A note on projective Levi flats and minimal sets of algebraic foliations


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1. Introduction.

A non trivial minimal set of a singular foliation $\mathcal{F}$ on a complex compact manifold $M$, is a closed set $\mathcal{M} \subset M$ with the following properties:

a) $\mathcal{M}$ is invariant for $\mathcal{F}$.

b) $\mathcal{M} \neq \emptyset$.

c) $\mathcal{M}$ does not contain singular points of $\mathcal{F}$.

d) $\mathcal{M}$ is minimal with respect to properties a), b) and c).

We shall use the abbreviation n.t.m.s. to denote "non trivial minimal set".

In [CLS1] the problem of the existence or not of n.t.m.s. for codimension one foliations of $\mathbb{CP}^2$ was studied. In particular it was proved that for any $k \geq 2$, the space of foliations of degree $k$ contains an open non empty set, say $A_k$, such that any foliation $\mathcal{F} \in A_k$ has no n.t.m.s. It was proved also that if a foliation has an algebraic leaf then it has no n.t.m.s. Since foliations of degree 0 or 1 have always algebraic leaves, they cannot have n.t.m.s. In general, the question of the existence of foliations in $\mathbb{CP}^2$ having n.t.m.s. remains open. Concerning this problem we will prove in §2.1 the following result:

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Theorem 1. — Codimension 1 foliations on $\mathbb{CP}^n$, $n \geq 3$, have no n.t.m.s.

In fact Theorem 1 will be a consequence of the following result:

Theorem 1'. — Any closed invariant set of a holomorphic codimension one foliation of $\mathbb{CP}^n$, $n \geq 3$, contains a singularity of the foliation.

Another problem that we will consider is the existence of Levi flats in projective spaces. Let $M$ be a complex manifold of complex dimension $n$ and $L$ be a $C^1$ submanifold of real codimension 1. Given $p \in L$, the tangent space $T_p(L)$ contains an unique complex subspace of complex dimension $n - 1$ that we shall denote $C_p$. This defines a distribution $C$ on $L$.

We say that $L$ is a Levi flat if the distribution $C$ is integrable. The integrability of $C$ implies that $L$ has a $C^{k-1}$ foliation $\mathcal{F}$, whose leaves are tangent to the subspaces $C_p, p \in L$. Since the subspaces $C_p$ are complex the leaves of $\mathcal{F}$ are holomorphic immersed submanifolds of complex dimension $n - 1$.

In §2.2 we shall prove the following result:

Theorem 2. — For $n \geq 3$ there are no real analytic Levi flats on $\mathbb{CP}^n$.

In §3 we will generalize Theorem 2. In order to state the main result that will be used, we consider the following situation:

Let $M$ be a holomorphic manifold of complex dimension $n \geq 2$. Let $V$ be an open set of $M$ satisfying the following properties:

a) All connected components of $V$ are Stein.

b) The closure $\overline{V}$ of $V$, is compact and connected.

We will denote by $K$ the boundary $\overline{V} \setminus V$ of $V$. Given a neighborhood $U$ of $K, 0 \leq j \leq 2n$, and $p \in K$, let

(*) $h_j : H_j(U, \mathbb{Z}) \to H_j(\overline{V}, \mathbb{Z})$ and $i_j : \Pi_j(U, p) \to \Pi_j(\overline{V}, p)$

be the homomorphisms induced by the inclusion $i : U \to \overline{V}$ in the homology and homotopy groups respectively. The following result is a kind of generalization of Lefschetz Theorem on hyperplane sections (cf. [M]):

Theorem 3. — For any neighborhood $A$ of $K$ in $\overline{V}$ there is a neighborhood $U$ of $K$ such that $U \subset A$ and $h_j$ and $i_j$ as in (*) are isomorphisms for $j \leq n - 2$ and are onto for $j = n - 1$. In particular we have the following:
(i) $K$ is connected.

(ii) If $K$ is a $C^1$ real submanifold of $M$ then the homomorphisms below are isomorphisms for $j \leq n - 2$ and are onto for $j = n - 1$:

$$h_j : H_j(K, \mathbb{Z}) \rightarrow H_j(\bar{V}, \mathbb{Z}) \text{ and } i_j : \Pi_j(K, p) \rightarrow \Pi_j(\bar{V}, p).$$

(iii) If $K$ is a $C^1$ real submanifold of $M$ and $n \geq 3$ then $\Pi_1(K, p)$ and $\Pi_1(\bar{V}, p)$ are isomorphic.

It is not difficult to see that Lefschetz Theorem on hyperplane sections is a consequence of Theorem 3. Another consequence is the following:

**Corollary** — Let $M$ be a compact complex manifold of complex dimension $n \geq 3$, with finite fundamental group. Then $M$ cannot contain a real analytic Levi flat $L$ such that all connected components of $M \setminus L$ are Stein.

The above corollary follows from (iii) of Theorem 3 and Haefliger’s Theorem, which says that a real analytic manifold with finite fundamental group admits no real analytic foliations (cf. [Ha]).

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2. Non trivial minimal sets and Levi flats.

In this section we prove Theorems 1’ and 2. In the proof we will use the following:

**Theorem [T].** — Let $U$ be an open connected subset of $\mathbb{CP}^n$, $n \geq 2$, satisfying the following properties:

(i) The boundary $\partial U$ of $U$ is not empty.

(ii) For any $p \in \partial U$ there exists a holomorphic embedding $f_p : B^{n-1} \rightarrow \mathbb{CP}^n$, where $B^{n-1}$ is the unit ball in $\mathbb{C}^{n-1}$, such that $f_p(0) = p$ and $f_p(B^{n-1}) \cap U = \emptyset$.

Then $U$ is Stein.

