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GALOIS CO-DESCENT FOR ÉTALE WILD KERNELS AND CAPITULATION

by M. KOLSTER* and A. MOVAHHEDI

Introduction.

For a number field $F$, the classical wild kernel - denoted by $WK_2(F)$ - is the kernel of all local power norm residue symbols on $K_2(F)$, in other words it fits into Moore’s exact sequence

$$0 \to WK_2(F) \to K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where $v$ runs through all finite and real infinite primes of $F$, and $\mu(F_v)$ and $\mu(F)$ denote the groups of roots of unity of the local field $F_v$ and of $F$, respectively. For a fixed prime number $p$, the $p$-primary part $WK_2(F)\{p\}$ of $WK_2(F)$ has another description in terms of étale cohomology: For any finite set $S$ of primes in $F$ containing the $p$-adic primes and the real infinite primes, we have

$$WK_2(F)\{p\} = \ker(H^2_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_p(2)) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(2))).$$

This property immediately leads to the definition of the higher étale wild kernels for $i \geq 2$:

$$WK_{2i-2}^{\text{ét}}(F) := \ker(H^2_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_p(i)) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))).$$

The étale wild kernels play a similar role in étale cohomology, étale K-theory and Iwasawa-theory as the $p$-primary parts $A'_F$ of the $S$-class groups

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of \(F\). For these, Galois co-descent is classically described by genus theory. The main result of this paper proves an analogous genus formula for the étale wild kernels of a cyclic extension \(L/F\) of degree \(p\), \(p\) odd. Let \(G = \text{Gal}(L/F)\). We first show that the transfer map \(WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)\) is onto except in a very special situation, and we determine its kernel as the cokernel of a certain cup-product which is obtained as follows: Let \(E = F(\mu_p)\), where \(\mu_p\) consists of the \(p\)-th roots of unity and let \(\Delta = \text{Gal}(E/F)\). We associate with the extension \(LE/E\) a certain set \(T_{LE/E}\) of primes of \(E\), consisting of all tamely ramified primes and some undecomposed \(p\)-adic primes. Let \(\text{Br}^T(E)\) denote the subgroup of the Brauer-group which is supported only at primes in \(T_{LE/E}\), and let \(p\text{Br}^T(E)\) denote the subgroup of all the elements in \(\text{Br}^T(E)\) of exponent \(p\). The target of the cup-product is the \((1-i)-\) eigenspace \(p\text{Br}^T(E)^{[1-i]}\), under the action of the Teichmüller character \(w\). Now, let \(S\) be the set of primes in \(E\) consisting of the \(p\)-adic primes, the real infinite primes as well as all primes ramified in \(LE\) and denote by \(\omega_E^S\) the ring of \(S\)-integers in \(E\). The étale cohomology group \(H^1(\omega_E^S, \mathbb{Z}/p\mathbb{Z})\) injects into the \((i-1)-\)fold Tate twist of the module \(E^*\) and hence is isomorphic to \(D_{E}^{(1)}\), where \(D_{E}^{(1)} \subset E^*\) can be viewed as the analog of the Tate kernel \((i = 2)\). The cup-product is now given by

\[
(D_{E}^{(1)}/E^*)^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to (p\text{Br}^T(E))^{[1-i]}.
\]

We illustrate the method by finding all Galois \(p\)-extensions of \(\mathbb{Q}\) for which the \(p\)-part of the classical wild kernel is trivial.

We also discuss the Galois co-descent situation for \(p = 2\) in the classical case \(i = 2\).

Let \(E_\infty\) denote the cyclotomic \(\mathbb{Z}_p\)-extension of \(E\) with finite layers \(E_n\). If we assume the Gross Conjecture for \(E_n\) with \(n\) large, for instance if \(E\) is abelian over \(\mathbb{Q}\), then the groups \(D_{E_n}^{(1)}/E_n^*\) can be described in terms of local conditions at \(p\)-adic primes, and are independent of \(i\).

Let \(F_\infty\) denote the cyclotomic \(\mathbb{Z}_p\)-extension of \(F\) with finite layers \(F_n\) and let \(A'_n\) denote the \(p\)-part of the \(p\)-class group of \(F_n\). The classical capitulation kernel is defined as

\[
\text{Cap}_0(F_\infty) = \ker(A'_n \to A'_\infty) \quad \text{for } n \text{ large}.
\]

The study of capitulation kernels under Galois extensions is an essential ingredient in the more general problem of comparing Iwasawa-invariants (cp. e.g. [28]). In Section 3 we introduce similar capitulation kernels.
Cap_{i-1}(F_\infty) for all i \geq 2 using étale K-theory, and show that they have properties similar to Cap_0(F_\infty).

Assume now that F is totally real, and let E^+ denote the maximal real subfield of E = F(\mu_p). A conjecture of Greenberg predicts that \lim A'_n(E^+) is finite. Under this assumption we show that for all odd i \geq 3:

\[ \text{Cap}_{i-1}(F_\infty) \cong A'_n(E^+)^{[1-i]} \cong WK_{2i-2}^{\text{ ét}}(F_n) \quad \text{for } n \text{ large.} \]

Therefore the co-descent results from Section 2 imply similar results for Cap_{i-1}(F_\infty) and for the eigenspaces A'_n(E^+)^{[1-i]}, when n is large.

In Section 4, we briefly discuss how our approach can be applied to the simpler problem of Galois co-descent for étale tame kernels. This has already been studied by Assim ([1], [2]) under Leopoldt's conjecture.

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1. Preliminaries.

