Symplectic subvarieties of projective fibrations over symplectic manifolds


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1. Introduction.

Suppose that $(M, \omega)$ is a compact symplectic manifold of dimension $2n$, such that the cohomology class $[\omega] \in H^2(M, \mathbb{R})$ lies in the integral lattice $H^2(M, \mathbb{Z})/\text{Torsion}$; we shall say that $(M, \omega)$ is almost-Hodge. It has been recently proved by Donaldson that for any sufficiently large integer $k$ there exists a symplectic submanifold $W \subset M$ representing the Poincaré dual of any fixed integral lift of $[k\omega]$, [D].

In this paper, we specialize this result to the case of a symplectic fibration $p : E \to M$ whose fibre is a projective manifold $F$ with a fixed Hodge form $\sigma$ on it. For instance, $E$ could be the relative projective space, or a relative flag space, associated to a complex vector bundle on $M$. Then, as follows from well-known symplectic reduction techniques ([W], [GLS]) $E$ has an almost Hodge structure $\tilde{\omega}$ restricting to $\sigma$ on each fibre of $p$, [MS]. We adapt Donaldson's arguments to show that the symplectic divisor guaranteed by his theorem may be chosen compatibly with the vertical holomorphic structure. More precisely,

**Theorem 1.1.** — Let $(M, \omega)$ be an almost Hodge manifold. Let $F \subseteq \mathbb{P}^N$ be a connected complex projective manifold and set $L = \mathcal{O}_F(1)$.
the restriction to $F$ of the hyperplane bundle on $\mathbb{P}^N$. Denote by $\sigma$ the restriction to $F$ of the Fubini-Study form on $\mathbb{P}^N$. Suppose that $G$ is a compact group of automorphisms of $\mathbb{P}^N$ preserving $F$. Let $p : E \to M$ be a fibre bundle with fibre $F$ and structure group $G$, so that in particular there is a line bundle $L_E \to E$ extending $L \to F$. Then $E$ admits an almost Hodge structure $\tilde{\omega}$ vertically compatible with $\sigma$. Furthermore, perhaps after replacing $\tilde{\omega}$ by $kp^*(\omega_M) + \tilde{\omega}$ for $k \gg 0$, any integral lift of $[\tilde{\omega}]$ is Poincaré dual to a codimension-2 symplectic submanifold $W \subset E$, meeting any fibre $F_m = p^{-1}(m) \ (m \in M)$ in a complex subvariety.

In general the submanifold $W$ may not be transverse to every fibre. For example, if $E$ is a rank-2 complex vector bundle on $M$ and $E = \mathbb{P}E^*$ with general fibre $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, then $W$ is the blow-up of $M$ along the zero locus $Z$ of a section of a suitable twist of $\mathcal{E}$, and therefore contains all the fibres over $Z$.

In practice one may have a fibre bundle $E \to M$ with fibre a complex projective manifold $(F, J_F)$ and structure group $G$ preserving the complex structure $J_F$ and some fixed Hodge form $\sigma$ on $F$, and complexification $\tilde{G} \subseteq \text{Aut}(F, J_F)$. If $L$ is a line bundle on $F$ such that $c_1(L) = [\sigma]$, then by general principles from geometric invariant theory a lifting to $L^\otimes k$ of the action of $G$ exists if $k \gg 0$. Therefore,

**Corollary 1.1.** — Suppose that $(F, \sigma)$, $M$ and $E$ are as just described. Then for $r \gg 0$ and $k > k(r)$ any integral lift of $[r\tilde{\omega} + kp^*(\omega_M)]$ is Poincaré dual to a codimension-2 symplectic submanifold intersecting each fibre $F_m$ in a divisor of the linear series $|L^\otimes r|$.

Again, $W$ is not transversal to every fibre. In the case of a $\mathbb{P}^1$-bundle $E = \mathbb{P}E^* \to M$, the projection $W \to M$ is a branched cover with non-empty ramification locus.

The theorem also yields that top Chern classes of symplectically very positive vector bundles have symplectic representatives, as already shown by Auroux, [A]:

**Corollary 1.2.** — Let $(M, \omega)$ be a $2n$-dimensional almost Hodge manifold and let $\mathcal{E}$ be a complex vector bundle on $M$ of complex rank $r < n$. Let $H$ be a complex line bundle on $M$ with $c_1(H) = [\omega]$. Then for $k \gg 0$ there is a transverse section $s$ of $\mathcal{E} \otimes H^\otimes k$ whose zero locus $Z$ is a connected symplectic submanifold of $M$; in fact, $H_j(M, Z) = 0$ if $j \leq n - r$. 
As we shall see, these sections are also asymptotically almost holomorphic in the sense of [A].

**Notation.** — For any integer \( r > 0 \), we shall denote by \( \omega_0^{(r)} = (i/2) \sum_{\alpha=1}^{r} dz_\alpha \wedge d\bar{z}_\alpha \) the standard symplectic structure on \( \mathbb{C}^r \). Furthermore, by \( C \) we shall often indicate an appropriate constant, appearing in various estimates, which is allowed to vary from line to line.

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## 2. Proof of the theorem and corollaries.