Note that condition (ii) implies that $U$ is locally pseudo-convex, that is, if $p \in \partial U$, then there exists a neighborhood $V$ of $p$ such that $V \cap U$ is
Stein. Therefore Theorem [T] is a consequence of a Theorem of Takeuchi (cf. [T] and [E]).

As a consequence we have the following:

**Corollary** — Let $K \subset \mathbb{C}P^n, n \geq 2$, be either a Levi flat or a closed invariant set of a holomorphic singular foliation of codimension 1 of $\mathbb{C}P^n$, which does not contain singular points of the foliation. Then every connected component of $U = \mathbb{C}P^n \setminus K$ is Stein.

### 2.1 Proof of Theorem 1'.

The idea is to prove that the singular set of a holomorphic foliation of codimension 1 on $\mathbb{C}P^n$ must have at least one irreducible component of complex codimension 2. This fact together with the corollary of Theorem [T] implies Theorem 1'. In fact, if a foliation $\mathcal{F}$ on $\mathbb{C}P^n, n \geq 3$, had a closed invariant set $K$ such that $K \cap \text{sing}(\mathcal{F}) = \emptyset$, then, by the corollary of Theorem [T], the connected components of the open set $U = \mathbb{C}P^n \setminus K$ would be Stein. On the other hand the singular set of $\mathcal{F}$ contains some irreducible component, say $S$, of dimension $\geq 1$. Since a Stein open set cannot contain a compact analytic subset of dimension greater than zero, this would imply that $S \cap K \neq \emptyset$, which is a contradiction.

We will consider the following situation:

Let $\mathcal{F}$ be a foliation of degree $k$ on $\mathbb{C}P^2$, with finite singular set $\text{sing}(\mathcal{F})$. Given $p \in \text{sing}(\mathcal{F})$ let $X$ be a holomorphic vector field in a neighborhood $V$ of $p$ which is tangent to $\mathcal{F}$. Suppose that the linear part $A = DX(p)$ of $X$ at $p$ is non singular. The Baum-Bott index of $\mathcal{F}$ at $p$ is defined in this case by

$$BB(\mathcal{F}, p) = \frac{(\text{trace}(A))^2}{\det(A)}.$$ 

The Baum-Bott index can be defined for any isolated singularity (cf. [BB] and [MB]), but we will use it only in the non degenerate case.

We will use the following result, which is a consequence of a theorem of Baum and Bott (cf. [AL-N]):

**Proposition 1.** — Let $\mathcal{F}$ be a foliation of degree $k$ on $\mathbb{C}P^2$, with singular set $\text{sing}(\mathcal{F})$ of complex codimension 2. Then

$$\sum_{p \in \text{sing}(\mathcal{F})} BB(\mathcal{F}, p) = (k + 2)^2.$$
In particular \[ \sum_{p \in \text{sing}(\mathcal{F})} BB(\mathcal{F}, p) \] is positive for any foliation \( \mathcal{F} \) on \( \mathbb{CP}^2 \).

Now let \( \mathcal{G} \) be a codimension one holomorphic foliation on \( \mathbb{CP}^n \), \( n \geq 3 \), and suppose by contradiction that all irreducible components of \( \text{sing}(\mathcal{G}) \) have complex codimension \( \geq 3 \). Let \( E \subset \mathbb{CP}^n \) be a 2-plane in general position with respect to \( \mathcal{G} \). The 2-plane \( E \) is a linear embedding of \( i: \mathbb{CP}^2 \hookrightarrow \mathbb{CP}^n \), where \( E = i(\mathbb{CP}^2) \), with the following properties:

a) \( E \) is not contained in any leaf of \( \mathcal{G} \).

b) \( E \) intersects transversally all smooth strata of \( \text{sing}(\mathcal{G}) \).

c) Outside \( E \cap \text{sing}(\mathcal{G}) \), the set of tangencies of \( \mathcal{G} \) with \( E \) has codimension at most 2 in \( E \).

In Proposition 1 of \[ \text{[CLS2]} \] it is proved that the set of all 2-planes satisfying (a), (b) and (c) is open and dense in the Grassmanian of 2-planes in \( \mathbb{CP}^n \). It follows from (a) that the \( i^*(\mathcal{G}) = \mathcal{F} \) is a codimension one foliation on \( \mathbb{CP}^2 \). Condition (b) and the fact that all irreducible components of \( \text{sing}(\mathcal{G}) \) have codimension \( \geq 3 \), imply that \( E \cap \text{sing}(\mathcal{G}) = \emptyset \), so that the singularities of \( \mathcal{F} \) correspond to the tangencies of \( E \) with some leaves of \( \mathcal{G} \), and so (c) implies that \( \text{sing}(\mathcal{F}) \) is finite.

Let \( p \in \text{sing}(\mathcal{F}) \). Since \( i(p) = p' \) is not a singular point of \( \mathcal{G} \), this foliation has a holomorphic first integral, say \( g \), defined in a neighborhood \( U \) of \( p' \), where \( dg(p') \neq 0 \). Let \( f = g \circ i \). Observe that \( f \) is a local first integral of \( \mathcal{F} \). Since \( p' \) is a tangency of \( \mathcal{G} \) with \( E \), we must have \( df(p) = 0 \), so that \( p \) is an isolated singularity of \( f \) and of \( \mathcal{F} \). We say that \( p' \) is a tangency of Morse type if \( p \) is a Morse singularity for \( f \), that is in some coordinate system \( (x, y) \) around \( p \) such that \( x(p) = y(p) = 0 \) we have \( f(x, y) = f(p) + xy \). If this is the case, then the foliation \( \mathcal{F} \) is defined in a neighborhood of \( p = (0,0) \) by the vector field \( x \partial/\partial x - y \partial/\partial y \), so that \( BB(\mathcal{F}, p) = 0 \).