In this section we briefly recall some of the basic properties of étale K-theory and étale cohomology which are subsequently needed. A more detailed account can be found in [3], [20]. Let F be a number field and p a fixed prime number. Let S be a finite set of primes in F, containing the set S_p of primes above p and the set S_\infty of infinite primes. As usual, G_S(F) denotes the Galois group over F of the maximal algebraic extension of F, which is unramified outside S. We note that the condition on infinite primes only intervenes if p = 2 and F is not totally imaginary. Let o_S^\infty denote the ring of S-integers of F. As is well-known, the étale cohomology groups H^{k}_{\text{ ét}}(\text{spec}(o_F^S), \mathbb{Z}/p^n\mathbb{Z}(i)) of \text{spec}(o_F^S) coincide with the Galois-cohomology groups H^{k}(G_S(F), \mathbb{Z}/p^n\mathbb{Z}(i)), and will be denoted by H^{k}_{\text{ ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i)). Here, as usual, \mathbb{Z}/p^n\mathbb{Z}(i) denotes the i-fold Tate twist of \mathbb{Z}/p^n\mathbb{Z}. Furthermore, let

\[ H^{k}_{\text{ ét}}(o_F^S, \mathbb{Z}_p(i)) = \lim H^{k}_{\text{ ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i)) \]
and
\[ H^k_{\text{ét}}(\mathcal{O}_F, \mathbb{Q}_p/\mathbb{Z}_p(i)) = \lim_{\rightarrow} H^k_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}/p^n\mathbb{Z}(i)). \]
Assume now that \( p \) is either odd or that \( p = 2 \) and \( F \) contains \( \sqrt{-1} \). Then for \( i \geq 2 \) and \( k = 1, 2 \) the étale cohomology groups \( H^k_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}_p(i)) \) are isomorphic to the higher étale \( K \)-theory groups \( K^\text{ét}_{2i-k}(\mathcal{O}_F) \), introduced by Dwyer-Friedlander ([8]). Moreover, the relation to Quillen’s \( K \)-theory groups \( K^i_k(F) \) is provided by a Chern character, which yields split surjective maps with finite kernels
\[ K^i_{2i-k}(\mathcal{O}_F) \otimes \mathbb{Z}_p \to K^\text{ét}_{2i-k}(\mathcal{O}_F) \]
(cp. [8], [15]), which conjecturally are isomorphisms (recall that for \( p = 2, \ F \) contains \( \sqrt{-1} \)). Borel’s results (cp. [4]) then imply that the groups \( K^\text{ét}_{2i-2}(\mathcal{O}_F) \) are finite and that the groups \( K^\text{ét}_{2i-1}(\mathcal{O}_F) \) are finitely generated of rank \( r_1 + r_2 \) if \( i \) is odd, and of rank \( r_2 \) if \( i \) is even, where as usual \( r_1 \) and \( r_2 \) denote the number of real and pairs of conjugate complex embeddings of \( F \), respectively. We note that the odd étale \( K \)-theory groups are independent of the choice of the set \( S \) of primes: If \( H^*(F) \) denotes the absolute Galois cohomology groups of \( F \) then, in fact, the localization sequence in étale cohomology (cp. [36, Proposition 1]) implies that
\[ H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}_p(i)) \cong H^1(F, \mathbb{Z}_p(i)) \quad \forall i \geq 2. \]
We therefore simply denote the odd étale \( K \)-theory groups by \( K^\text{ét}_{2i-1}(F) \).

The torsion subgroup of \( K^\text{ét}_{2i-1}(F) \) is isomorphic to \( H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(i)) \).

In the special case \( i = 2 \) more is known: There exist isomorphisms
\[ K_2(\mathcal{O}_F) \otimes \mathbb{Z}_p \to H^2_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}_p(2)) \]
and
\[ K^\text{nd}_3(F) \otimes \mathbb{Z}_p \to H^1(F, \mathbb{Z}_p(2)) \]
without any restrictions on the prime \( p \) and the number field \( F \) (cp. [36], [22]). Here \( K^\text{nd}_3(F) \) denotes the indecomposable \( K_3 \)-group of \( F \), i.e. \( K_3(F) \) divided by the image of the Milnor group \( K^M_3(F) \), which is 2-torsion.

More recently, Kahn ([18]) and Rognes-Weibel ([32]) have determined the kernel and cokernel of the 2-adic Chern character
\[ K^i_{2i-k}(\mathcal{O}_F) \otimes \mathbb{Z}_2 \to H^k_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}_2(i)), \]
which in general are non-trivial.
The following result is due to B. Kahn (cp. [16, Theorem 2.1, Proposition 6.1]):

**Theorem 1.1.** — Let $L/F$ be a Galois extension of number fields with Galois group $G$ and let $S$ be a finite set of primes in $F$, containing the primes which are ramified in $L$. There is an exact sequence

$$0 \to H^1(G, K_3^{nc}(L)) \to K_2(o_F^S) \to K_2(o_L^S)^G \to H^2(G, K_3^{nc}(L)) \to 0.$$ 

The following étale analog is well-known (cp. [1], [5]), however we include a proof, since the sources are not easily accessible:

**Theorem 1.2.** — Let $p$ be an odd prime and let $L/F$ be a Galois $p$-extension of number fields with Galois group $G$. Let $S$ be a finite set of primes, containing the primes above $p$ and the primes which ramify in $L$. Then for $i \geq 2$ there is an exact sequence

$$0 \to H^1(G, K_2^{et}(L)) \to K_2^{et}(o_F^S) \to K_2^{et}(o_L^S)^G \to H^2(G, K_2^{et}(L)) \to 0.$$

**Proof.** — Consider the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G, H^q_{et}(o_L^S, \mathbb{Z}_p(i))) \Rightarrow H^{p+q}_{et}(o_F^S, \mathbb{Z}_p(i)).$$

Since $H^0_{et}(o_L^S, \mathbb{Z}_p(i)) = 0$ ([36, Lemme 7]), all terms $E_2^{pq}$ vanish. On the other hand, $cd_p(G_S(F)) = 2$ and hence $H^q_{et}(o_F^S, \mathbb{Z}_p(i)) = H^q_{et}(o_L^S, \mathbb{Z}_p(i)) = 0$ for all $q \geq 3$. The spectral sequence therefore yields

$$E_1 \cong E_{\infty}^{01} \cong E_2^{01},$$

i.e. an isomorphism

$$K_{2i-1}^{et}(F) \cong K_{2i-1}^{et}(L)^G,$$

as well as the exact sequence

$$0 \to E_2^{11} \to E^2 \to E_2^{02} \to E_2^{21} \to 0,$$

which is precisely the claim. 

As a by-product, we obtained the fact that the odd étale $K$-groups satisfy Galois descent. Note that this, in the form

$$H^1(F, \mathbb{Z}_p(i)) \cong H^1(L, \mathbb{Z}_p(i))^G,$$

remains true for $p = 2$. 

