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ALGEBRAIC EQUIVALENCE OF REAL ALGEBRAIC CYCLES

by M. ABÁNADES & W. KUCHARZ

1. Introduction.

Let X be a nonsingular, n -dimensional, quasiprojective variety over \mathbb{R} (that is, an irreducible, n -dimensional, quasiprojective scheme over \mathbb{R} , smooth over \mathbb{R}). We endow the set $X(\mathbb{R})$ of \mathbb{R} -rational points of X with the topology induced by the usual metric topology on \mathbb{R} , and assume that $X(\mathbb{R})$ is nonempty and compact. Thus $X(\mathbb{R})$ is a C^∞ , closed, n -dimensional manifold. Given a nonnegative integer k , we let $Z^k(X)$ denote the group of algebraic $(n - k)$ -cycles on X (that is, the free Abelian group on the set of closed, $(n - k)$ -dimensional subvarieties of X). There exists a unique group homomorphism

$$\text{cl}_{\mathbb{R}} : Z^k(X) \rightarrow H^k(X(\mathbb{R}), \mathbb{Z}/2)$$

such that for every closed, $(n - k)$ -dimensional subvariety V of X , the cohomology class $\text{cl}_{\mathbb{R}}(V)$ is Poincaré dual to the homology class in $H_{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ determined by $V(\mathbb{R})$ (cf. [5] for the definition of this homology class). In the present paper we study the cohomology classes of the form $\text{cl}_{\mathbb{R}}(z)$, where z is a cycle in $Z^k(X)$ algebraically equivalent to 0 (we refer to [7] for the theory of algebraic equivalence of cycles). Such cohomology classes need not be trivial, but as we shall see below they must satisfy quite restrictive conditions.

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The extreme cases, $k = 0$ and $k = n$, are easy to analyze. Obviously, a cycle z in $Z^0(X)$ is algebraically equivalent to 0 if and only if $z = 0$. On the other hand, every cycle in $Z^n(X)$ of the form $x_0 - x_1$, where x_0 and x_1 are points in $X(\mathbb{R})$, is algebraically equivalent to 0. We have $\text{cl}_{\mathbb{R}}(x_0 - x_1) \neq 0$ whenever x_0 and x_1 belong to distinct connected components of $X(\mathbb{R})$. It follows that a cohomology class u in $H^n(X(\mathbb{R}), \mathbb{Z}/2)$ can be written as $u = \text{cl}_{\mathbb{R}}(z)$ for some cycle z in $Z^n(X)$ algebraically equivalent to 0 if and only if the homology class in $H_0(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to u is represented by an even number of points of $X(\mathbb{R})$. In view of these facts, we concentrate our attention on the intermediate cases, $1 \leq k \leq n - 1$.

Given a continuous map $f : M \rightarrow N$ between topological spaces, we denote by $H^k(f) : H^k(N, \mathbb{Z}/2) \rightarrow H^k(M, \mathbb{Z}/2)$ the homomorphism induced by f . Recall that a cohomology class u in $H^k(M, \mathbb{Z}/2)$ with $k \geq 1$ is said to be *spherical* if $u = H^k(f)(c)$, where $f : M \rightarrow S^k$ is a continuous map into the unit k -sphere S^k , and c is the generator of $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$. We denote, as usual, by \cup and $\langle -, - \rangle$ the cup product of cohomology classes and the Kronecker index (pairing) of cohomology and homology classes, cf. [11]. If M is a C^∞ , closed manifold of dimension n , we denote by $w_k(M)$ the k th Stiefel-Whitney class of M and by μ_M the fundamental homology class of M in $H_n(M, \mathbb{Z}/2)$.

THEOREM 1.1. — *Let X be a nonsingular, n -dimensional, quasiprojective variety over \mathbb{R} with $X(\mathbb{R})$ nonempty and compact. Let z be a cycle in $Z^k(X)$ that is algebraically equivalent to 0. Then the cohomology class $\text{cl}_{\mathbb{R}}(z)$ in $H^k(X(\mathbb{R}), \mathbb{Z}/2)$ satisfies $\text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = 0$ in $H^{2k}(X(\mathbb{R}), \mathbb{Z}/2)$ and*

$$\langle \text{cl}_{\mathbb{R}}(z) \cup w_{i_1}(X(\mathbb{R})) \cup \dots \cup w_{i_r}(X(\mathbb{R})), \mu_{X(\mathbb{R})} \rangle = 0$$

for all nonnegative integers i_1, \dots, i_r with $i_1 + \dots + i_r = n - k$. Furthermore, if $k = 1$ or if $k = n - 1 \geq 1$ with $X(\mathbb{R})$ connected, then the cohomology class $\text{cl}_{\mathbb{R}}(z)$ is spherical.