Now observe that the 2-plane \( E \) can be deformed a little bit to a 2-plane \( E' \) in such a way that all tangencies of \( E' \) with \( \mathcal{G} \) are of Morse type. This assertion is an easy consequence of the following facts:

(i) Morse type singularities are stable by small perturbations.

(ii) Let \( g: B^n(0,2) \rightarrow \mathbb{C} \) be a holomorphic function, where \( g(0) = 0 \) and \( dg(0) \neq 0 \), say \( \partial g/\partial x_n(0) \neq 0 \). Let \( f(x, y) = g(x, y, 0, \ldots, 0) \). Assume that \( df(0,0) = 0 \) and that \( df(x,y) \neq 0 \) for \( (x,y) \in B^2(0,2) \setminus \{0\} \). Then, given \( \epsilon > 0 \), there are \( a, b, c \in \mathbb{C} \), with \( |a|, |b|, |c| < \epsilon \) and such that all
singularities of $h(x, y) = g(x, y, \ldots, ax + by + c)$ in $B^2(0, 1)$ are of Morse type.

We leave the details of the proof for the reader.

Finally, observe that we have obtained a foliation $\mathcal{F}' = \mathcal{G}|_{E'}$ such that for any $p \in \text{sing}(\mathcal{F}')$ we have $BB(\mathcal{F}', p) = 0$. This contradicts Proposition 1, so that $\text{sing}(\mathcal{G})$ must have some component of codimension 2.

2.2. Proof of Theorem 2.

The idea is to use Theorem 1', and the following result:

**Theorem 4.** Let $L$ be a real analytic Levi flat in $\mathbb{C}P^n$, $n \geq 2$. Then there exists a holomorphic codimension one foliation $\mathcal{F}$ on $\mathbb{C}P^n$ such that $L$ is $\mathcal{F}$-invariant.

Clearly Theorem 4 follows from the corollary of Theorem [T] and of the following lemmas:

**Lemma 1.** Let $M$ be a complex manifold and $L \subset M$ be a real analytic Levi flat. Then, there exists a neighborhood $U$ of $L$ and a holomorphic codimension one foliation $\mathcal{G}$ on $U$ such that $L$ is $\mathcal{G}$-invariant.

Observe that, if $\mathcal{F}$ is the foliation on $L$ defined by the integrable distribution $\mathcal{C}$ of complex hyperplanes in $L$, then $\mathcal{G}|_L = \mathcal{F}$.

**Lemma 2.** Let $V$ be a Stein manifold and $K \subset V$ be a compact set such that $U = V \setminus K$ is connected. Then any holomorphic codimension one foliation $\mathcal{G}$ on $U$, such that $\text{cod}(\text{sing}(\mathcal{G})) \geq 2$, can be extended to a holomorphic foliation on $V$.

2.2.1 - Proof of Lemma 1.

We will use the following fact in the proof:

**Assertion.** For any $p \in L$ there exists a holomorphic function $H$, defined in a neighborhood $U$ of $p$, such that $dH(p) \neq 0$ and $L \cap U = v^{-1}(0)$, where $v = \mathfrak{S}(H)$.

The above assertion is well known and can be proved by using Frobenius Theorem and the results of [To]. Since its proof is not long we will give it at the end of §2.2.1. Let us prove Lemma 1 from the assertion.
Observe first that the assertion implies that for any \( p \in L \) there exists a holomorphic coordinate system
\[
\phi = (x, y) : U \to \mathbb{C}^n, \quad \text{where } x : U \to \mathbb{C}^{n-1} \text{ and } y : U \to \mathbb{C}
\]
such that \( L \cap U = y_2^{-1}(0) \), where \( y_2 = \Re(y) \).

It follows from the above observation that it is possible to find a holomorphic atlas of a neighborhood \( V \) of \( L \)
\[
U = \{ U_j, \phi_j = (x_j, y_j) \}_{j \in J} \text{ where } (x_j, y_j) : U_j \to \mathbb{C}^{n-1} \times \mathbb{C}
\]
such that

(i) If \( i, j \in J \) and \( i \neq j \), then \( U_{i,j} = U_i \cap U_j \) is connected and homeomorphic to a ball in \( \mathbb{C}^n \), if not empty.

(ii) For any \( j \in J \) we have \( U_j \cap L \neq \emptyset \) and is homeomorphic to a ball in \( L \).

(iii) For any \( i, j \in J \) such that \( U_{i,j} \neq \emptyset \) then \( L \cap U_{i,j} \neq \emptyset \) and is homeomorphic to a ball in \( L \).

(iv) For any \( j \in J \) we have \( U_j \cap L = \{ \Im(y_j) = 0 \} \).

Now let \( i, j \in J \) be such that \( U_i \cap U_j \neq \emptyset \) and consider the change of chart
\[
\phi = \phi_{i,j} = \phi_i \circ (\phi_j)^{-1} : \phi_j(U_{i,j}) \to \phi_i(U_{i,j}).
\]
In order to simplify the notations let us call \( x_i = x, y_i = y, x_j = z \) and \( y_j = w \), so that \( \phi(x, y) = (z(x, y), w(x, y)) \). It follows from (i), (ii) and (iii) above, that the domain of \( w, \phi_j(U_{i,j}) \), is homeomorphic to a ball and contains a ball \( B \subset \mathbb{C}^{n-1} \times \mathbb{R} \subset \mathbb{C}^{n-1} \times \mathbb{C} \). Moreover, condition (iv) implies that if \( (x, t) \in B \) \( (t \in \mathbb{R}) \), then \( w(x, t) \in \mathbb{R} \), so that the holomorphic function \( x \mapsto w(x, t) \) \( (t \text{ fixed}) \) must be constant. This implies that the map \( (x, t) \mapsto w(x, t) \) does not depend on \( x \). Since \( w \) is holomorphic it follows that \( (x, y) \mapsto w(x, y) \) does not depends on \( x \).