Let \( \pi : P \to M \) be the principal \( G \)-bundle associated with the fibration. Given a connection for \( \pi \), the existence of a compatible almost Hodge form on \( E \) follows from well-known symplectic reduction arguments, [MS]. In fact, minimal coupling produces a compatible closed 2-form \( \vartheta = \vartheta_{\text{min}} \) on \( E \), [GS]. Explicitly, let the induced connection be given by the horizontal distribution \( \mathcal{H}(E/M) \subset TE \) and denote by \( V(E/M) \subset TE \) the vertical tangent space. Let \( g \) be the Lie algebra of \( G \) and view the curvature \( F \) as a \( g \)-valued 2-form on \( M \). Let \( \mu : F \to g^* \) be the moment map for the action. If \( e \in E \) and \( x = p(e) \), let \( U \subseteq M \) be an open subset over which \( P \) trivializes and let \( \gamma : U \times F \to p^{-1}(U) \) be the corresponding trivialization. Then \( \mathcal{H}(E/M) \) and \( V(E/M) \) are mutually orthogonal for \( \sigma \). Furthermore, with abuse of language, \( \vartheta|_{(P/M)} = \sigma \), while if \( X, Y \in T_xM \) and \( X^h, Y^h \) are their horizontal lifts at \( e = \gamma(x, f) \), then \( \vartheta_e(X^h, Y^h) = \left( \mu(f), F_x(X, Y) \right) \). Therefore \( \tilde{\omega}(k) = \vartheta + kp^*(\omega) \) is a compatible symplectic structure on \( E \) if \( k \gg 0 \). However, in order to adapt Donaldson’s construction we shall need to describe \( -2\pi i \vartheta \) as the curvature of a connection on a suitable line bundle on \( E \).

Clearly, the action of \( G \) lifts to \( L \) and preserves the unit circle bundle \( S_L \subset L \). Let \( \nabla_L \) be the unique covariant derivative on \( L \) compatible with the complex and hermitian structures, that is, the restriction to \( F \) of the connection on \( O_{\mathbb{P}^n}(1) \). Let \( \mathcal{H}(S_L/F) \subset TS_L \) be the corresponding \( S^1 \)-invariant horizontal distribution, which by uniqueness is also \( G \)-invariant. The line bundle \( L_E := P \times_G L \) over \( E \) restricts to \( L \) on every fibre of \( p \) and has an hermitian metric extending that of \( L \). Then the unit circle
bundle $S_{LE} = P \times_G S_L \subset L_E$ has a connection over $E$, as follows. Let $p' : S_{LE} \to M$ be the projection, a fibre bundle over $M$ with general fibre $S_L$. Given $s \in S_{LE}$ mapping to $e \in E$, set $x = p(e)$ and choose as above a trivialization of $P$ in a neighbourhood $U$ of $x$, with induced trivializations $\gamma : U \times F \to p^{-1}(U)$ and $\gamma' : U \times S_L \to p'^{-1}(U)$. If $e = \gamma(x, f)$ and $s = \gamma'(x, \ell)$ ($\ell \in S_L$ lies over $f \in F$), then the horizontal space of $S_{LE}$ at $s$ is $\mathcal{H}(S_{LE}/E) = \mathcal{H}(S_{LE}/M) \oplus d\gamma'_{(x, \ell)}\left(\mathcal{H}_\ell(S_L/F)\right)$. This gives a well-defined connection $\nabla_{LE}$ on $L_E$, and we leave it to the reader to check that $\theta_{\text{min}}$ may also be obtained as the normalized curvature of $\nabla_{LE}$:

**Lemma 2.1.** — Let $\theta$ be the normalized curvature form on $E$ of the connection $\mathcal{H}(S_E/E)$. Then for $k > 0$ the 2-form $\tilde{\omega}(k) = \theta + kp^*(\omega)$ is a compatible symplectic structure, and $\mathcal{H}(E/M)$ is the symplectic complement of $V(E/M)$ for $\tilde{\omega}$. In particular, the subbundle $\mathcal{H}(E/M) \subset TE$ is symplectic with respect to $\tilde{\omega}$.