TOME 50 (2000), FASCICULE 1
On the other hand we have Galois co-descent for the even étale $K$-theory groups $K^{\text{ét}}_{2i-2}(o^S_F)$:

**Proposition 1.3.** — Let $p$ be odd and $L/F$ a Galois $p$-extension of number fields with Galois-group $G$. If $S$ contains the primes above $p$ and the ramified primes of $L/F$, then

$$K^{\text{ét}}_{2i-2}(o^S_L)_G \cong K^{\text{ét}}_{2i-2}(o^S_F).$$

**Proof.** — This follows as above using the Tate spectral sequence (cp. [35], [17], [26]).

Now $K^{\text{ét}}_{2i-2}(o^S_L)$ is finite, and hence this proposition together with Theorem 1.2 yields

**Corollary 1.4.** — For any cyclic $p$-extension $L/F$ ($p$ odd) of number fields with Galois group $G$, the quotient

$$\frac{|H^2(G, K^{\text{ét}}_{2i-1}(L))|}{|H^1(G, K^{\text{ét}}_{2i-1}(L))|}$$

is trivial.

**Remark 1.5.** — The previous results depended only upon two facts:

$$cd_p(G_S(F)) \leq 2 \quad \text{and} \quad H^0(G_S(F), \mathbb{Z}_p(i)) = 0.$$ 

Therefore analogous results also hold for example for finite extensions of local fields, thus, for a Galois $p$-extension $E/F$ of local fields with Galois group $G$, we have an exact sequence

$$0 \to H^1(G, H^1(E, \mathbb{Z}_p(i))) \to H^2(F, \mathbb{Z}_p(i)) \to H^2(E, \mathbb{Z}_p(i))^G \to H^2(G, H^1(E, \mathbb{Z}_p(i))) \to 0,$$

and an isomorphism

$$H^2(E, \mathbb{Z}_p(i))_G \cong H^2(F, \mathbb{Z}_p(i)).$$

Again, in the case $i = 2$, more information on co-descent is available, i.e. no restrictions on $F$ are necessary to also include results concerning the 2-primary part.
The following result is easily obtained from [16, Théorème 5.1]:

**Proposition 1.6.** — Let $L/F$ be a finite Galois extension of number fields with Galois group $G$ and let $S$ be a finite set of primes in $F$, containing the primes which ramify in $L/F$. Then, there is a short exact sequence

$$0 \to K_2(o_L^S)_G \to K_2(o_F^S) \to \bigoplus_{v \in S_\infty^r} \mu_2 \to 0,$$

where $S_\infty^r$ consists of the real infinite primes in $F$ which ramify in $L$.

Finally, we recall the definition of the higher étale wild kernels (cp. [3], [20], [29]):

$$WK^{et}_{2i-2}(F) = \ker (H^2_{et}(o_F^S, \mathbb{Z}_p(i)) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))).$$

The definition is independent of the choice of the set $S$ containing $S_p$, and part of the Poitou-Tate duality sequence yields the exact sequence

$$0 \to WK^{et}_{2i-2}(F) \to K^{et}_{2i-2}(o_F^S) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \to H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to 0,$$

where $*$ indicates the Pontrjagin dual. Moreover by local duality

$$H^2(F_v, \mathbb{Z}_p(i)) \cong H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*.$$

The groups $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ and $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ are finite cyclic for $i \neq 1$.

The étale wild kernels are the analogs of the $p$-part of the classical wild kernel $WK_2(F)$ - defined for any number field $F$ - which occurs in Moore’s exact sequence of power norm symbols (cp. [23]):

$$0 \to WK_2(F) \to K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where $v$ runs through all finite primes and all real infinite primes of $F$, and $\mu(F_v)$ and $\mu(F)$ denote the group of roots of unity of $F_v$ and of $F$ respectively. If $S$ is a finite set of primes in $F$ containing $S_p$ and $S_\infty$, then we obtain an exact sequence of finite groups

$$0 \to WK_2(F)\{p\} \to K_2(o_F^S)\{p\} \to \bigoplus_{v \in S} \mu(F_v)\{p\} \to \mu(F)\{p\} \to 0.$$

Here, for an abelian group $A$, we use the notation $A\{p\}$ for the $p$-primary part of $A$. 
2. Galois co-descent for the étale wild kernel.

Let \( p \) be an odd prime and let \( L/F \) be a cyclic extension of number fields of degree \( p \) with Galois group \( G \). In this section, for any local or global field \( K \), we denote by \( K_\infty \) the cyclotomic \( \mathbb{Z}_p \)-extension of \( K \) with finite layers \( K_n \). We also assume that \( i \geq 2 \). We obtain necessary and sufficient conditions for the étale wild kernel \( WK_{2i-2}(L) \) to satisfy Galois co-descent. This approach also yields a genus"-formula comparing the sizes of \( WK_{2i-2}(L)^G \) and \( WK_{2i-2}(F) \). Let \( S \) be the finite set of primes in \( F \), containing the set \( S_p \) of all primes above \( p \), as well as all primes which ramify in \( L \). We denote by \( S_L \) the set of primes in \( L \) above \( S \). Moreover, let \( \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \) be the kernel of the surjection

\[
\bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \to H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1 - i)^*).
\]

Then by definition of the étale wild kernel from section 1, we have the following short exact sequence:

\[
0 \to WK_{2i-2}^\text{ét}(F) \to K_{2i-2}^\text{ét}(\sigma_F^S) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \to 0.
\]

By Proposition 1.3 the group \( K_{2i-2}^\text{ét}(\sigma_F^S) \) satisfies Galois co-descent. The following commutative diagram:

\[
\begin{array}{ccc}
WK_{2i-2}^\text{ét}(L)^G & \to & K_{2i-2}^\text{ét}(\sigma_L^S)_G \\
& \downarrow & \downarrow \\
0 & \to & WK_{2i-2}^\text{ét}(F) \to K_{2i-2}^\text{ét}(\sigma_F^S) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \to 0
\end{array}
\]

then shows that

\[
\text{coker}(WK_{2i-2}^\text{ét}(L)_G \to WK_{2i-2}^\text{ét}(F))
\]

\[
\cong \ker \left( \bigoplus_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i))_G \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \right)
\]

and

\[
\ker(WK_{2i-2}^\text{ét}(L)_G \to WK_{2i-2}^\text{ét}(F))
\]

\[
\cong \text{coker} \left( K_{2i-2}^\text{ét}(\sigma_L^S)_G \to \left( \bigoplus_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i)) \right)^G \right).
\]

Before we compute the first group we need a preliminary result: Let \( M/N \) be a cyclic extension of degree \( p \), \( p \) odd, of global or local fields of

ANNALES DE L'INSTITUT FOURIER
characteristic \neq p$, and let $G$ denote the Galois group of $M/N$. Furthermore, let $N_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $N$.