The above argument implies the coordinate changes \( \phi_{i,j} \), are of the form
\[
\phi_{i,j}(x_j, y_j) = (x_i(x_j, y_j), y_i(y_j)).
\]
Therefore the atlas \( U \) defines a codimension one foliation on the neighborhood \( V = \bigcup_j U_j \) of \( L \).

**Proof of the assertion.** — Let \( \mathcal{F} \) be the foliation on \( L \) defined by the distribution of complex hyperplanes \( \mathcal{C} \). Since \( L \) is real analytic \( \mathcal{F} \) is
also real analytic. Fix a point \( p \in L \) and a holomorphic coordinate system \((\phi = (x, y), U)\) with the following properties:

\begin{itemize}
  \item[a)] \( x = (x_1, \ldots, x_{n-1}) : U \to \mathbb{C}^{n-1} \) and \( y : U \to \mathbb{C} \).
  \item[b)] \( p \in U \) and \( \phi(p) = 0 \).
  \item[c)] The surface \( \{ x = 0 \} \) is transversal to the leaves of \( \mathcal{F} \).
\end{itemize}

Condition (c) implies that \( L \cap \{ x = 0 \} \) is a real analytic curve \( \gamma(t) = (0, y(t)) \). After a holomorphic change of variables in the coordinate \( y \) we can suppose that \( y(t) = t \), which means that \( L \cap \{ x = 0 \} \) is the real axis in the \( y \)-plane. Let \( F_t \) be the leaf of \( \mathcal{F} \) through \( \gamma(t) \). Since \( F_t \) is holomorphic it can be written locally as the graph of a function, say \( y = \varphi(x, t) \), where \( \varphi \) is real analytic, holomorphic with respect to \( x \) and \( \varphi(0, t) = t \). If we set \( \varphi = u + iv \), where \( u = \Re(\varphi) \) and \( v = \Im(\varphi) \), then the hypersurface \( L \) can be defined locally around \( p \) by eliminating \( t \) in the equation \( y_1 - u(x, t) = 0 \) \((y = y_1 + iy_2)\), say \( t = h(x, y_1) \), and substituting \( h \) in \( y_2 - v(x, t) \), so that in a neighborhood of \( p \), \( L \) is given by \( y_2 = v(x, h(x, y_1)) \).

Now, let \( \varphi(x, t) = \sum_{n=1}^{\infty} a_n(x) t^n \) be the Taylor series in \( t \) of \( \varphi \) around \((0,0)\). Extend \( \varphi \) to a neighborhood of \((0,0)\) in \( \mathbb{C}^{n-1} \times \mathbb{C} \) by setting \( \varphi(x, z) = \sum_{n=1}^{\infty} a_n(x) z^n \), \( z \in \mathbb{C} \). Let \( F(x, y, z) = y - \varphi(x, z) \). Since \( F(0, 0, 0) = 0 \) and \( \partial F/\partial z(0, 0, 0) \neq 0 \), by the implicit function theorem, there exists a holomorphic function \( H(x, y) \), defined in a neighborhood of \((0,0)\), such that \( \varphi(x, H(x, y)) = y \). Finally, observe that \( L \) can be defined in a neighborhood of \((0,0)\) by \( \Im(H(x, y)) = 0 \). We leave the proof of this fact for the reader. This ends the proof of Lemma 1.

2.2.2. Proof of Lemma 2.

Let \( f : V \to \mathbb{R} \) be a strictly-pluri-subharmonic \( C^\infty \) exhaustion of \( V \). Since \( \lim_{p \to \infty} f(p) = +\infty \), the sets \( M_t = \{ p \in V ; f(p) \leq t \} \) are compact and \( K \subseteq M_t \) for \( t \geq t_0 \), so that the foliation \( \mathcal{G} \) is defined on \( V_t = V \setminus M_t \), for \( t \geq t_0 \). The idea is to prove the following:

\( (*) \) Suppose that \( \mathcal{G}_t \) is a codimension one holomorphic foliation defined on \( V_t \), such that \( \operatorname{cod}(\operatorname{sing}(\mathcal{G}_t)) \geq 2 \). Then there exists \( \epsilon > 0 \) such that \( \mathcal{G}_t \) can be extended to a foliation on \( V_{t-\epsilon} \).

Since \( m = \inf\{ f \} > -\infty \), it is clear that \( (*) \) implies Lemma 2. On the other hand, since \( f^{-1}(t) \) is compact, it is not difficult to see that \( (*) \) is a consequence of the following:
Let $G_t$ be as in $(*)$ and $p \in f^{-1}(t)$. Then $G_t$ can be extended to $V_t \cup W$, where $W$ is a neighborhood of $p$.

In order to prove $(**)$ we use the following results:

**Lemma A** (cf. [ST]). — Given $p \in f^{-1}(t)$ there exists a biholomorphic function $f : W \rightarrow W' \subset \mathbb{C}^n$, where $W$ is a neighborhood of $p$, and a Hartog's domain $H \subset W'$ such that $f^{-1}(H) \subset V_t$ and $p \in \phi^{-1}(H)$.

A Hartog's domain is an open set $H$ of $\mathbb{C}^n$, $n \geq 2$ of the form $H = (U' \times \Delta(r)) \cup (U \times (\Delta(r) \setminus \Delta(r')))$, where $U$ and $U'$ are open sets of $\mathbb{C}^{n-1}$, $U$ connected, $U \supset U' \neq \emptyset$, $\Delta(r)$ is the disk of radius $r$ in $\mathbb{C}$ and $0 < r' < r$. The set $H$ is, by definition $U \times \Delta(r)$. We observe that in Lemma A, we can suppose that $U$ and $U'$ are polydisks.

**Levi's Theorem** (cf. [S]). — Let $H$ be a Hartog's domain and $f$ be a meromorphic function on $H$. Then $f$ can be extended to a meromorphic function on $H$.