We shall need an auxiliary non-degenerate 2-form $\omega_{\text{aux}}$ on $E$. The vertical tangent bundle $V(E/M)$ has an obvious symplectic structure, the restriction of $\tilde{\omega}$, that we shall also indicate by $\sigma$, and an obvious complex structure $J_{\text{vert}}$, inherited by that of $TF$. The horizontal distribution $\mathcal{H}(E/M)$, on the other hand, carries the symplectic structure $p^*\omega$. Then $\omega_{\text{aux}} \in \Omega^2(E)$ will denote the orthogonal direct sum of $\sigma$ and $p^*\omega$. In general $\omega_{\text{aux}}$ will not be closed, and in view of the minimal coupling horizontal component of $\nabla$ we see that $\omega_{\text{aux}} \neq \tilde{\omega}(1)$ when $P$ is not flat. Let us pick some $J_M \in J(M, \omega)$ and view it in a natural manner as a complex structure on $\mathcal{H}(E/M)$; then $J_{\text{aux}} := J_M \oplus J_{\text{vert}} \in J(E, \omega_{\text{aux}})$. Thus $g_{\text{aux}}(\cdot, \cdot) = \omega_{\text{aux}}(\cdot, J_{\text{aux}}\cdot)$ is a riemannian metric on $E$. On the other hand, we have $\tilde{\omega}(k) = \tilde{\omega}^h(k) + \tilde{\omega}^v(k)$, where $\tilde{\omega}^h(k)$ and $\tilde{\omega}^v(k) = \sigma$ denote, respectively, the horizontal and vertical components. Now $\alpha_k := (1/k)\tilde{\omega}^h(k)$ is a sequence of symplectic structures on the vector bundle $\mathcal{H}(E/M)$, converging to $p^*\omega$ in the $C^1$-topology, namely $\|\alpha_k - p^*\omega\| < C/k$ and $\|\nabla(\alpha_k - p^*\omega)\| < C/k$. Given a vector bundle $\mathcal{F}$ on a manifold and any symplectic structure $\eta$ on $\mathcal{F}$, there is a retraction $r_\eta : \text{Met}(\mathcal{F}) \to \mathcal{J}(\mathcal{F}, \eta)$ depending pointwise analytically on $\eta$, where $\text{Met}(\mathcal{F})$ is the space of all riemannian metrics on $\mathcal{F}$, and $\mathcal{J}(\mathcal{F}, \eta)$ denotes the space of all complex structures on $\mathcal{F}$ compatible with $\eta$ ([MS], ch. 2). Denote by $g^h_{\text{aux}}$ the restriction of $g_{\text{aux}}$ to $H(E/M)$, and let $J^h_k := r_{\alpha_k}(g^h_{\text{aux}}) \in \mathcal{J}(H(E/M), \alpha_k)$ for each $k$; then $\|J^h_k - J_M\| < C/k$, $\|\nabla(J^h_k - J_M)\| < C/k$. Therefore $J_k := J^h_k \oplus J_{\text{vert}} \in \mathcal{J}(E, \tilde{\omega}_k)$ and
\[ \|J_k - J_{aux}\| < C/k, \|\nabla(J_k - J_{aux})\| < C/k. \]

Let \( \bigwedge_{J_{aux}}^{(1,0)} T_E^* \) and \( \bigwedge_{J_{aux}}^{(0,1)} T_E^* \) denote, respectively, the \( \mathbb{C} \)-linear and \( \mathbb{C} \)-antilinear complex functionals on \((T_E, J_{aux})\), and let \( \mu_k : \bigwedge_{J_{aux}}^{(1,0)} T_E^* \to \bigwedge_{J_{aux}}^{(0,1)} T_E^* \) be the morphism of vector bundles relating \( J_k \) to \( J_{aux} \), [D]. Then \( \|\mu_k\| < C/k \) and \( \|\nabla\mu_k\| < C/k. \)

The riemannian metric \( g_M = \omega(\cdot, J_M \cdot) \) on \( M \) induces a distance function \( d \); for \( k \) a positive integer, let \( d_k \) denote the distance function associated to the pair \((k\omega, J_M)\), that is to the metric \( kg_M \). Similarly, let \( d_F \) be the distance function on \( F \) associated to the pair \((\sigma, J_F)\).

Furthermore, on \( M \) there is an hermitian line bundle \( H \) together with a unitary connection on it having curvature form \( -2\pi i\omega \). Replacing \( \tilde{\omega} \) by \( \tilde{\omega}_{(k)} \) amounts to replacing \( L_E \) by \( B = p^*(H^\otimes k) \otimes L_E \) with the tensor product connection. Thus we are looking for a section \( s \) of \( B \) for some \( k \gg 0 \) whose zero locus is a symplectic submanifold \( Z \subset E \) with respect to \( \tilde{\omega} \), meeting each fibre \( F_x \) in a complex subvariety.

Let \( \nabla_B \) be the covariant derivative on \( B \). Given the almost complex structure \( J_E \), we have a decomposition \( \nabla_B = \partial_B + \overline{\partial}_B \). The zero locus \( Z = Z(s) \) of a smooth section \( s \) of \( B \) will be symplectic if \( |\overline{\partial}_{J_k,B}s| < |\partial_{J_k,B}s| \) at every point of \( Z \) ([D]; Lemma 4.30 of [MS]); the two latter terms represent, respectively, the \((0,1)\) and \((1,0)\) components of \( \nabla_B s \) with respect to the almost complex structure \( J_k \). Following the path of Donaldson’s construction, we shall produce such a section as a linear combination of certain “concentrated” building blocks. In order for \( Z \cap F_x \) to be a complex subvariety of \( F_x \) for every \( x \in M \), these basic pieces must be chosen in an appropriate way.

**Definition 2.1.** — If \( U \subset E \) is an open set, a smooth function \( f : U \to \mathbb{C} \) will be called **vertically holomorphic** (in short, \( v \)-holomorphic) if its restriction to \( U \cap F_x \) is holomorphic, whenever the latter set is non-empty. Let \( A \) be any complex line bundle on \( E \). A \( v \)-holomorphic structure on \( A \) is the datum of an open cover \( \mathcal{U} = \{U_\alpha\} \) of \( A \), together with \( v \)-holomorphic transition functions \( g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^* \). With such an assignment, \( H \) will be called a \( v \)-holomorphic line bundle. There is a natural notion of equivalence of \( v \)-holomorphic structures. Clearly, the restriction of \( A \) to any fibre \( F_x \) is a holomorphic line bundle \( A_x \). A local section of \( A \) on \( U \subset E \) is called \( v \)-holomorphic if it restricts to a holomorphic local section of \( A_x \) for every \( x \in M \) for which \( U \cap F_x \neq \emptyset \). Let \( \mathcal{O}_E^v \) denote the sheaf of rings of \( v \)-holomorphic functions on \( E \); the sheaf of \( v \)-holomorphic sections

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\[ \nabla(J_k - J_{aux}) \]
of $A$, denoted $\mathcal{O}_E^\nu(A)$, is a sheaf of $\mathcal{O}_E^\nu$-modules.

