,\beta)$ and (a,b) must hence be $\frac{1}{2}$ and a sketch on a sheet of paper (or a short computation) shows that we have $(x,y) = (\alpha,\beta) + (a,b)$.

Assertion (ii) follows from a short computation. □

2.7. LEMMA. — *Let $\Lambda = (\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{k-1})$ be the k-halfstar sequence associated to a k-halfstar τ. Let τ' be the $(k+1)$-halfstar obtained by elementary subdivision of the triangle Δ having interior edges associated to the curvature flows λ_i and λ_{i+1}. The $(k+1)$-halfstar sequence Λ' of τ' is then given by*

$$\Lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, 0, \lambda_{i+1} - 1, \lambda_{i+2}, \ldots, \lambda_{k-1}).$$

Proof of Lemma 2.7. — Denote by A and B the two vertices of Δ which are different from $(0,0)$. Suppose that the oriented edge $e^{\rightarrow}$ from A to $(0,0)$ has curvature flow λ_i. The unique triangle distinct from Δ with common oriented edge $e^{\rightarrow}$ has then vertices $(0,0), A$ and $-B + (1-\lambda_i)A = -(A+B) + (1-(\lambda_i - 1))A$. This shows that the curvature flow of the edge e in the elementary subdivision of Δ takes the value $\lambda_i - 1$. For the adjacent triangle with common oriented edge from B to $(0,0)$ we get similarly $-A+(1-\lambda_{i+1})B = -(A+B)+(1-(\lambda_{i+1}))B$ which yields a curvature flow of $\lambda_{i+1} - 1$ of the corresponding (oriented) edge in the elementary subdivision. Finally we have $B = -A + (1-0)(A+B)$ which shows that the curvature flow between the two triangles Δ' and Δ'' replacing Δ is zero. This shows the result. □

Proof of Theorem 2.2. — There exists (up to action of the affine group $\mathrm{Aff}^+(\mathbb{Z}^2)$) a unique 2-halfstar and it is easy to check that the associated halfstar sequence is (1).

By induction, we can suppose that Theorem 2.2 describes all k-halfstar sequences for a given integer $k \geq 2$. By Lemma 2.6 every $(k+1)$-halfstar sequence is associated to an elementary subdivision of some k-halfstar τ and Lemma 2.7 relates the corresponding halfstar-sequences.

The equivalence of conditions (ii) and (ii′) follows from Lemma 2.6 (ii) and Lemma 2.7. □

Proof of Corollary 2.5. — The corresponding result is well known for rooted planar 3-regular subtrees of FT (see for instance exercice 14, p. 85, of [A] for the equivalent problem of counting parenthesis systems or reprove it using generating series).

The bijection (Theorem 2.4) between generalized Farey sequences (which correspond to finite 3-regular rooted subtrees of FT) and equivalence classes of halfstars implies then the result. □

3. Stars and star-cycles.

Stars have many common features with halfstars. This explains the scent of "déjà vu" in this section.

3.1. *Definitions.* — A *k-star* is a maximal integral triangulation of a neighbourhood of some integral point v (called the *center* of the star) consisting of k triangles all containing v among their vertices.

A k-star gives rise to a cyclic sequence of k integers by reading cyclically the values of the curvature flow of the k edges pointing to its center v. We call such a sequence a *k-star-cycle.*

Given a k-star σ with center v, a straight line through v which intersects every triangle of the star either at v or along an edge, is a *cut.* The number $\mathrm{cut}(\sigma)$ of such cuts is the *cut number* of σ.

A k-star is *exceptional* if it has cut number > 1.

Stars (analogously to half-stars) are determined (up to action of $\mathrm{Aff}^+(\mathbb{Z}^2)$) by their star-cycles.

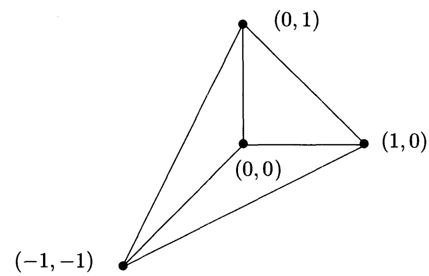

Figure 3.1. A 3-star of star-cycle $(2,2,2)$.