Let us finish the proof of Lemma 2. Consider the biholomorphism $\phi$ as in Lemma A and let $G'$ be the restriction of $\phi_* (G_t)$ to $H \subset \mathbb{C}^n$. We will prove that there exists an integrable holomorphic 1-form $\omega$ on $H$ such that $G'$ is defined by the differential equation $\omega |_H = 0$. This will prove the lemma.

The foliation $G'$ is defined locally by integrable 1-forms, so that there exist a covering of $H$ by open sets $U = (U_j)_{j \in J}$ and collections $(\omega_j)_{j \in J}$, $(h_{i,j})_{i,j \neq \varnothing} (U_{i,j} = U_i \cap U_j)$, such that

a) $\omega_j$ is an integrable 1-form on $U_j$ such that $\text{cod} (\text{sing}(\omega_j)) \geq 2$ and $G' |_{U_j}$ is defined by $\omega_j = 0$.

b) If $U_{i,j} \neq \emptyset$ then $h_{i,j} \in \mathcal{O}^\ast(U_{i,j})$ and on $U_{i,j}$ we have $\omega_i = h_{i,j} \cdot \omega_j$.

Since $\mathcal{H} \subset \mathbb{C}^n$ we can write

$$\omega_j = \sum_{i=1}^n g^j_i dx_i , \text{ where } g^j_i \in \mathcal{O}(U_j).$$

Observe that condition (b) implies that if $U_{j,t} \neq \emptyset$ then

$$g^j_i = h_{j,t} g^j_i \text{ for all } i = 1, \ldots, n.$$

Since $\mathcal{H}$ is connected, it follows that for some $i \in \{1, \ldots, n\}$ we must have $g^j_i \neq 0$ for all $j \in J$. We suppose $i = n$, so that $g^j_n / g^j_n$ defines a
meromorphic function \( f_i^j \) on \( U_j \) for all \( i = 1, \ldots, n - 1 \). Now, (*) implies that if \( U_{j,t} \neq \emptyset \), then \( f_i^j = f_i^t \) on \( U_{j,t} \), so that, for all \( i = 1, \ldots, n - 1 \), there exists a meromorphic function \( f_i \) on \( H \), such that \( f_i |_{U_j} = f_i^j \). It follows from Levi’s Theorem that \( f_i \) can be extended to a meromorphic function on \( \widehat{H} \), which we call still \( f_i \).

Consider the meromorphic 1-form \( \eta \) defined on \( \widehat{H} \) by

\[
\eta = dx_n + \sum_{i=1}^{n-1} f_i \cdot dx_i
\]

Since \( \widehat{H} \) is a polydisk, it follows that there exists \( h \in \mathcal{O}(\widehat{H}) \) and a holomorphic 1-form \( \omega \) on \( \widehat{H} \) such that \( \text{cod}(\text{sing}(\omega)) \geq 2 \) and \( \eta = \frac{1}{h} \cdot \omega \). It is not difficult to see that for all \( j \in J \) we have \( \omega |_{U_j} = g_j \cdot \omega_j \) for some \( g_j \in \mathcal{O}^*(U_j) \), so that \( \omega \) is integrable and the foliation defined by \( \omega = 0 \) on \( \widehat{H} \) extends \( \mathcal{G}' \). This ends the proof of Lemma 2.

### 3. Theorem 3 and its corollary.

In this section we prove Theorem 3 and its corollary.

Let \( V, \overline{V} \) and \( K = V \setminus V \), be as in Theorem 3. Fix a neighborhood \( A \) of \( K \) in \( \overline{V} \). We need a lemma.

**Lemma 3.** — There exist neighborhoods \( U_1 \) and \( U \) of \( K \) in \( \overline{V} \) and a flow \( \varphi : \mathbb{R} \times \overline{V} \rightarrow \overline{V} \) such that

(a) \( U_1 \subset \overline{U}_1 \subset U \subset A \).
(b) \( \varphi_t(p) = p, \ \forall \ p \in \overline{U}_1, \) where \( \varphi_t(p) = \varphi(t,p) \).
(c) \( \varphi |_{\mathbb{R} \times V} \) is \( C^{\infty} \). Let \( X \) be the \( C^{\infty} \) vector field on \( V \) which generates \( \varphi \).
(d) All singularities of \( X \) in \( V \setminus \overline{U}_1 \) are hyperbolic.
(e) If \( p \in V \setminus \overline{U}_1 \) is a singularity of \( X \), then its stable manifold, \( W^S(p) \), has (real) dimension at most \( n \).
(f) If \( p \in V \setminus U \) is a singularity of \( X \), then \( W^S(p) \) is contained in \( V \setminus U \).
(g) If \( F \subset V \) is a compact subset of \( \overline{V} \) such that \( F \cap W^S(q) = \emptyset \) for any singularity \( q \) of \( X \) in \( V \setminus U \), then there exists \( s_0 > 0 \) such that \( \varphi_t(F) \subset U \) for \( t \geq s_0 \).
3.1. Proof of Lemma 3.

Let $N$ be a connected component of $V$. Since $N$ is Stein, there exists on $N$ a $C^\infty$ strictly-pluri-subharmonic exhaustion, say $f$. We will use the following facts:

1. The set of exhaustions of $N$, is open in the Whitney $C^0$ topology of $C^0(N, \mathbb{R})$.

2. The set of $C^2$ Morse functions is open and dense in the Whitney $C^2$ topology of $C^2(N, \mathbb{R})$.

3. The set of $C^\infty$ functions is dense in the Whitney $C^2$ topology of $C^2(N, \mathbb{R})$.

4. The set of strictly-pluri-subharmonic functions of class $C^2$ on $N$, say $SPSH^2(N, \mathbb{R})$, is open in the Whitney $C^2$ topology of $C^2(N, \mathbb{R})$.