Let $f : U \to \mathbb{C}$ be a smooth function on an open subset $U \subset E$, and let $(df)_{\text{vert}} \in V(E/M)^* \otimes \mathbb{C}$ be the restriction of its differential to the vertical tangent bundle. Let $j$ denote the complex structure of $\mathbb{C}$. Then $f$ is $\nu$-holomorphic if and only if $\overline{\partial}_{\text{vert}} f := (df)_{\text{vert}} + j \circ (df)_{\text{vert}} \circ J_{\text{vert}} = 0$; the left hand side is the $\mathbb{C}$-antilinear component of $(df)_{\text{vert}}$. Now the line bundle $L_E$ is naturally $\nu$-holomorphic, and restricts to $L$ on each fibre. Thus Theorem 1.1 is a consequence of the following:

**Proposition 2.1.** — For $k \gg 0$ there is a $\nu$-holomorphic section $s$ of $B$ such that $|\partial_{J_k,B}s| < |\partial_{J_k,B}s|$ at all points of the zero locus of $s$.

To prove the proposition, we shall first produce a suitable choice of compactly supported $\nu$-holomorphic sections, peaked at points of $E$ in an appropriate sense, to be used as the basic building blocks in Donaldson's construction. Next we shall give an appropriate open cover of $E$ on which to perform the inductive part of his argument.

Fix $e_0 \in E$ and let $U_0 \subseteq M$ be an open neighbourhood of $x_0 = p(e_0)$ over which $P$ is trivial; perhaps after replacing $\omega$ by some multiple, there is a Darboux coordinate chart $\chi : B^{2n} \to U_0 \subseteq M$ centred at $x_0$ for $\omega$, which is $\mathbb{C}$-linear at the origin. Let $\eta$ be a unitary section of $H$ over $U_0$ such that the connection matrix $\theta_M$ of $H$ on $U_0$ with respect to $\eta$ satisfies $\chi^* \theta_M = A$, where $A = (1/4) \sum_{\alpha=1}^n (\bar{z}_\alpha dz_\alpha - z_\alpha d\bar{z}_\alpha)$, [D]. We have an induced trivialization $\gamma : U_0 \times F \to p^{-1}(E|U_0)$, under which $\gamma^*(L_E) \cong q_\nu^*(L)$, where $q_\nu$ is the projection on the second factor; suppose $e_0 = \gamma(x_0,f_0)$. We may assume that $\forall f \in F$ the local section $\gamma_f(y) = \gamma(y,f)$ defined over $U$ satisfies $d_{x_0} \gamma_f(T_{x_0}M) = H_\nu$, where $e = \gamma_f(x_0)$. The product map $\phi = \gamma \circ (\chi,\text{id}_P) : B^{2n} \times F \to E$ is holomorphic along $F_{x_0}$ with respect to $J_{\text{aux}}$, i.e. $d_{(0,f)}\phi : \mathbb{C}^n \times T_F \to T_{\gamma(x_0,f)} E$ is $\mathbb{C}$-linear for all $f \in F$.

The picture may be rescaled on the base. If $\delta_k(z) = z/\sqrt{k}$ for $z \in \mathbb{C}^n$, define $\bar{\chi}_k = \chi \circ \delta_k : \sqrt{k}B^{2n} \to U_0$, [D]. There are product maps

$$\bar{\phi}_k : \sqrt{k}B^{2n} \times F (\bar{\chi}_k,\text{id}_F) U_0 \times F \xrightarrow{\gamma} E.$$  

The function $\bar{\phi}_k$ maps diffeomorphically onto $p^{-1}(U_0)$, and is holomorphic along $F_{x_0}$ and on $B^{2n} \times F$ we have $\bar{\phi}_k^*(\omega) = \omega_0 + \sigma + O(1/k)$. One can check arguing as in [D] that it is approximately holomorphic, in the following sense.
LEMMA 2.2. — Let $J_{pr}$ denote the product complex structure $J_0 \times J_F$ on $\sqrt{k}B^{2n} \times F$, and let $\mu'_k(z, f) : \bigwedge^{0,1} (C^n \times T_F) \to \bigwedge^{1,0} (C^n \times T_F)$, 
\[(z, f) \in \sqrt{k}B^{2n} \times F,
\]
be the bundle morphism relating $\tilde{\phi}'_k(J_k)$ to $J_{pr}$. Then $|\mu'_k| \leq C|z|/\sqrt{k}$, $|\nabla \mu'_k| \leq C/\sqrt{k}$.

If $\nu \in H^0(F, L)$, the product $\eta^{\otimes k} \otimes \nu$ may be regarded as a $v$-holomorphic section of $B$ on $p^{-1}(U_0)$. We may choose $\nu_0 \in H^0(F, L)$ and an open neighbourhood $V_0 \ni f_0$ so that $1/2 \leq |\nu_0| \leq 1$ on $V$, $|\nu_0| \leq 1/2$ on $F \setminus V_0$ and $|\nu_0(f)| = 1 \iff f = f_0$. The connection matrix $\theta$ of $\nabla_L$ with respect to the trivialization $\nu_0$ satisfies $\theta(f_0) = 0$.