Let us note that, in general, the cohomology class $\text{cl}_{\mathbb{R}}(z)$ of Theorem 1.1 need not be spherical. Indeed, suppose $X = X' \times X''$ (product over $\text{Spec} \mathbb{R}$), where X' and X'' are nonsingular, projective varieties over \mathbb{R} such that $X'(\mathbb{R})$ is nonempty and $X''(\mathbb{R})$ is disconnected. Let z' be any algebraic cycle on X' . Choose two points p_0 and p_1 in $X''(\mathbb{R})$ that belong to distinct connected components. Since the 0-cycle $z'' = p_0 - p_1$ on X'' is algebraically equivalent to 0, the cycle $z' \times z''$ on X is algebraically equivalent to 0 as well. Furthermore, the cohomology class $\text{cl}_{\mathbb{R}}(z' \times z'') = \text{cl}_{\mathbb{R}}(z') \times \text{cl}_{\mathbb{R}}(z'')$ is

spherical if and only if the cohomology class $\text{cl}_{\mathbb{R}}(z')$ is spherical (for p_0 and p_1 belong to distinct connected components of $X''(\mathbb{R})$). Taking $X' = \mathbb{P}_{\mathbb{R}}^m$, we have $\text{cl}_{\mathbb{R}}(Z^k(X')) = H^k(X'(\mathbb{R}), \mathbb{Z}/2)$, and the unique nontrivial cohomology class in $H^k(X'(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2$ is not spherical, provided that $1 \leq k \leq m - 1$ and m is even. In particular, “connected” cannot be omitted in the last part of Theorem 1.1.

If $\text{cl}_{\mathbb{R}}(z)$ is spherical, then $\text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = 0$ is automatically satisfied, and Theorem 1.1 is in some sense the best possible result. More precisely, we have the following.

THEOREM 1.2. — *Let M be a C^∞ , closed, n -dimensional manifold and let u be a spherical cohomology class in $H^k(M, \mathbb{Z}/2)$ with $1 \leq k \leq n - 1$. Then the following conditions are equivalent :*

(a) *There exist a nonsingular, projective algebraic variety X over \mathbb{R} and a C^∞ diffeomorphism $\varphi : X(\mathbb{R}) \rightarrow M$ such that $H^k(\varphi)(u) = \text{cl}_{\mathbb{R}}(z)$ for some cycle z in $Z^k(X)$ algebraically equivalent to 0;*

(b) *$\langle u \cup w_{i_1}(M) \cup \dots \cup w_{i_r}(M), \mu_M \rangle = 0$ for all nonnegative integers i_1, \dots, i_r with $i_1 + \dots + i_r = n - k$.*

Let us mention that Theorem 1.2 is an improvement upon inefficient [10], Theorem 2.4.

2. Proofs.

Let X be a nonsingular, n -dimensional, quasiprojective algebraic variety over \mathbb{R} with $X(\mathbb{R})$ nonempty and compact. Recall that if an algebraic cycle z in $Z^k(X)$ is rationally equivalent to 0, then $\text{cl}_{\mathbb{R}}(z) = 0$ (cf. [5], 5.13) and hence $\text{cl}_{\mathbb{R}}$ induces a homomorphism, also denoted by $\text{cl}_{\mathbb{R}}$, from the Chow group $A^k(X)$ of X into $H^k(X(\mathbb{R}), \mathbb{Z}/2)$. It is known that $\text{cl}_{\mathbb{R}} : A^*(X) \rightarrow H^*(X(\mathbb{R}), \mathbb{Z}/2)$ is a homomorphism of graded rings [5], p. 495. Thus

$$H_{\text{alg}}^*(X(\mathbb{R}), \mathbb{Z}/2) = \text{cl}_{\mathbb{R}}(Z^*(X)) = \text{cl}_{\mathbb{R}}(A^*(X))$$

is a graded subring of $H^*(X(\mathbb{R}), \mathbb{Z}/2)$. We shall need the following result [10], Theorem 2.1 :

$$(1) \quad \langle \text{cl}_{\mathbb{R}}(z) \cup v, \mu_{X(\mathbb{R})} \rangle = 0$$

for all cycles z in $Z^k(X)$ algebraically equivalent to 0 and all v in $H_{\text{alg}}^{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$.