The definition of the Whitney topology and the proofs of (1), (2) and (3) can be found in [H]. Let us prove (4).

Let $h \in C^2(N, \mathbb{R})$ and let $\mathcal{L}_h$ be the Levi form of $h$, which is defined in a holomorphic coordinate system $(x = (x_1, \ldots, x_n), U)$ of $N$ by

$$\mathcal{L}_h(p)v = \sum_{i,j} \frac{\partial^2 h}{\partial x_i \partial \overline{x}_j}(x) v_i \overline{v}_j,$$

where $(p, v) \in TU$ and $(x, (v_1, \ldots, v_n)) = T_x(p, v)$.

By definition, $h$ is strictly-pluri-subharmonic if, and only if, $\mathcal{L}_h(p)v > 0$ for all $(p, v) \in TN$, with $v \neq 0$. Let us fix a riemannian metric $g$ on $N$ with norm $|\cdot|$. Given $h \in SPSH^2(N, \mathbb{R})$ define $k_h : N \to \mathbb{R}$ by

$$k_h(p) = \inf\{\mathcal{L}_h(p)v ; v \in T_pN \text{ and } |v|_p = 1\}$$

so that $k_h > 0$ on $N$. Now, for a fixed $h_0 \in SPSH^2(N, \mathbb{R})$ let

$$U = \{h \in C^2(N, \mathbb{R}) ; |\mathcal{L}_h(p)v - \mathcal{L}_{h_0}(p)v| < 1/2 . k_{h_0}(p) \forall (p, v) \in N \text{ with } |v|_p = 1\}.$$ 

Since $\mathcal{L}_h$ and $k_h$ depend only on the second jet of $h$, it follows from the definition of the Whitney topology that $U$ is a neighborhood of $h_0$ in $C^2(N, \mathbb{R})$. Moreover if $h \in U$, then $k_h > 0$, so that $h \in SPSH^2(N, \mathbb{R})$, which proves (4).

It follows from (1), (2), (3) and (4) that we can suppose that $f$ is a Morse function. Let $g$ be the riemannian metric on $N$ fixed before. Let
Y = \text{grad}_g(f), which is defined by \( df_p.v = g_p(v, Y(p)) \), for any \((p, v) \in TN\).

Let \( Y_t \) be the flow of \( Y \).

The following facts are well known (cf. [M] and [Sm]):

(5) \( f \) is strictly increasing along non singular orbits of \( Y \).

(6) The singularities of \( Y \) are the points \( p \in N \) for which \( df_p = 0 \).

(7) Given \( p \) such that \( df_p = 0 \), there exists a \( C^\infty \) coordinate system \( x = (x_1, \ldots, x_{2n}) \) around \( p \) such that

\[
(*) \quad f(x) = f(p) + \sum_{j=1}^{2n} b_j(x_j)^2 + \text{h.o.t., where } b_j \in \mathbb{R} \setminus \{0\}.
\]

The number \( i(p) \) of negative \( b_j \)'s in \( (*) \), is an invariant of \( f \) and \( p \) and is called the Morse index of \( f \) at \( p \).

(8) If \( p \) and \( i(p) \) are as in (7), then \( p \) is a hyperbolic singularity of \( Y \) and its stable manifold, \( W^s(p) \), has (real) dimension \( i(p) \).

We need the following:

**Assertion.** — For any singularity \( p \) of \( Y \), we have \( i(p) \leq n \).

**Proof.** — It follows from Theorem 1.4.15, pg. 29 of [HL], that there exists a holomorphic coordinate system \( (z = (z_1, \ldots, z_n), W) \) around \( p \), such that \( z(p) = 0 \) and the expression of \( f \) in \( W \) is of the form

\[
(*) \quad f(z) = f(p) + \sum_{j=1}^{n} [(1 + a_j).x_j^2 + (1 - a_j).y_j^2] + \text{h.o.t.},
\]

where \( z_j = x_j + i y_j \) and \( 1 \neq a_j > 0 \) (because \( f \) is a Morse function). This implies the assertion.

Let us consider now the open set \( A' = A \cap N \). Since \( \overline{V} \) is compact, it follows that \( N \setminus A' \) is compact, which implies that there exists \( t_0 \in \mathbb{R} \) such that \( f^{-1}((-\infty, t]) \supset N \setminus A', \) for \( t \geq t_0 \). Fix \( t_2 > t_1 > t_0 \) and let \( N_j = f^{-1}(t_j, +\infty) \), \( j = 1, 2 \), so that \( A' \supset \overline{N_1} \supset N_1 \supset \overline{N_2} \). We choose \( t_1 \) in such a way that \( f^{-1}(t_1) \) does not contain singularities of \( Y \), which implies that \( Y \) is transverse to \( f^{-1}(t_1) \) and points inward \( N_1 \). Observe also that \( Y \) has finitely many singularities on \( N \setminus \overline{N_2} \).

Now, let \( p \in N \setminus \overline{N_j}, \) \( j = 1, 2 \) be a singularity of \( Y \). It follows from (5), from the definition of \( N_j \) and from the fact that

\[
W^s(p) = \{ q \mid \lim_{t \to +\infty} Y_t(q) = p \}
\]
that

\[(9) \ W^s(p) \subset N \setminus \overline{N_j}, \text{ for } j = 1, 2.\]

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^\infty$ function such that $\phi(t) = 0$ for $t \geq t_2$, $\phi(t) = 1$ for $t \leq t_1$ and $\phi(t) > 0$ for $t \in (t_1, t_2)$.