Let $\theta_{L_E}$ and $\tilde{\theta}$ be the connection matrices of $\nabla_{L_E}$ and $\nabla_B$ with respect to the trivializations $\nu_0$ and $\eta^{\otimes k} \otimes \nu_0$, respectively. We may assume that $\theta_{L_E}(e_0) = 0$; let $\zeta_0$ denote the resulting section of $B$ over $U_0$. If the $t_i$'s are local coordinates on $F$ centred at $f_0$ and the $x_1, \ldots, x_{2n}$ are the local coordinates on $M$ centred at $x_0$ given by the chart $\chi$, in the resulting trivialization on $\chi_k(B^{2n} \times F)$ we have $\tilde{\phi}'_k \theta_B = \theta + A + \beta_k$, where $|\beta_k| = O(1/\sqrt{k})$.

The function $g(z) = \exp(-|z|^2/4)$ is a holomorphic section of the trivial line bundle $\xi$ on $\mathbb{C}^n$ with the connection $A$, [D]. If $\beta$ is the standard cut-off function centred at the origin and $\beta_k(z) = \beta(k^{-1/6}|z|)$, then $\varphi_k = \beta_k g$ is the compactly supported, approximately holomorphic section of $(\xi, A)$ constructed in [D]. The following lemma shows that $\varphi_0(e) = \varphi_k(\chi_k^{-1}(x))\zeta_0(e)$, where $e = \gamma(x, f)$, is a good candidate for the sought concentrated $v$-holomorphic section of $B$.

Let us consider, as in [D], the following real function on $M \times M$:

\[
\ell_k(x, x') = \begin{cases} 
e^{-d_k(x, x')^2/5} & \text{if } d_k(x, x') \leq k^{1/4} \\ 0 & \text{if } d_k(x, x') > k^{1/4}. \end{cases}
\]

LEMMA 2.3. — If $x = p(e)$ then $|\varphi_0(e)| \leq \ell_k(x, x_0)$. If $d_k(x, x_0) \leq k^{1/6}/4$, then $|\varphi_0(e)| \geq \exp(-d_k(x, x_0)^2/3)|\nu_0(f)|$; in particular, for a fixed $R > 0$ and all $k \geq 0$, if $\|d_k(x, x_0) \leq R$ and $f \in V_0$ then $|\varphi_0(e)| \geq 1/C$. For all $e \in E$, we have

\[
|\nabla_B \varphi_0(e)| \leq C(1 + d_k(x_0, x))\ell_k(x_0, x),
\]

\[
|\nabla_{J_k,B} \varphi_0(e)| \leq Ck^{-1/2}(1 + d_k(x_0, x) + d_k(x_0, x)^2)\ell_k(x_0, x),
\]

and

\[
|\nabla_B \nabla_{J_k,B} \varphi_0(e)| \leq Ck^{-1/2}(1 + d_k(x_0, x) + d_k(x_0, x)^2 + d_k(x_0, x)^3)\ell_k(x_0, x).
\]
Proof of Lemma 2.3. — We may introduce an additional almost Kähler structure on $E|_U$, as follows. Given the trivialization $\gamma : U \times F \cong E|_U$, for each $e = \gamma(x, f) \in E|_U$ we have $T_x E \cong d_x \gamma_f(T_x F) \oplus V_e$. We define a horizontal distribution $H' \subset TE$ over $U$ by setting $H'_e = d_x \gamma_f(T_x E)$, so that $TE \cong H' \oplus V$. Let us pull back the almost complex structure $J_M$ to an almost complex structure $J'_M$ on $H'$ and then set $J' = J'_M \oplus J_{\text{vert}}$, where $\oplus$ is the direct sum with respect to the latter decomposition. By construction $H'_e = H_e$ and so $J_{\text{aux}}(e) = J'(e) \forall e \in F_{x_0}$. Similarly set $\omega' := \omega \oplus \sigma$, where $\omega$ is implicitly pulled-back to $H'$. Then $\omega'$ is a nondegenerate 2-form on $E|_U$ and $J' \in \mathcal{J}(E|_U, \omega')$. Hence $g' := \omega'(\cdot, J')$ is a riemannian metric on $E|_U$ and $g'_* = g_{aux}$ on $F_{x_0}$. Let $\mu' = \mu'(x, t) : \bigwedge^k TE \to \bigwedge^{0,1} TE$ be the morphism of vector bundles relating $J_{aux}$ to $J'$. Thus $\mu'(e) = 0 \forall e \in F_{x_0}$ and so $|\mu'| \leq C|x|$. Let $\mu'_k$ be the vector bundle morphism relating $\phi^*_k J_{aux}$ to $\phi^*_k J'$; then $\mu'_k = \delta^k \mu_1$, hence $|\mu'_k| \leq C d_k(x, x_0)/\sqrt{k}$ and $|\nabla \mu'_k| < C/\sqrt{k}$. Similarly, replacing $\omega$ by $\omega$ in the above construction but leaving the vertical component $\sigma$ unchanged, we get non-degenerate 2-forms $\omega^{aux}_k$ and $\omega^{(k)}$, and riemannian metrics $g^{aux}_k$ and $g^{(k)}$; perhaps after restricting $U$ for $k \gg 0$ the corresponding quadratics forms $q^{aux}_k$ and $q^{(k)}$ are equivalent on $E|_U$. In turn, $q^{aux}_k$ is equivalent to $q^{(k)}$ (the quadratic form associated to $g_k$). On the upshot the claimed estimates may be proved using $q^{(k)}$, by an adaptation of the arguments in [D]. Let us give some detail for $\vartheta_0$ and $\nabla_B \vartheta_0$. As to the former, the claim follows directly from the definition. As to the latter, the proof is straightforward on the region $T$ where $d_k(x_0, x) \leq k^{1/6}/4$ and $f \in V_0$. Fix $e_1 \notin T$. Let $\vartheta_1$ be a section constructed as above, but with reference point $e_1$. Then $\vartheta_0 = s \vartheta_1$ near $e_1$ for a suitable $\nu$-holomorphic function $s$, and therefore $|\nabla_B \vartheta_0(e_1)| = |ds(e_1)|$. The claim easily follows from this.