There are two maps relating the cohomology groups $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$, where we assume $k \in \mathbb{Z}$, $k \neq 0$: The natural map $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and the norm map $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$. The first one induces an isomorphism

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \xrightarrow{\sim} H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))^G,$$

which implies immediately that either both groups are trivial or both groups are non-trivial. Assume now that $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ is non-trivial. Then the order of $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ is the maximal power $p^m$, such that the Galois group $\text{Gal}(N(\mu_{p^m})/N)$ has exponent $k$. If $M \not\subset N_\infty$, then $[M(\mu_{p^m}) : M] = [N(\mu_{p^m}) : N]$, and therefore $G$ acts trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$. Therefore, in this case, the natural map is an isomorphism, and hence the norm map has both kernel and cokernel of order $p$. On the other hand, if $M \subset N_\infty$ and say $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \cong (\mathbb{Z}/p^m\mathbb{Z})(k)$, then $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k)) \cong (\mathbb{Z}/p^{m+1}\mathbb{Z})(k)$, and $p$ being odd - the norm

$$(\mathbb{Z}/p^{m+1}\mathbb{Z})(k) \to (\mathbb{Z}/p^m\mathbb{Z})(k)$$

is surjective, and therefore induces an isomorphism

$$H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))^G \xrightarrow{\sim} H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)).$$

We summarize:

**Lemma 2.1.** — Let $k \in \mathbb{Z}$, $k \neq 0$ and $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \neq 0$.

i) If $M \not\subset N_\infty$, then $G$ acts trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$, and hence the natural map

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$$

is an isomorphism, whereas the norm map has kernel and cokernel of order $p$.

ii) If $M \subset N_\infty$, then $G$ acts non-trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and the norm induces an isomorphism

$$H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))^G \xrightarrow{\sim} H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)).$$
The non-vanishing of \( H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \) can be characterized as follows: Let \( d = [N(\mu_p) : N] \). Then
\[
H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \neq 0 \iff k \equiv 0 \mod d.
\]

Let us now study the question of co-descent for \( \oplus_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i)) \). Using local duality the problem is equivalent to computing the cokernel of the map
\[
\oplus_{v \in S} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \rightarrow \left( \oplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)^G.
\]
As we noted above, we have isomorphisms
\[
H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \cong H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^G
\]
and
\[
\oplus_{v \in S} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \cong \left( \oplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)^G,
\]
hence the above cokernel is isomorphic to the kernel of
\[
H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^G \rightarrow \left( \oplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)^G.
\]
We consider the commutative diagram
\[
\begin{array}{ccc}
H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^G & \rightarrow & \left( \oplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)^G \\
0 & \rightarrow & \oplus_{v \in S} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))
\end{array}
\]
induced by the norm maps. It is now clear that the map in the top row is not injective, if and only if Galois co-descent fails globally for \( H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \), but holds locally for all \( w \in S_L \), in which case the kernel is of order \( p \). If \( v \in S \) is decomposed in \( L \), then obviously co-descent holds. We now define \( T_{L/F} \) to be the set of undecomposed primes \( v \in S \), such that Galois co-descent fails for \( H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \). By Lemma 2.1, an undecomposed prime \( v \) lies in \( T_{L/F} \) if and only if \( H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \neq 0 \) and \( L_w \not\subset F_{v, \infty} \). Let \( d = [F(\mu_p) : F] \). Then it is clear from the definition that
\[
T_{L/F}^{(i)} = T_{L/F}^{(j)} \text{ if } i \equiv j \mod d.
\]
Let us analyze this set a little further:
Lemma 2.2. — i) $T^{(i)}_{L/F}$ contains all tamely ramified primes:

\[ S \setminus S_p \subset T^{(i)}_{L/F} \subset S. \]

ii) Assume that $L \nsubseteq F_\infty$ and $i \equiv 1 \mod d$. Then, for large $n$, the set $T^{(i)}_{L_n/F_n}$ contains all undecomposed $p$-adic primes.

Proof. — Let $v$ be any prime in $S \setminus S_p$. Then $F_v$ contains the $p$-th roots of unity $\mu_p$, which shows that $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \neq 0$. Moreover, $F_{v,\infty}$ is the maximal unramified pro-$p$-extension of $F_v$, which shows that $L_v \nsubseteq F_{v,\infty}$. This proves i). To prove ii), it suffices to choose $n$ large enough so that no $p$-adic prime of $L_n$ decomposes in $L_{n+1}$.

We can now formulate our first result in terms of the set $T^{(i)}_{L/F}$.

Proposition 2.3. — The canonical map

\[ WK^{et}_{2i-2}(L)_G \rightarrow WK^{et}_{2i-2}(F) \]

induced by the corestriction is surjective precisely in the following situations:

i) $T^{(i)}_{L/F} \neq \emptyset$;

ii) $T^{(i)}_{L/F} = \emptyset$ and either $i \neq 1 \mod d$ or $L \subset F_\infty$.

In the exceptional case where $T^{(i)}_{L/F} = \emptyset$, $i \equiv 1 \mod d$ and $L \nsubseteq F_\infty$, the cokernel of $WK^{et}_{2i-2}(L)_G \rightarrow WK^{et}_{2i-2}(F)$ is cyclic of order $p$.