Let $X^N$ be the $C^\infty$ vector field on $N$ defined by $X^N(p) = \phi(f(p)).Y(p)$, and denote by $X^N_t$ its flow. It is not difficult to verify the following facts:

\[(10) \ X^N_t(p) = p, \ \forall \ p \in \overline{N_2}.\]

\[(11) \ \text{If } p \in N \setminus \overline{N_2} \text{ and } o_X(p), o_Y(p) \text{ are the orbits of } X^N \text{ and } Y \text{ through } p, \text{ respectively, then } o_X(p) = o_Y(p) \cap (N \setminus \overline{N_2}).\]

\[(12) \ f \text{ is strictly increasing along non singular orbits of } X^N.\]

\[(13) \ The \ singularities \ of \ X^N \ on \ N \setminus \overline{N_2} \text{ are hyperbolic and are singularities of } Y.\]

\[(14) \ If \ p \in N \setminus \overline{N_j}, \ j = 1, 2, \text{ is a singularity of } X^N, \text{ then its stable manifold coincides with } W^s(p), \text{ the stable manifold of } p \text{ with respect to } Y. \text{ In particular (13) is true for } X^N.\]

\[(15) \ If \ p \in N \setminus \overline{N_1} \text{ does not belongs to the stable manifold of some singularity of } X^N \text{ in } N \setminus \overline{N_1}, \text{ then there exists } s_0 > 0 \text{ and a neighborhood } W_p \text{ of } p \text{ such that if } t > s_0, \text{ then } X^N_t(W_p) \subset N_1.\]

This last assertion follows from the continuity of the flow and from (12).

Let us finish the proof of Lemma 3. Consider the decomposition of $V$ in connected components, $V = \bigcup_{j \in J} N^j$. For each $j \in J$, let $f^j$ be a Morse strictly-pluri-subharmonic exhaustion of $N^j$. Let $A^j = A \cap N^j$ and $N^j_2 \subset N^j_1$ be like $N_1$ and $N_2$ considered before. Let $X^N_j = X^j$ be a vector field on $N^j$ which satisfies properties (10), . . . , (15). Define the flow $\varphi : \mathbb{R} \times \overline{V} \rightarrow \overline{V}$ by $\varphi(t, p) = p$ if $p \in K$ and $\varphi(t, p) = X^j_t(p)$ if $p \in N^j$. Observe that $\varphi$ satisfies (c) of Lemma 3.

Since $f^j$ is an exhaustion of $N^j$ for any $j \in J$, it is not difficult to see that $U = \bigcup_{j} N^j_1 \cup K$ and $U_1 = \bigcup_{j} N^j_2 \cup K$ are neighborhoods of $K$ and satisfy (a) of Lemma 3. Observe that this fact and (10) imply that $\varphi$ is continuous and satisfies (b), (d) and (f) of Lemma 3. On the other hand property (g) of Lemma 3 can be easily checked from (15). We leave the details for the reader. This ends the proof of Lemma 3.
3.2. Proof of Theorem 3.

Let $V, \overline{V}, K$ and $A$ be as in Theorem 3. Let $U, U_1$ and $\varphi$ be as in Lemma 3. Fix $p \in K$ and consider the homomorphisms induced by the inclusion $i : U \longrightarrow \overline{V}$:

\begin{equation}
(*) \ h_q : H_q(U, \mathbb{Z}) \longrightarrow H_q(\overline{V}, \mathbb{Z}) \text{ and } i_q : \Pi_q(U, p) \longrightarrow \Pi_q(\overline{V}, p).
\end{equation}

Let us consider first the homotopy case. Consider $i_q$ as above and let us prove that it is onto for $1 \leq q \leq n - 1$. We will consider a class $[g]$ in $\Pi_q(\overline{V}, p)$ represented by a continuous map $g : S^q \longrightarrow \overline{V}$, where $S^q$ is the unit sphere in $\mathbb{R}^{q+1}$ and $g(e) = p$ for some fixed $e \in S^q$. Let $V_1 = V \setminus U_1$ and consider the open set $A = g^{-1}(V_1) \subset S^q$. It follows from standard arguments of differential topology that it is possible to find a continuous map $h : S^q \longrightarrow \overline{V}$ such that

1. $h$ coincides with $g$ in $S^q \setminus A$.
2. $h$ is homotopic to $g$.
3. $h$ is $C^\infty$ in $A$.

Let $p_1, \ldots, p_m$ be the singularities of $X$ in $V \setminus U$, and $W = \bigcup_{j=1}^m W^s(p_j)$. Since for all $j = 1, \ldots, m$ we have $q + \dim \mathbb{R}(W^s(p_j)) \leq 2n - 1 < 2n = \dim \mathbb{R}(V)$, it follows from transversality theory that it is possible to find $h$ in such a way that

4. $h(A) \cap W = \varnothing$, so that $h(S^q) \cap W = \varnothing$ (by (f) of Lemma 3 and (1)).

It follows from (g) of Lemma 3 that there exists $t > 0$ such that $\varphi_t(h(S^q)) \subset U$. Since $\varphi_t$ is homotopic to the identity of $\overline{V}$, this implies that $i_q$ is onto.

Let us prove that $i_q$ is injective if $1 \leq q \leq n - 2$. Let $[g] \in \Pi_q(U, p)$ be such that $i_q[g] = 0$ and $g : S^q \longrightarrow U$ be a representative of $[g]$. Let $B^{q+1}$ be the closed unit ball in $\mathbb{R}^{q+1}$ and $G : B^{q+1} \longrightarrow \overline{V}$ be a continuous map such that $G|_{S^{q+1}} = g$. Let $A = G^{-1}(V_1)$. It follows from standard arguments of differential topology that it is possible to find a continuous map $H : B^{q+1} \longrightarrow \overline{V}$ such that

5. $H$ coincides with $G$ in $B^{q+1} \setminus A$.
6. $H$ is homotopic to $G$.
7. $H$ is $C^\infty$ in $A$. 
Moreover, since for all \( j = 1, \ldots, m \) we have \( q + 1 + \dim \mathbb{R}(W^S(p_j)) \leq 2n - 1 < 2n = \dim \mathbb{R}(V) \), it follows from transversality theory that it is possible to find \( H \) in such a way that

\[
(8) \ H(S^q) \cap W = \emptyset.
\]

It follows from (g) of Lemma 3 that there exists \( t > 0 \) such that \( \varphi_t(H(\mathbb{B}^q + 1)) \subset U \). This implies that \( [g] = 0 \) in \( \Pi_q(U, p) \), so that \( i_q \) is injective.