The estimates on $\bar{\partial}_J \vartheta_0$ and $\nabla_B \bar{\partial}_J \vartheta_0$ also follow by similar arguments, in view of the fact that, up to $(1 - \bar{\mu}' \mu'^{-1})$ etc,

$$
\bar{\partial}_J \vartheta_0 = \bar{\partial}_{J_{aux}} \vartheta_0 - \mu_k(\bar{\partial}_{J_{aux}} \vartheta_0),
$$

$$
\bar{\partial}_{J_{aux}} \vartheta_0 = \bar{\partial}_{J'} \vartheta_0 - \mu'_k(\bar{\partial}_{J_{aux}} \vartheta_0),
$$

$$
\partial_{J_{aux}} \vartheta_0 = \partial_{J'} \vartheta_0 - \mu'_k(\partial_{J_{aux}} \vartheta_0),
$$

[D]. \hfill \Box

We now need to describe a suitable open cover of $E$. This is obtained by locally taking products of open sets in an open cover of $M$ depending on $k$ as in [D] and in a suitable fixed open cover of $F$. For $k \gg 0$ let $U = \{U_i\}$ be an open cover of $M$ by a collection of $g_k$-unit balls $U_i$, with centres $x_i$, 

i = 1, \ldots, M_k, satisfying the properties stated in Lemmas 12 and 16 of loc. cit. In particular, for every \( e \in E \) and \( r = 0, 1, 2, 3 \) one has

\[
\sum_{i=1}^{M_k} d_k(x_i, x)^r \ell_k(x_i, x) \leq C.
\]

For \( D > 0 \), let \( N = CD^{2n} \) and the partition of \( I = \bigcup_{\alpha=1}^N I_{\alpha} \), where \( I = \{1, \ldots, M_k\} \) be as in the statement of Lemma 16 of loc. cit.

For each \( i \) fix a trivialization \( \gamma_i : U_i \times F \cong E|_{U_i} \). Consider an open cover \( V = \{V_j\}_{j \in J} \) of \( F \), \( J = \{1, \ldots, R\} \), by balls of a suitable \( g_r \)-radius \( \delta > 0 \) centred at points \( f_j \in V_j \), so that for each \( j \) there exists \( \nu_j \in H^0(F, L) \) satisfying \( 1/2 \leq |\nu_j|_{V_j} \leq 1 \) and \( |\nu_j(f)| = 1 \) if and only if \( f = f_j \). We thus obtain an open cover \( W = \{W_{ij}\} \) of \( E \), where \( W_{ij} = \gamma_i(U_i \times V_j) \). For each \( (i, j) \) there is a \( \nu \)-holomorphic section \( \theta_{ij} \) of \( B \) supported near \( F_{x_i} \) and peaked at \( e_{ij} = \gamma_i((x_i, f_j)) \). Partition the index set \( I \times J \) as \( I \times J = \bigcup_{\beta=1}^{NR} S_{\beta} \), where \( S_{kN+\alpha} = I_{\alpha} \times \{k+1\}, k = 0, \ldots, R-1, 1 \leq \alpha \leq N \). Now let us insert the \( \theta_{ij} \)’s in Donaldson’s construction. Given any \( \bar{w} \in \mathbb{C}^{NR} \), with \( |w_{\beta}| \leq 1 \), set \( s_{\bar{w}} = \sum_i w_{ij}\theta_{ij} \); since \( s_{\bar{w}} \) is \( \nu \)-holomorphic, its zero locus \( Z_{\bar{w}} \) meets any fibre \( F_x \) in a complex subvariety. For any \( (i, j) \in I \times J \), the local functions \( f_{ij} = s_{\bar{w}}/\theta_{ij} \) are defined on \( W_{ij} \), and by Lemma 2.2, when viewed as functions on a suitable multidisc \( \Delta^+ \) of fixed radius in \( \mathbb{C}^{n+d} \), they satisfy properties as in lemmas 18 and 19 of [D]. We may then proceed by adjusting the coefficients \( w_{\beta} \)’s in \( NR \) steps to obtain a \( \bar{w}_f \in \mathbb{C}^{NR} \), such that \( s_{\bar{w}_f} \) satisfies \( |\theta_{B}s_{\bar{w}_f}| > |\bar{\partial}_B s_{\bar{w}_f}| \) on \( Z_f \), so that \( Z_f \) is a symplectic submanifold of \( E \).