Assume now that X is projective. Then the set $X(\mathbb{C})$ of \mathbb{C} -rational points of X is a compact complex manifold of complex dimension n . There exists a unique group homomorphism

$$\text{cl}_{\mathbb{C}} : Z^k(X) \rightarrow H^{2k}(X(\mathbb{C}), \mathbb{Z})$$

such that for every closed, $(n - k)$ -dimensional subvariety V of X , the cohomology class $\text{cl}_{\mathbb{C}}(V)$ is Poincaré dual to the homology class in $H_{2n-2k}(X(\mathbb{C}), \mathbb{Z})$ determined by $V(\mathbb{C})$ (cf. [5] for the definition of this homology class). In other words, if $\pi : X_{\mathbb{C}} = X \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C} \rightarrow X$ is the canonical projection, then $\text{cl}_{\mathbb{C}}(z)$ is the cohomology class corresponding to the pullback algebraic cycle $\pi^*(z)$ on $X_{\mathbb{C}}$, cf. [5], 4.2 or [7], Chapter 19. In particular,

$$(2) \quad \text{cl}_{\mathbb{C}}(z) = 0$$

for all cycles z in $Z^k(X)$ algebraically equivalent to 0, cf. [5], 4.14 or [7], Proposition 19.1.1. Furthermore, it follows from the proof of [2], Theorem A that

$$(3) \quad \text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = \text{the reduction modulo 2 of } r(\text{cl}_{\mathbb{C}}(z))$$

for all z in $Z^k(X)$, where $r : H^{2k}(X(\mathbb{C}), \mathbb{Z}) \rightarrow H^{2k}(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

Proof of Theorem 1.1. — By Hironaka’s resolution of singularities theorem [8], 3, we may assume that X is projective.

We obtain $\text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = 0$ directly from (2) and (3).

It follows from [5], p. 498 that $w_i(X(\mathbb{R}))$ is in $H_{\text{alg}}^i(X(\mathbb{R}), \mathbb{Z}/2)$, and hence if i_1, \dots, i_r are nonnegative integers with $i_1 + \dots + i_r = n - k$, then the cohomology class

$$v = w_{i_1}(X(\mathbb{R})) \cup \dots \cup w_{i_r}(X(\mathbb{R}))$$

belongs to $H_{\text{alg}}^{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$. In view of (1), we have $\langle \text{cl}_{\mathbb{R}}(z) \cup v, \mu_{X(\mathbb{R})} \rangle = 0$, which completes the proof of the first part of the theorem.

Given an invertible sheaf \mathcal{L} on X , we denote by $\mathcal{L}_{\mathbb{R}}$ (resp. $\mathcal{L}_{\mathbb{C}}$) the topological real (resp. complex) line bundle on $X(\mathbb{R})$ (resp. $X(\mathbb{C})$) determined by \mathcal{L} in the usual way. If \mathcal{L} corresponds to a Weil divisor D on X , then

$$w_1(\mathcal{L}_{\mathbb{R}}) = \text{cl}_{\mathbb{R}}(D) \quad \text{and} \quad c_1(\mathcal{L}_{\mathbb{C}}) = \text{cl}_{\mathbb{C}}(D),$$

where $w_1(-)$ and $c_1(-)$ stand for the first Stiefel-Whitney class and the first Chern class, respectively, cf. [5], p. 498, p. 489. Note that the restriction $\mathcal{L}_{\mathbb{C}}|_{X(\mathbb{R})}$ of $\mathcal{L}_{\mathbb{C}}$ to $X(\mathbb{R})$ is the complexification of $\mathcal{L}_{\mathbb{R}}$, and hence

$$c_1(\mathcal{L}_{\mathbb{C}}|_{X(\mathbb{R})}) = \beta(w_1(\mathcal{L}_{\mathbb{R}})),$$

where $\beta : H^1(X(\mathbb{R}), \mathbb{Z}/2) \rightarrow H^2(X(\mathbb{R}), \mathbb{Z})$ is the Bockstein homomorphism that appears in the long exact sequence

$$\dots \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^1(X(\mathbb{R}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^2(X(\mathbb{R}), \mathbb{Z}) \rightarrow \dots,$$

cf. [11], Problems 15-C and D. The last equality can be written in an equivalent form

$$(4) \quad r(\text{cl}_{\mathbb{C}}(D)) = \beta(\text{cl}_{\mathbb{R}}(D)),$$

where $r : H^2(X(\mathbb{C}), \mathbb{Z}) \rightarrow H^2(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