Remark 2.4. — The possibility of the failure of Galois co-descent in Proposition 2.3 was already observed in [2]. The situations where this happens are easily described: First of all we must have $i \equiv 1 \mod d$ and $L \nsubseteq F_\infty$, in which case, for any $n$, the set $T^{(i)}_{L/F} = \emptyset$ if and only if $T^{(i)}_{L_n/F_n} = \emptyset$. Now, choose $n$ large enough, such that no $p$-adic prime in $L_n$ decomposes in $L_{n+1}$. By Lemma 2.2, the set $T^{(i)}_{L_n/F_n} = \emptyset$ precisely when $L_n/F_n$ is unramified and all $p$-adic primes of $F_n$ split in $L_n$. Thus, the exceptional case occurs for $L/F$ if and only if the following two conditions hold:

i) $i \equiv 1 \mod d$;

ii) $\lim_{n \to \infty} A_n \neq 0$ and $L_\infty/F_\infty$ is an unramified cyclic extension of degree $p$, in which all primes above $p$ split.
Example 2.5. — Assume that the prime $p$ is irregular and let $F = \mathbb{Q}(\mu_p)$. Then $F$ possesses a cyclic extension $L$ of degree $p$ inside the Hilbert $p$-class field, which is disjoint from $F_\infty$. Therefore the canonical map $WK_{2i-2}(L)_G \to WK_{2i-2}(F)$ is not surjective for any $i \geq 2$.

We recall that by Lemma 2.1, the natural map

$$H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1 - i)) \to H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1 - i))$$

is an isomorphism for $v \in T^{(i)}_L/F$, and hence the norm map

$$H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1 - i)) \to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1 - i))$$

can be identified with the $p$-th power map on $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1 - i))$, which is induced by the $p$-th power map on $\mathbb{Q}_p/\mathbb{Z}_p(1 - i)$. Hence we have an exact sequence for $v \in T^{(i)}_L/F$:

$$0 \to H^0(F_v, \mathbb{Z}/p\mathbb{Z}(1 - i)) \to H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1 - i)) \to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1 - i))/p \to 0.$$

The dual sequence then reads:

$$0 \to pH^2(F_v, \mathbb{Z}_p(i)) \to H^2(F_v, \mathbb{Z}_p(i)) \to H^2(L_w, \mathbb{Z}_p(i))^G \to H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \to 0.$$

If we compare this sequence with the one mentioned in Remark 1.5, we see that we have an isomorphism

$$H^2(G, H^1(L_w, \mathbb{Z}_p(i))) \cong H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

We make this isomorphism more explicit in Proposition 2.9.

Let us now consider the problem of the surjectivity of the homomorphism

$$K_{2i-2}^{\text{et}}(\mathcal{O}_L^\infty)^G \to \left( \bigoplus_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i)) \right)^G.$$

If $T^{(i)}_{L/F} = \emptyset$, then we assume that either $i \not\equiv 1 \mod d$, or that $L \subset F_\infty$, so that we have Galois co-descent for $(\bigoplus_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i)))$, i.e. $WK_{2i-2}(L)_G \to WK_{2i-2}(F)$ is surjective. In particular this implies that
the map $\beta$ in the following commutative diagram is surjective:

\[
\begin{array}{cccccc}
0 & & 0 & &  & \\
\oplus_{v \in T_L^{(i)}} (pH^2(F_v, \mathbb{Z}_p(i))) & \rightarrow & B & \\
\downarrow & & \downarrow & & \downarrow & \\
K_{2i-2}(\sigma^S_F) & \rightarrow & \oplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) & \rightarrow & H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
K_{2i-2}(\sigma^S_L) & \rightarrow & \oplus_{v \in S} (\oplus_{w|v} H^2(L_w, \mathbb{Z}_p(i)))^G & \rightarrow & (H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*)^G & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
H^2(G, K_{2i-1}(L)) & \rightarrow & \oplus_{v \in T_L^{(i)}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) & \rightarrow & C & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & &  & \\
\end{array}
\]

Here we define $B$ and $C$ to be the kernel and cokernel of the homomorphism

\[H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \rightarrow (H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*)^G\]

respectively, hence both $B$ and $C$ are either trivial or of order $p$. More precisely, by Lemma 2.1, they are non-trivial if and only if $i \equiv 1 \mod d$ and $L \not\subset F_\infty$. In this diagram the columns are exact and also the rows, except possibly at $\oplus_{v \in S} H^2(L_w, \mathbb{Z}_p(i)))^G$ and $\oplus_{v \in T_L^{(i)}} H^2(F_v, \mathbb{Z}/p\math{Z}(i))$. Note that

\[\ker \beta/\text{im } \alpha = \text{coker} \left( K_{2i-2}(\sigma^S_L)^G \rightarrow \left( \oplus_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i)) \right)^G \right) \]

is precisely the cokernel we want to study.

An easy diagram chase shows:

**Lemma 2.6.** — The surjection

\[\ker \beta/\text{im } \alpha \rightarrow \ker \beta'/\text{im } \alpha'\]

is an isomorphism if the map

\[\oplus_{v \in T_L^{(i)}} pH^2(F_v, \mathbb{Z}_p(i)) \rightarrow B\]

is surjective (otherwise, its kernel is of order at most $p$).

In particular, this settles the case $T_L^{(i)} = \emptyset$:
COROLLARY 2.7. — If $T_{L/F}^{(i)} = \emptyset$, then $WK_{2i-2}(L)_G \cong WK_{2i-2}(F)$ if and only if either $i \not\equiv 1 \mod d$ or $L \subset F_\infty$.

Thus, for example, in the cyclotomic $\mathbb{Z}_p$-extension, the wild kernels satisfy Galois codescent, whereas, in general, the $p$-class groups do not.

Let us assume now that $T_{L/F}^{(i)} \neq \emptyset$. Then $L$ is disjoint from $F_\infty$, and therefore the kernel $B$ is non-trivial if and only if $i \equiv 1 \mod d$. In this case $B$ is clearly isomorphic to $pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$, and we can characterize the surjectivity of the map $\bigoplus_{v \in T_{L/F}^{(i)}} pH^2(F_v, \mathbb{Z}_p(i)) \to pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$ as follows:

**Lemma 2.8.** — If $T_{L/F}^{(i)} \neq \emptyset$ and $i \equiv 1 \mod d$, then

$$\bigoplus_{v \in T_{L/F}^{(i)}} pH^2(F_v, \mathbb{Z}_p(i)) \to pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$$

is surjective if and only if at least one of the primes in $T_{L/F}^{(i)}$ is undecomposed in the first layer $F_1$ of the cyclotomic $\mathbb{Z}_p$-extension $F_\infty/F$.