3.2. Theorem. — *A cyclic sequence $C=(\lambda_1,\dots,\lambda_k)$ is a k-star-cycle if and only if one (at least) of the following conditions holds:*

(i) $C=(2,2,2)$;

(ii) $C=(1,k+1,1,-k+1)$;

(iii) *there exist $1\leq i\leq k$ with $\lambda_i=0$ such that*

$$(\lambda_1,\dots,\lambda_{i-1}+1,\lambda_{i+1}+1,\dots,\lambda_k)$$

is a $(k-1)$-star-cycle (indices are modulo k, i.e. λ_0 is identified with λ_k and λ_1 with λ_{k+1}).

Moreover, condition (iii) *is equivalent to condition*

(iii′) *For every* $1 \leq i \leq k$ *with* $\lambda_i = 0$ *the cyclic sequence*

$$(\lambda_1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1} + 1, \ldots, \lambda_k)$$

is a $(k-1)$*-star-cycle.*

It is straightforward to check that Theorem 3.2 implies Theorem 1.3.

3.3. PROPOSITION. — *Every* k*-star with* $k \geq 4$ *has a cut.*

The number of equivalence classes of k-stars is not finite for $k \geq 4$. Indeed, cutting a k-star along a cut, one gets k_1- and k_2-halfstars for some integers $k_1, k_2 \geq 2$ with $k_1 + k_2 = k$. Conversely, a k_1- and a k_2-halfstar can always be glued together thus producing a $(k_1 + k_2)$-star and this can be done in infinitely many non-equivalent ways.

There exists however only finitely many exceptional k-stars (up to equivalence) for any given natural number k.

More precisely, one can give a kind of "mass-formula" for exceptional k-stars. We need some definitions before stating the result.

Given a k-halfstar τ with base $[(-1,0),(1,0)]$ which is contained in the upper halfplane $\{(x,y) \mid x \in \mathbb{R},\ y \geq 0\} \subset \mathbb{R}^2$, the second coordinate y (called the *height*) of a vertex $(x,y) \in \tau$ is well-defined. Let

$$h_i = \{(x,i) \in \mathbb{Z}^2 \cap \tau\}$$

denote the number of vertices in τ with second coordinate equal to i. We call the vector

$$Y_\tau = (h_1, h_2, h_3, h_4, \ldots) \in \mathbb{Z}^{\mathbb{N}}$$

the *height vector* of the halfstar τ. The sum

$$Y_k = \sum_\tau Y_\tau$$

of all height-vectors of non-equivalent k-halfstars τ is then the *total height-vector of* k*-halfstars.* We denote by $\langle Y, Y' \rangle$ the usual scalar product of height vectors or total height vectors (no convergence problems arise since only a finite number of coordinates are non-zero).

Given a k-star σ, we denote by $\mathrm{Aut}(\sigma) \subset \mathrm{Aff}^+(\mathbb{Z}^2)$ the subgroup of all orientation-preserving integral affine transformations which preserve σ. This group, called the *automorphism group* of σ, is cyclic of order $1, 2, 3, 4$ or 6.

3.4. THEOREM. — *One has for $k \geq 4$*

$$2\sum_{\sigma} \frac{(\mathrm{cut}(\sigma))(\mathrm{cut}(\sigma)-1)}{|\mathrm{Aut}(\sigma)|} = \sum_{j=2}^{k-2} \langle Y_j, Y_{k-j} \rangle$$

where the sum at the left side is over representants σ of all equivalence classes of k-stars (only exceptional k-stars yield non-zero contributions).