Suppose now that $k = 1$, that is, z is a Weil divisor on X . By (2) and (4), we have $\beta(\text{cl}_{\mathbb{R}}(z)) = 0$, which means that $\text{cl}_{\mathbb{R}}(z)$ is the reduction modulo 2 of a cohomology class in $H^1(X(\mathbb{R}), \mathbb{Z})$. This last fact implies that the cohomology class $\text{cl}_{\mathbb{R}}(z)$ is spherical, cf. [9], p. 49.

Let us now assume that $k = n - 1 \geq 1$ and $X(\mathbb{R})$ is connected. We already know that $\langle \text{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})), \mu_{X(\mathbb{R})} \rangle = 0$, which in view of the connectedness of $X(\mathbb{R})$ is equivalent to $\text{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})) = 0$. The last condition implies that the homology class in $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to $\text{cl}_{\mathbb{R}}(z)$ can be represented by a C^∞ , closed curve in $X(\mathbb{R})$, with trivial normal vector bundle, cf. for example [4], p. 599. This in turn implies that $\text{cl}_{\mathbb{R}}(z)$ is spherical, cf. [12], Théorème II.1. Thus the proof is complete. \square

Proof of Theorem 1.2. — By Theorem 1.1, (a) implies (b), and we show below that (b) implies (a).

Choose a nonsingular, irreducible algebraic subset W of \mathbb{R}^{k+1} , which has precisely two connected components W_0 and W_1 , each diffeomorphic to the unit k -sphere S^k (for example, $W = \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^4 - 4x_1^2 + 1 + x_2^2 + \dots + x_{k+1}^2 = 0\}$). Let c be the unique generator of the group $H^k(W_0, \mathbb{Z}/2) \cong \mathbb{Z}/2$, viewed as a subgroup of $H^k(W, \mathbb{Z}/2)$. Since the cohomology class u is spherical, there exists a C^∞ map $h : M \rightarrow W$ such that $h(M) \subseteq W_0$ and $u = H^k(h)(c)$. Choose a regular value y_0 of h in W_0 . Then u is Poincaré dual to the homology class in $H_{n-k}(M, \mathbb{Z}/2)$ represented by the C^∞ submanifold $h^{-1}(y_0)$ of M , cf. [5], 2.15. Clearly, there exists a unique C^∞ map $f : M \rightarrow W$ such that for every connected component S of M and every point x in S , we have $f(x) = h(x)$ if $S \cap h^{-1}(y_0) \neq \emptyset$ and $f(x) = y_0$ if $S \cap h^{-1}(y_0) = \emptyset$. The map f satisfies

$$(5) \quad f(M) \subseteq W_0 \quad \text{and} \quad u = H^k(f)(c).$$

Furthermore, each connected component of $f^{-1}(y_0)$ is a C^∞ submanifold of M of dimension either $n - k$ or n . Also, each connected component of M contains a connected component of $f^{-1}(y_0)$. Since $n - k \geq 1$, we can find a C^∞ closed curve C in M such that

$$(6) \quad f(C) = \{y_0\},$$

the normal vector bundle of C in M is trivial, and each connected component of M contains a connected component of C . Choose an integer d with $2n + 1 \leq d$ and let D be a compact, nonsingular, irreducible, 1-dimensional algebraic subset of \mathbb{R}^d that has the same number of connected components as C . Replacing M by its image under a suitable C^∞ embedding into \mathbb{R}^d , we may assume that

$$(7) \quad D = C \subseteq M \subseteq \mathbb{R}^d.$$

By Tognoli's theorem [13] or [1], Corollary 2.8.6, there exists a nonsingular real algebraic subset A of \mathbb{R}^p , for some p , diffeomorphic to M . Consider the disjoint union $N = M \amalg A$ and the C^∞ map $F : N \rightarrow W$ defined by $F(x) = f(x)$ for x in M and $F(x) = y_0$ for x in A . We assert that if w is a cohomology class in $H^\ell(W, \mathbb{Z}/2)$ and if j_1, \dots, j_s are nonnegative integers with $j_1 + \dots + j_s = n - \ell$, then

$$\langle H^\ell(F)(w) \cup w_{j_1}(N) \cup \dots \cup w_{j_s}(N), \mu_N \rangle = 0.$$