**Proof.** — It is clear that the map in question is surjective if and only if $|H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))| = |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|$ for at least one prime $v \in T_{L/F}^{(i)}$. On the other hand $|H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))| > |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|$ if and only if $v$ splits in $F_1$. \hfill \Box

We note that any finite place $v$ in $F$ is finitely decomposed in $F_\infty$. Therefore, if $n$ is large enough, all the primes in $T_{L/F}^{(i)}$ will be undecomposed in $F_{n+1}$. If $i \equiv 1 \mod d$, we will assume that $T_{L/F}^{(i)}$ contains at least one prime, which is undecomposed in $F_1$. We are then left with the determination of $|\ker \beta'/\text{im} \alpha'|$.

The order of $\ker \beta'$ is clearly equal to

$$|\ker \beta'| = \frac{p|T_{L/F}^{(i)}|}{|H^0(F, \mathbb{Z}/p\mathbb{Z}(1-i))|}.$$

To determine the order of $\text{im} \alpha'$ we construct a canonical homomorphism

$$H^2(G, H^1(L, \mathbb{Z}_p(i))) \to H^2(F, \mathbb{Z}/p\mathbb{Z}(i)),$$ 

which gives rise to a commutative diagram

$$
\begin{array}{ccc}
H^2(G, H^1(L, \mathbb{Z}_p(i))) & \to & \bigoplus_{v \in T_{L/F}^{(i)}} H^2(G, H^1(L_v, \mathbb{Z}_p(i))) \\
\downarrow & & \downarrow \\
H^2(F, \mathbb{Z}/p\mathbb{Z}(i)) & \to & \bigoplus_{v \in T_{L/F}^{(i)}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))
\end{array}
$$

and will factor the map $\alpha'$. 

ANNALES DE L'INSTITUT FOURIER
PROPOSITION 2.9. — Let $M/N$ be a cyclic extension of degree $p$ of local or global fields of characteristic $\neq p$, where $p$ is an arbitrary prime. Let $G = \text{Gal}(M/N)$. There is a canonical map

$$H^2(G, H^1(M, \mathbb{Z}_p(i))) \to H^2(N, \mathbb{Z}/p\mathbb{Z}(i))$$

with kernel isomorphic to

$$(H^1(N, \mathbb{Z}_p(i))/p \cap N_{M/N}(H^1(M, \mathbb{Z}/p\mathbb{Z}(i)))) / N_{M/N}(H^1(M, \mathbb{Z}_p(i))/p).$$

Proof. — We first note that the exact sequence

$$0 \to \mathbb{Z}_p(i) \to \mathbb{Z}_p(i) \to \mathbb{Z}/p\mathbb{Z}(i) \to 0$$

induces an injection

$$H^1(N, \mathbb{Z}_p(i))/p \hookrightarrow H^1(N, \mathbb{Z}/p\mathbb{Z}(i)),$$

and therefore we can view $H^1(N, \mathbb{Z}_p(i))/p$ as a subgroup of $H^1(N, \mathbb{Z}/p\mathbb{Z}(i))$, and similarly for $M$. Since $G$ is cyclic, we have a canonical isomorphism

$$H^2(G, H^1(M, \mathbb{Z}_p(i))) \cong \hat{H}^0(G, H^1(M, \mathbb{Z}_p(i))) \otimes H^2(G, \mathbb{Z}_p)$$

given by the cup-product. Here $\hat{H}$ denotes Tate-cohomology. Now the group $H^1(M, \mathbb{Z}_p(i))$ satisfies Galois descent as we have seen in the proof of Theorem 1.2, even in the case $p = 2$. Hence

$$\hat{H}^0(G, H^1(M, \mathbb{Z}_p(i))) \cong H^1(N, \mathbb{Z}_p(i))/N_{M/N}(H^1(M, \mathbb{Z}_p(i)))$$

$$\cong (H^1(N, \mathbb{Z}_p(i))/p)/N_{M/N}(H^1(M, \mathbb{Z}_p(i))/p).$$

Now $H^2(G, \mathbb{Z}_p) \cong H^1(G, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1(G, \mathbb{Z}/p\mathbb{Z})$, since $G$ is cyclic of order $p$, and we have the cup-product

$$H^1(N, \mathbb{Z}/p\mathbb{Z}(i)) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to H^2(N, \mathbb{Z}/p\mathbb{Z}(i))$$

whose kernel is equal to

$$N_{M/N}(H^1(M, \mathbb{Z}/p\mathbb{Z}(i))) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}).$$

To see this, we may assume without loss of generality that $N$ contains $\mu_p$, in which case this product is just a twisted version of the standard cup-product into the Brauer group of $F$. Restricting the last morphism to the subgroup $H^1(N, \mathbb{Z}_p(i))/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z})$ yields the result. \qed
Let us return now to the situation considered before: $p$ is odd and $L/F$ is a cyclic extension of number fields of degree $p$ with Galois group $G$. We are going to compare the global and local maps constructed in Proposition 2.9. Let $C_v := \ker (H^2(F_v, \mathbb{Z}_p(i)) \to (\oplus_{w|v} H^2(L_w, \mathbb{Z}_p(i)))^G)$. Then by definition

$$C_v = 0 \iff v \notin T_{L/F}^{(i)}$$

and $C_v = H^2(G, H^1(L_w, \mathbb{Z}_p(i)) \cong H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ if $v \in T_{L/F}^{(i)}$. The following commutative diagram:

$$
\begin{array}{ccc}
H^2(G, K_{2i-1}^{ét}(L)) & \to & \prod_v C_v \\
\downarrow & & \downarrow \\
0 & \to & H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \\
\end{array}
$$

then shows that the image of $H^2(G, K_{2i-1}^{ét}(L))$ in $\prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ is in fact contained in $\oplus_{v \in T_{L/F}^{(i)}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$. The injectivity of the localization map $H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \to \prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ is proved for instance in [33, Section 2, Lemma 7]. We can now conclude:

**Lemma 2.10.** — The canonical map $H^2(G, K_{2i-1}^{ét}(L)) \to H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ induces the map

$$\alpha' : H^2(G, K_{2i-1}^{ét}(L)) \to \oplus_{v \in T_{L/F}^{(i)}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

Furthermore:

$$\ker \alpha' \cong [K_{2i-1}^{ét}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i)))/N_{L/F}(K_{2i-1}^{ét}(L)/p)$$

and

$$|\text{im } \alpha'| = [K_{2i-1}^{ét}(F)/p : K_{2i-1}^{ét}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i))]].$$

Combining this with the calculation of $\ker \beta'$ provides the main result of this section, a “genus formula” for the étale wild kernels for cyclic extensions of degree $p$:

**Theorem 2.11.** — Let $L/F$ be a cyclic extension of number fields of degree $p$, $p$ odd, with Galois group $G$. Assume that $T_{L/F}^{(i)} \not= \emptyset$ and that some $v \in T_{L/F}^{(i)}$ is undecomposed in $F_1$ if $i \equiv 1 \mod d$. Then the natural map $WK_{2i-2}^{ét}(L)_G \to WK_{2i-2}^{ét}(F)$ is surjective and its kernel has order

$$|H^0(F, \mathbb{Z}/p\mathbb{Z}(1-i))| \cdot [K_{2i-1}^{ét}(F)/p : K_{2i-1}^{ét}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i))]].$$
Remark 2.12.— Let us consider the special case that \( i \equiv 1 \mod d \), and that all \( p \)-adic primes of \( L \) are undecomposed in \( L_\infty \). Then \( T_{L/F}^{(i)} \) contains all undecomposed \( p \)-adic primes as well as all tamely ramified primes. Hence we can rewrite the formula in the preceding theorem as

\[
\frac{|WK_{2i-2}^\text{ét}(L)^G|}{|WK_{2i-2}^\text{ét}(F)|} = \frac{\prod_{p | dp} d_p(L/F) \cdot \prod_{p | ep} e_p(L/F)}{[L : F] \cdot [K_{2i-1}^\text{ét}(F)/p : K_{2i-1}^\text{ét}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i)))].}
\]

Here \( d_p(L/F) \) and \( e_p(L/F) \) denote the local degrees and the ramification indices, respectively. If we replace the étale \( K \)-theory index by the index \([U_p : U_p \cap N_{L/F}(L^*)]\) for the \( p \)-units \( U_p \), then this becomes precisely the genus formula for the \( p \)-class groups. We will return to this peculiarity later on.

Example 2.13.— 1) Take \( p = i = 3 \) and \( F = \mathbb{Q} \) the field of rationals. Since \( K^p(\mathbb{Z}) \) is trivial (cp. [30], [31], [32]), so is \( WK_{3}^\text{ét}(\mathbb{Z}) \). We are going to give an infinite family of cubic fields \( L \) such that \( WK_{3}^\text{ét}(L) = 0 \). For this, consider the set of primes (see also [37, Remarks page 182])

\[
P = \{ \ell \mid \ell \equiv 1 \mod 3 \text{ and } 3 \equiv 1 \mod \ell \} = \{ \ell \mid \ell \equiv 1 \mod 3 \text{ and } \sqrt[3]{3} \in \mathbb{Z}/\ell\mathbb{Z} \}.
\]

Obviously, by Hensel’s lemma, we have

\[
P = \{ \ell \mid \mu_3 \subset \mathbb{Q}_\ell \text{ and } \sqrt[3]{3} \in \mathbb{Q}_\ell \} = \{ \ell \mid \ell \text{ splits in } \mathbb{Q}(\mu_3, \sqrt[3]{3}) \}.
\]

We are interested in the infinite family (of density \( \frac{1}{6} - \frac{1}{18} \)) of the primes \( \ell \) in \( P \) which do not split in \( \mathbb{Q}(\mu_3, \sqrt[3]{3}) \). Now let \( L \) be the cubic extension of \( \mathbb{Q} \) contained in \( \mathbb{Q}(\mu_3) \) and \( G = G(L/\mathbb{Q}) \). Then \( T_{L/Q}^{(3)} = \{ \ell \} \) and, according to Theorem 2.11, the wild kernel \( WK_3^\text{ét}(L) = 0 \).

2) In this example, we are going to determine the Galois \( p \)-extensions \( M \) of \( \mathbb{Q} \), for which the \( p \)-part of the classical wild kernel is trivial. The two cases \( p = 3 \) and \( p \geq 5 \) are completely different due to the fact that in the latter case the considered fields do not contain the maximal real subfield \( \mathbb{Q}(\mu_p)^+ \) of the cyclotomic field \( \mathbb{Q}(\mu_p) \). For \( p \geq 5 \), the Galois \( p \)-extensions \( M \) of \( \mathbb{Q} \) for which \( WK_2(M)\{p\} = 0 \) are exactly the layers \( \mathbb{Q}_n \) of the \( \mathbb{Z}_p \)-extension \( \mathbb{Q}_\infty/\mathbb{Q} \). Indeed, since \( \mathbb{Q}_\infty \) is the maximal \( p \)-ramified
pro-$p$-extension of $\mathbb{Q}$, we see that the maximal $p$-ramified extension of $\mathbb{Q}$ contained in $M$ is a layer $\mathbb{Q}_n$ of $\mathbb{Q}_\infty$. If $M = \mathbb{Q}_n$ then, by Corollary 2.7, $WK_2(M)\{p\} = 0$. Otherwise, choose a tower of degree $p$ cyclic extensions

$$\mathbb{Q}_n = M_0 \subset M_1 \subset \cdots \subset M_r = M.$$ 

Since $T_{M_1/\mathbb{Q}_n}^{(2)} \neq \varnothing$, we have $WK_2(M_1)\{p\} \neq 0$ (Theorem 2.11). Moreover, for each intermediate extension $M_{\nu+1}/M_\nu$, the canonical map

$$WK_2(M_{\nu+1})\{p\} \rightarrow WK_2(M_\nu)\{p\}$$

is surjective (Proposition 2.3), which shows that $WK_2(M)\{p\} \neq 0$. A number field $M$ for which $H_2^G(o_M, \mathbb{Z}/p\mathbb{Z}) = 0$, is called $p$-rational [25], [24]. Moreover, if $M$ contains $\mathbb{Q}(\mu_p)^+$, then it is also called $p$-regular [10].