The proof in the homology case is similar. Let us sketch it.

We will work in the singular homology theory, with the notations of \([G]\). If \( c = \sum_{j=1}^{k} \nu_j \sigma_j \) is a q-chain in \( \overline{V} \) then each simplex \( \sigma_j \) is a continuous map from the standard simplex

\[
\Delta_q = \left\{ p \in \mathbb{R}^q; \ p = \sum_{i=0}^{q} t_i E_i, \ 0 \leq t_i \leq 1, \sum t_i \leq 1 \right\}
\]

into \( \overline{V} \). We will use the notation \( \text{supp}(c) = \bigcup_j \sigma_j(\Delta_q) \). A sub-simplex \( \sigma^I_j \), \( I = (i_0 < i_1 < \ldots < i_r) \), \( r \leq q \), is, by definition, the restriction of \( \sigma_j \) to \( \Delta^I_q \), where

\[
\Delta^I_q = \left\{ p \in \mathbb{R}^q; \ p = \sum_{j=0}^{r} t_j E_i_{i_j}, \ 0 \leq t_j \leq 1, \sum t_j \leq 1 \right\}.
\]

We will say that \( c \) is \( C^\infty \) if for all \( j \) and all \( I \) the restriction of \( \sigma^I_j \) to the open subsets \( (\sigma^I_j)^{-1}(V) \) of \( \Delta^I_q \) is \( C^\infty \).

Let us prove that \( h_q \) is onto if \( 0 \leq q \leq n - 1 \). Let \([c] \in H_q(\overline{V}, \mathbb{Z})\) and \( c = \sum_{j=1}^{k} \nu_j \sigma_j \) be a representative of \([c]\). It follows from standard arguments of homology theory that we can suppose that all \( \sigma^I_j \) are \( C^\infty \) and transversal to \( W \). Since \( 0 \leq q \leq n - 1 \) this implies that \( W \cap \text{supp}(c) = \emptyset \). On the other hand, (g) of Lemma 3 implies that there exists \( t > 0 \) such that \( \varphi_t(\text{supp}(c)) \subset U \). Since \( \varphi_t \) is homotopic to the identity, it follows that \( h_q \) is onto.

Let us prove that \( h_q \) is injective if \( 0 \leq q \leq n - 2 \). Let \([c] \in H_q(U, \mathbb{Z})\) be such that \( h_q[c] = 0 \). Let \( c \) be a \( C^\infty \) representative of \([c]\). It follows from standard arguments of homology theory that there exists a \( C^\infty \) \((q+1)\)-chain \( c' = \sum_j \nu_j \sigma_j \) on \( \overline{V} \), such that \( \partial c' = c \) and all \( \sigma^I_j \) are transversal to \( W \). Since
this implies that $W \cap \text{supp}(c') = \emptyset$. On the other hand, (g) of Lemma 3 implies that there exists $t > 0$ such that $\varphi_t(\text{supp}(c')) \subset U$. Since $\varphi_t$ is homotopic to the identity, it follows that $h_q$ is injective.

It remains to prove (i) and (ii) of Theorem 3 (since (iii) follows from (ii)).

**Proof of (i).** — Since $n \geq 2$, it follows from the theorem that there exists a collection $\{U_n\}_{n=1}^{\infty}$ of open neighborhoods of $K$ such that $\bigcap_n U_n = K$ and for all $n$

$$h_0 : H_0(U_n, \mathbb{Z}) \rightarrow H_0(\overline{V}, \mathbb{Z})$$

is an isomorphism. On the other hand, this implies that $\overline{U_n}$ is compact and connected for all $n$. Therefore $K$ is connected.

**Proof of (ii).** — Let us suppose now that $K$ is a $C^1$ submanifold of $M$. Let $B$ be a tubular neighborhood of $K$ with projection $\pi : B \rightarrow K$. We have two possibilities:

1st - $B \setminus K$ has one connected component.

2nd - $B \setminus K$ has two connected components, say $B_1$ and $B_2$ (if $K$ has real codimension 1).

In the first case $\overline{V} \cap B = B$ and $B$ is a neighborhood of $K$ in $\overline{V}$. In the second case we have two possibilities: either $\overline{V} \cap B = B_j \cup K$ for $j = 1$ or 2 and $B_j \cup K$ is a neighborhood of $K$ in $\overline{V}$. In any case we will set $A = \overline{V} \cap B$, so that $A$ is a neighborhood of $K$ in $\overline{V}$. It is well known that the homomorphisms induced by the inclusion $K \rightarrow A$

$$h'_q : H_q(K, \mathbb{Z}) \rightarrow H_q(A, \mathbb{Z}) \text{ and } i'_q : \Pi_q(K, p) \rightarrow \Pi_q(A, p)$$

are isomorphisms for all $q$.

On the other hand there exists a neighborhood $U$ of $K$ in $\overline{V}$ such that $U \subset A$ and the homomorphisms induced by the inclusion $U \rightarrow \overline{V}$

$$h''_q : H_q(U, \mathbb{Z}) \rightarrow H_q(\overline{V}, \mathbb{Z}) \text{ and } i''_q : \Pi_q(U, p) \rightarrow \Pi_q(\overline{V}, p)$$

are onto if $q \leq n - 1$ and injectives if $q \leq n - 2$. It is not difficult to see that this implies (ii). This finishes the proof of Theorem 3.
BIBLIOGRAPHY


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