3.5. LEMMA. — *Up to action of $\mathrm{Aff}^+(\mathbb{Z}^2)$ there exists a unique 3-star with associated star-cycle $(2,2,2)$.*

We leave the easy proof of this lemma to the reader. □

3.6. LEMMA. — *The set of all equivalence classes of 4-star-cycles is represented by the set*

$$\{(1, k+1, 1, -k+1)\}_{k\in\mathbb{Z}}.$$

Proof. — We can suppose that a given 4-star σ has center $(0,0)$ and contains the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

First case: σ contains the triangle with vertices $(0,0)$, $(0,1)$ and $(-1,0)$. This implies that the remaining two triangles of σ share a common vertex of the form $(k,-1)$ and an easy computation ends the proof.

Second case: σ contains the triangle with vertices $(0,0)$, $(0,1)$ and $(-1,-k)$ for some integer k. In this case, a rotation of angle $\frac{1}{2}\pi$ around the origin sends the triangulation σ into a triangulation of the first case. □

Elementary subdivisions of stars (with respect to some triangle) are defined in the obvious way.

Proof of Proposition 3.3. — The proof is by induction on k. Lemma 3.6 shows that every 4-star has a cut.

Let now σ be a k-star with center $(0,0)$. Proceed as in the proof of lemma 2.6 and choose a vertex (x,y) of σ which has maximal Euclidean norm. An inspection of the proof of Lemma 2.6 shows that the curvature flow λ of the oriented edge from (x,y) to $(0,0)$ is either 0, 1 or 2.

• If $\lambda = 0$, then the k-star σ is obtained from a $k-1$ star σ' by an elementary subdivision (same proof as for (ii) of Lemma 2.6). Since elementary subdivisions preserve cuts, we get a cut for σ' by induction.

- If $\lambda = 1$ then the line through (x, y) and $(-x, -y)$ is a cut of σ.

- If $\lambda = 2$, then $k \geq 4$ and a little thought show that $(-x, -y)$ is among the vertices of σ. The line through (x, y) and $(-x, -y)$ is hence a cut. □

3.7. LEMMA. — *If Theorem 3.2 holds for some k-star σ, then it holds for any elementary subdivision σ' of σ.*

The proof coincides with the proof of Lemma 2.7.

Proof of Theorem 3.2. — It holds for $k = 3, 4$ by Lemma 3.5 and Lemma 3.6. Proposition 3.3 shows that every k-star σ with $k \geq 5$ has a cut and one of the two halfstars obtained by cutting open has at least 3 triangles. This implies by Lemma 2.6 that σ is obtained by elementary subdivision from a $(k-1)$-star $\widetilde{\sigma}$ for which the result holds by induction. It holds hence for σ by Lemma 3.7.

Edges with 0-curvature flow in a k-star σ are associated with elementary subdivisions of suitable $(k-1)$-stars leading to σ. This implies (iii′). □

Proof of Theorem 3.4. — Given a k-star σ there exist $2\,\mathrm{cut}(\sigma)/|\mathrm{Aut}(\sigma)|$ non-isomorphic ways to cut it open into an ordered pair of halfstars (some of the obtained ordered pairs may be isomorphic but then they have been glued together non-isomorphically). In each such ordered pair τ, τ' of halfstars there are exactly $(\mathrm{cut}(\sigma) - 1)$ pairs of vertices $(v \in \tau, v' \in \tau')$ which corresponds to vertices on cuts of σ. The vertices $v \in \tau$ and $v' \in \tau'$ must have identical height ≥ 1 in the halfstars τ and τ' (otherwise they can never yield a cut).

On the other hand, given vertices $v \in \tau$ and $v' \in \tau'$ of the same height on two halfstars τ, τ' these halfstars can be uniquely glued together to form a star σ having a cut through the vertices v and v'. Hence the formula. □

4. Conclusion.

4.1. *Definition.* — A function $\psi : E_i^{\rightarrow} \longrightarrow \mathbb{Z}$ on the set $E_i^{\rightarrow}$ of oriented interior edges is called a *local curvature flow* if it satisfies the following conditions:

(i) $\psi(a^{\rightarrow}) = -\psi(a^{\leftarrow})$ for every interior edge a of Γ;