Let us prove Corollary 1.1. If \( L \) is a holomorphic line bundle on \( F \) with \( c_1(L) = [\sigma] \), there are an hermitian structure on \( L \) and a unitary connection on it whose normalized curvature form is \( \sigma \). For \( r \gg 0 \), the action of \( G \) on \( F \) admits a linearization \( \bar{v} : \bar{G} \times L^{\otimes r} \to L^{\otimes r} \) ([M], section 1.3). Let \( s \) be the section of \( B = L^{\otimes r} \otimes H^{\otimes k} \) for \( k > k(r) \) provided by the theorem, \( Z \) its zero locus. Given a \( \nu \)-holomorphic line bundle \( A \) on \( E \) we define its \( \nu \)-holomorphic direct image, \( p_\nu^*(A) \), as the sheaf of modules over the ring of smooth functions on \( M \) given by \( p_\nu^*(A)(U) = \mathcal{O}_E^r(p^{-1}U, A) \) for any open subset \( U \subseteq M \). Then \( F := p_\nu^*(B) \) is a smooth vector bundle on \( M \) of rank \( r = h^0(F, L^{\otimes r}) \) and \( \mathcal{O}_E^r(B) \cong A(M, F) \), the latter being the space of smooth sections of \( F \). Let \( V \) be the vector space of \( \nu \)-holomorphic
sections of $B$ spanned by the $\psi_i$'s and let $W \supseteq V$ be a finite dimensional space of $C^\infty$ sections of $F$ that globally generates $F$. Then $s \in W$ has an open neighbourhood $Q$ consisting of $\nu$-holomorphic sections of $B$ whose zero locus is a symplectic submanifold of $E$. On the other hand, except for those in a subset of $W$ of measure zero the elements of $W$ are transversal to the zero section and this is true in particular for some section $s' \in Q$. But for $r \gg 0$ certainly $\text{rank}(F) = h^0(F, L^{\otimes r}) > \dim(M)$ and therefore $s'$ is nowhere vanishing. \hfill \Box

Finally let us come to Corollary 1.2. Fix an hermitian metric on $E$ and thus an associated principal $U(r)$-bundle. With $E = \mathbb{P}\mathcal{E}^*$, $L_E$ is the relative hyperplane line bundle and $p_*^\nu(L_E) = \mathcal{E}$. Let $\mathcal{H}$ be the connection on $L_E$ induced by the compatible connection on $L = O_{\mathbb{P}^{r-1}}(1)$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes H^{\otimes k}$, $L_E$ changes to $L_E \otimes p^*(H^{\otimes k})$. When $k \gg 0$ the theorem yields a $\nu$-holomorphic section $\sigma$ of $B = L_E \otimes p^*(H^{\otimes k})$ with zero locus $D$ at each point of which $|\bar{\partial} J_{\nu k} B\sigma(e)|_k < Ck^{-1/2}|\partial J_{\nu k} B\sigma(e)|_k$, where $| \cdot |_k$ is the norm induced by $g_k$. By perturbing $\sigma$ slightly, the section $\tilde{\sigma}$ of $\mathcal{E} \otimes H^{\otimes k}$ corresponding to it may be assumed transverse, with smooth zero locus $Z \subseteq M$. Now $J_{\text{aux}}$ and $J_k$ differ by $O(1/k)$ and $q^{(k)}_{\text{aux}}$ is equivalent to $q^{(k)}$. Thus $|\bar{\partial} J_{\text{aux}, k} B\sigma(e)|_{\text{aux}, k} < |\partial J_{\text{aux}, k} B\sigma(e)|_{\text{aux}, k}$ at all $e \in D$, where $| \cdot |_{\text{aux}, k}$ denotes the norm associated to $q^{(k)}_{\text{aux}}$, and therefore $\omega^{(k)}_{\text{aux}}$ restricts to an everywhere non-degenerate 2-form on $D$. I claim that this implies that $Z$ is a symplectic submanifold of $M$. If not, there exist $x \in Z$ and $v \in T_xZ$ such that $\omega_x(v, w) = 0$ for all $w \in T_xZ$. The restriction $p|_D : D \to X$ is a $\mathbb{P}^{r-2}$-bundle off $Z$, while $D_Z = p_D^{-1}(Z)$ is $\mathbb{P}\mathcal{E}^*|_Z$. Identify a tubular neighbourhood of $Z$ in $M$ with a neighbourhood of the zero section in $\mathcal{E}|_Z$. If $v^\perp \subset T_xM$ is the symplectic annihilator of $v$ and $W = E(x) \cap v^\perp$, then $\dim W \geq 2r - 1$ and $\dim W \cap iW \geq 2r - 2$, where $i$ is the complex structure of $E(x)$. Thus there is a complex hyperplane $\Lambda$ of $E(x)$ with $\Lambda \subseteq v^\perp$. If $\lambda \in p^{-1}(x)$ is the corresponding point, $T_{\lambda}D$ is generated by $T_{\lambda}D_Z$ and $2(r-1)$ vectors $w_1, \cdots, w_{2r-2}$ projecting to a real basis of $\Lambda$. Let $v^\sharp \in H_{\lambda}$ be the horizontal lift of $v$; by construction $v^\sharp$ lies in the kernel of $\omega^{(k)}_{\text{aux}}|_{T_{\lambda}D}$, a contradiction.

