EDOARDO BALLICO

A splitting theorem for the Kupka component of a foliation of \(\mathbb{CP}^n\), \(n \geq 6\). Addendum to an addendum to a paper by Calvo-Andrade and Soares


<http://www.numdam.org/item?id=AIF_1999__49_4_1423_0>
A SPLITTING THEOREM
FOR THE KUPKA COMPONENT
OF A FOLIATION OF $\mathbb{C}P^n$, $n \geq 6$.
Addendum to an addendum to a paper
by Calvo-Andrade and Soares

by Edoardo BALLICO

A codimension one singular holomorphic foliation $F$ of $\mathbb{C}P^n$ is given by $\omega \in H^0(\mathbb{C}P^n, \Omega^k)$ (for some $k$) with $\omega \neq 0$, $\omega$ not vanishing on a hypersurface. The Kupka subset $K(F) := \{P \in \mathbb{C}P^n : \omega(P) = 0, d\omega(P) \neq 0\}$ of the singular set $S(F) := \{P \in \mathbb{C}P^n : \omega(P) = 0\}$ of $F$ has remarkable properties (e.g. if not empty it is a smooth submanifold of pure codimension 2 with strong stability properties with respect to deformations of $F$). For much more on this topic, see [GL] and [GS]. Let $K \neq \emptyset$ be a Kupka component of $F$, i.e. ([CS]) a connected component of $K(F)$. It was proved in [CL] that if $K$ is a complete intersection, then $F$ has a meromorphic first integral. Motivated by this result in [CS] it was conjectured and proved in some cases that every Kupka component, $K$, is a complete intersection. This conjecture was proved in [B] under the assumption that $\deg(K)$ is not a square. Here we will remove this restrictive assumption and prove the following result.

THEOREM. — Let $F$ be a codimension 1 singular holomorphic foliation of $\mathbb{C}P^n$, $n \geq 6$, induced by $\omega \in H^0(\mathbb{C}P^n, \Omega^1(k))$ and such that the

Keywords: Singular foliations – Codimension 1 foliations – Kupka component – Complete intersection – Unstable vector bundle – Rank 2 vector bundle – Splitting of a vector bundle – Meromorphic first integral – Barth-Lefschetz theorems.
codimension 2 component of the singular set of $F$ contains a Kupka component $K$. Then $K$ is a complete intersection and $F$ has a meromorphic first integral.

Let $N_K$ be the normal bundle of $K$ in $\mathbb{CP}^n$. By [CS], Corollary 3.5, $N_K$ is the restriction $E|K$ to $K$ of a rank 2 vector bundle $E$ on $\mathbb{CP}^n$. $K$ is a complete intersection if and only if $E$ is the direct sum of two line bundles ([OSS]). If $n \geq 6$ every line bundle on $K$ is the restriction of a line bundle on $\mathbb{CP}^n$ (see [FL]). Hence, by a very nice result of Faltings ([F]) if $n \geq 6$ and $N_K$ is the direct sum of two line bundles, $K$ is a complete intersection. Now we use the notations of [GL], page 320. There are two complex numbers (not both zero) such that the linear part of the foliation near each point of $K$ depends from these two complex numbers. Up to a normalization we may assume that one of these numbers is 1; call $\mu$ the other one. If $\mu \neq \pm 1$, then $N_K$ splits into the direct sum of two line bundles ([GL]), Remark 2 at p. 320). If $\mu = -1$, then there is a two-to-one unramified covering $\pi : Z \to K$ such that $\pi^*(N_K)$ is the direct sum of two isomorphic line bundles (see [GL], the two lines before eq. (2.6)). Since $K$ is simply connected ([FL], Cor. 6.3), $N_K$ splits into the direct sum of two isomorphic line bundles. Now assume $\mu = 1$. This is the only difference with respect to [B]. By [GL], eq. (2.12) and the two following lines, $N_K$ is an extension of a line bundle $L_1$ by itself. Since $n \geq 6$ we have $H^1(K, \mathbb{C}) \cong \mathbb{C}$ by a theorem of Barth's (see e.g. [FL], eq. (**)) at p. 83). Hence by standard Hodge theory ([GH], bottom of p. 105) we have $H^1(K, \mathcal{O}_K) = 0$. Thus any such extension of $L_1$, by itself splits and hence $N_K \cong L_1 \oplus L_1$, concluding the proof of the theorem.

The author was partially supported by MURST and GNSAGA of CNR (Italy).

BIBLIOGRAPHY


Edoardo BALLICO,
Department of Mathematics
University of Trento
38050 Povo (NT) (Italy).
ballico@science.unitn.it