Indeed, first note that $w = 0$, unless $\ell = 0$ or $\ell = k$. If $\ell = 0$, then either $H^0(F)(w) = 0$ or $H^0(F)(w) = 1$. In the latter case the assertion holds since M and A are diffeomorphic. If $\ell = k$, then either $H^k(F)(w) = 0$ or $H^k(F)(w) = u$ (we view $H^k(M, \mathbb{Z}/2)$ as a subgroup of $H^k(N, \mathbb{Z}/2)$). In the latter case the assertion follows from condition (b). Thus the assertion is proved. It implies that there exist a C^∞ compact manifold \tilde{N} with boundary $\partial\tilde{N} = N$ and a C^∞ map $\tilde{F} : \tilde{N} \rightarrow W$ satisfying $\tilde{F}|_N = F$, cf. [6], 17.3. In other words, the map $f : M \rightarrow W$ and the constant map $A \rightarrow W$, which sends A to y_0 , represent the same class in the unoriented bordism group of W . By construction, the normal bundle of D in M is trivial, so the restriction $\nu|_D$ to D of the normal bundle ν of M in \mathbb{R}^d admits an algebraic structure. Therefore, by [1], Theorem 2.8.4 and in view of (6) and (7), one can find a nonnegative integer e , a nonsingular algebraic subset V of $\mathbb{R}^d \times \mathbb{R}^e$, a C^∞ diffeomorphism $\varphi : V \rightarrow M$, and a regular map $g : V \rightarrow W$ (the latter designates the restriction to V of a rational map from $\mathbb{R}^d \times \mathbb{R}^e$ into \mathbb{R}^{k+1} which has no poles on V and maps V into W) such that g is homotopic to $f \circ \varphi$ and $D \times \{0\} \subseteq V$. Since D is irreducible and each connected component of V contains a connected component of

$D \times \{0\}$, it follows that V is irreducible as well (this is the only place where D is needed).

Irreducibility of V and W allows us to choose nonsingular, quasiprojective varieties T and Y over \mathbb{R} with $T(\mathbb{R}) = V$ and $Y(\mathbb{R}) = W$. By Hironaka's resolution of singularities theorem [8], 3, we may assume that T and Y are projective (and still nonsingular). Let $\tilde{g} : U \rightarrow Y$ be an algebraic morphism over \mathbb{R} , defined on a Zariski open neighborhood of $T(\mathbb{R}) = V$ in T , such that $\tilde{g}|_{T(\mathbb{R})} = g$. By applying Hironaka's theorem on removing points of indeterminacy [8], 3, we can find a nonsingular, projective algebraic variety X over \mathbb{R} and an algebraic morphism $G : X \rightarrow Y$ over \mathbb{R} satisfying $X(\mathbb{R}) = T(\mathbb{R})$ and $G|_{X(\mathbb{R})} = g$.

Let y_1 be a point in W_1 and let β be the class in $A^k(Y)$ of the 0-cycle $y_0 - y_1$ on Y . By (5), $u = H^k(f)(\text{cl}_{\mathbb{R}}(\beta))$ (although, of course, $\text{cl}_{\mathbb{R}}(\beta) \neq c$). Since $G|_{X(\mathbb{R})} = g$ is homotopic to $f \circ \varphi$, we obtain

$$\begin{aligned} H^k(\varphi)(u) &= H^k(\varphi)(H^k(f)(\text{cl}_{\mathbb{R}}(\beta))) = H^k(f \circ \varphi)(\text{cl}_{\mathbb{R}}(\beta)) \\ &= H^k(G|_{X(\mathbb{R})})(\text{cl}_{\mathbb{R}}(\beta)) = \text{cl}_{\mathbb{R}}(G^*(\beta)), \end{aligned}$$

where the last equality is a consequence of the functorial property of $\text{cl}_{\mathbb{R}} : A^* \rightarrow H^*$, cf. [5], 5.12. Let z be a cycle in $Z^k(X)$ that represents in the Chow group $A^k(X)$ the pullback class $G^*(\beta)$. Then $\text{cl}_{\mathbb{R}}(z) = \text{cl}_{\mathbb{R}}(G^*(\beta)) = H^k(\varphi)(u)$. The proof is now complete since the cycle $y_0 - y_1$ is algebraically equivalent to 0 on Y and hence the cycle z is algebraically equivalent to 0 on X . \square

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