Now essentially the same argument as in the proof of Proposition 39 of [D] (with $\omega^{(k)}$ in place of $k\omega$) shows that $E$ is obtained topologically from $D$ by attaching cells of dimension $\geq n+r-1$, so that by Lefschetz duality $H^k(E \setminus D) = 0$ for $k \geq n+r$. Since $E \setminus D$ is a $\mathbb{C}^{r-1}$-bundle over $M \setminus Z$, this implies $H_j(M, Z) = 0$ for $j \leq n-r$ (cf. [S] and [L], §1).

We now examine the almost complex geometry of the sections of $\mathcal{E} \otimes H^{\otimes k}$ produced in Corollary 1.2. Let us write $\mathcal{F}$ for $\mathcal{E} \otimes H^{\otimes k}$ and, in
the notation of the proof, fix $x \in Z$ and a unitary frame $f_1, \ldots, f_r$ for $\mathcal{F}$ in a neighbourhood $U$ of $x$. Then $\tilde{\sigma} = \sum a_i f_i$, where the $a_i$'s are smooth functions and $Z \cap U = \{a_i = 0 \ \forall i\}$. Therefore $\nabla_{\mathcal{F}} \tilde{\sigma}(x) = \sum_i d_x a_i \otimes f_i(x)$ and so $\partial J_{\mathcal{F}} \tilde{\sigma}(x) = \sum_i \partial J a_i(x) \otimes f_i(x)$, $\overline{\partial}_{J_{\mathcal{F}}} \tilde{\sigma}(x) = \sum_i \overline{\partial}_J a_i(x) \otimes f_i(x)$ whence

$$\|\partial_{J_{\mathcal{F}}} \tilde{\sigma}(x)\|^2 = \sum_i \|\partial J a_i(x)\|^2, \quad \|\overline{\partial}_{J_{\mathcal{F}}} \tilde{\sigma}(x)\|^2 = \sum_i \|\overline{\partial}_J a_i(x)\|^2.$$ Given that $B = \mathcal{O}_\mathcal{P}(F^*)$, we have on $\mathbb{P}^r(\mathcal{F}^*) = \mathbb{P}(\mathcal{F}^*)$ the short exact sequence $0 \to \Omega^1_{rel} \otimes B \to \pi^*(\mathcal{F}) \to B \to 0$, where $\Omega^1_{rel}$ is the relative cotangent bundle. In loose notation, on $\pi^{-1}(U)$ we have $\sigma = \alpha(\tilde{\sigma}) = \sum a_i F_i$, where $F_i = \alpha(f_i)$. At any $e \in \pi^{-1}(x)$, we have $\nabla_B \sigma(e) = \sum d_x a_i \otimes F_i(e)$, and therefore $\partial_{J_{aux}, B} \sigma(e) = \sum \partial_{J_{aux}, B} a_i(x) \otimes F_i(e)$, $\overline{\partial}_{J_{aux}, B} \sigma(e) = \sum \overline{\partial}_{J_{aux}, B} a_i(x) \otimes F_i(e)$. Now $\|\overline{\partial}_{J_{aux}, B} \sigma(e)\|_{aux, k} < C k^{-1/2} \|\partial_{J_{aux}, B} \sigma(e)\|_{aux, k}$ at every $e \in \mathbb{P}^r(\mathcal{F}^*)$. For $i = 1, \ldots, r$ let $e_i \in \mathbb{P}(\mathcal{F}^*)_x \cong \mathbb{P}^{r-1}$ be the point where all the $F_j$'s except $F_i$ vanish. Evaluating the latter inequality at $e_i$, we obtain $\|\overline{\partial}_{J_{aux}, B} a_i(x)\|_{aux, k} < C k^{-1/2} \|\partial_{J_{aux}, B} a_i(x)\|_{aux, k}$ and thus $\|\overline{\partial}_{J_M} a_i(x)\| < C k^{-1/2} \|\partial_{J_M} a_i(x)\|$ on $M$ for every $i$, whence $\|\overline{\partial}_J \tilde{\sigma}(x)\| < C k^{-1/2} \|\partial_{J_{\mathcal{F}}} \tilde{\sigma}(x)\|$. In fact, we also know that $\|\partial_{J_{aux}, B} \sigma(e)\|_{aux, k} > \eta$ at all $x \in D$ for some $\eta > 0$ independent of $k$, and the argument just given then shows that $\|\partial_{J_{\mathcal{F}}} \tilde{\sigma}(x)\| > \eta$ for all $x \in Z$.

Furthermore, these sections are asymptotically almost holomorphic in the sense of [A]. By construction, $\sigma = \sum w_{ij} e_j \otimes \sigma_i, \text{ where } |w_{ij}| \leq 1$ for all $i, j$, while the $\sigma_i$'s are compactly supported sections of $H^{\otimes k}$ as in Proposition 11 of [D], and the $e_j$'s are local sections of $\mathcal{E}$, chosen once for all and thus independent of $k$. A slight modification of the arguments proving Lemma 14 of [D] then leads to the estimates stated in Definition 1 of [A].

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