The $p$-regularity of $M$ is simply expressed by the triviality of the $p$-part of the tame kernel $K_2(o_M)$. As the $\mathbb{Q}_n$ are not the only $p$-extensions of $\mathbb{Q}$ which are $p$-rational, we notice that, for $p \geq 5$, among the $p$-extensions $M$ of $\mathbb{Q}$, some are $p$-rational but have a non-trivial $WK_2(M)\{p\}$. Now take $p = 3$. Then by Moore’s exact sequence $WK_2(M)\{3\} = 0$ if and only if the tame kernel $K_2(o_M)$ has no $3$-torsion. Hence the number field $M$ is $3$-rational or $3$-regular. In this case, $WK_2(M)\{3\} = 0$ if and only if outside the prime $3$, the $3$-extension $M/\mathbb{Q}$ is at most ramified at one prime $l$, which is inert in the $\mathbb{Z}_3$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ (cp. [10], [25], [24]).

Let us have a closer look at the cup-product

$$K_{2i-1}^{\text{ét}}(F)/p \otimes H^i(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2(F, \mathbb{Z}/p\mathbb{Z}(i)),$$

which occurred in the proof of Proposition 2.9, and describe the maps $\alpha'$ and $\beta'$. Let $E = F(\mu_p)$ and $\Delta = \text{Gal}(E/F)$. Over $E$ we have

$$H^2(E, \mathbb{Z}/p\mathbb{Z}(i)) \cong H^2(E, \mathbb{Z}/p\mathbb{Z}(1))(i - 1) \cong p\text{Br}(E)(i - 1),$$

where $\text{Br}(E)$ stands for the Brauer group of $E$. The set $T_{LE/E}^{(i)}$ is independent of $i$, and we simply denote it by $T_{LE/E}$. Obviously, every prime in $E$ which lies above a prime in $T_{L/F}^{(i)}$ belongs to $T_{LE/E}$. Conversely, let $v_E$ be a prime in $T_{LE/E}$, and let $v$ denote the prime of $F$ below $v_E$. Then $v \in T_{L/F}^{(i)}$ if and only if $H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \neq 0$. Let $\text{Br}^T(E)$ denote the subgroup of $\text{Br}(E)$ of all isomorphism classes of central simple $E$-algebras split outside $T_{LE/E}$. It is now easy to see that

$$\ker \beta' \cong (p\text{Br}^T(E)(i - 1))^{\Delta} \cong (p\text{Br}^T(E))^{[1-i]},$$
where $\omega$ denotes the Teichmüller character of $\Delta$, and $A^{[j]}$ denotes the $j$-th eigenspace of $\omega$ acting on a $\Delta$-module $A$.

Since $K_{2i-1}^{\text{ét}}(E)/p$ is contained in $H^1(E, \mathbb{Z}/p\mathbb{Z}(i)) \cong (E^*/E^{*p})(i - 1)$, there exists a subgroup $D_E^{(i)}$ of $E^*$ containing $E^{*p}$ - the analog of the Tate-kernel in case $i = 2$ - such that

$$K_{2i-1}^{\text{ét}}(E)/p \cong (D_E^{(i)}/E^{*p})(i - 1),$$

and hence

$$K_{2i-1}^{\text{ét}}(F)/p \cong ((D_E^{(i)}/E^{*p})(i - 1))^\Delta \cong (D_E^{(i)}/E^{*p})^{[1-i]}.$$ 

Note that for $i \equiv 1 \mod d$ we can similarly define $D_F^{(i)}$, and clearly in this case $(D_E^{(i)}/p)^\Delta \cong D_F^{(i)}/p$. The considerations after Proposition 2.9 now show that the cup-product over $E$ is explicitly given as

$$(D_E^{(i)}/E^{*p})(i - 1) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to p\text{Br}^T(E)(i - 1),$$

where

$$D_E^{(i)}/E^{*p} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to p\text{Br}^T(E),$$

is the classical cup-product $x \otimes \chi \mapsto (\chi, x)$ (cp. [34, Chap. XIV]). Descending to $F$, we see that the image of $\alpha'$ is precisely the image of the cup-product

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to (p\text{Br}^T(E))^{[1-i]}.$$ 

We can therefore reformulate the condition for Galois co-descent of the wild kernel as follows:

**Theorem 2.14.** — Let $L/F$ be a cyclic extension of number fields of degree $p$, $p$ odd, with Galois group $G$. Assume that $T_L^{(i)} \neq \varnothing$ and that some $v \in T_L^{(i)}$ is undecomposed in $F_1$ if $i \equiv 1 \mod d$. Then the étale wild kernel $WK_{2i-2}^{\text{ét}}(L)$ satisfies Galois co-descent if and only if the cup-product

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to (p\text{Br}^T(E))^{[1-i]}$$

is surjective.

In the special case where $i \equiv 1 \mod d$, the condition can be reformulated as: The cup-product

$$D_F^{(i)}/F^{*p} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to p\text{Br}^T_{L/F}(F)$$
is surjective. Moreover, in this special case the genus-formula simplifies to:

\[
\frac{|WK^{\text{ét}}_{2i-2}(L)|^G}{|WK^{\text{ét}}_{2i-2}(F)|} = \frac{p^{|T_L^{(i)}| - 1}}{|D_F^{(i)} : D_F^{(i)} \cap N_{L/F}(L^*)|}.
\]

Since \(i \geq 2\), we can reinterpret the cup-product as a Galois symbol: Let \(EL = E(\sqrt[3]{\delta})\). Then

\[(D_E^{(i)}/E^{*p})(i - 1) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) = (D_E^{(i)}/E^{*p}) \otimes H^1(G, \mu_p)(i - 2),\]

and therefore the cup-product over \(E\) is the \((i - 2)\)-th twist of the Galois symbol

\[D_E^{(i)}/E^{*p} \otimes H^1(G, \mu_p) \rightarrow \mu_p \otimes p\text{Br}^T(E)\]

The Kummer radical \(H^1(G, \mu_p)\) is generated by \(\delta\), and the map is given by (cp. [23])

\[x \otimes \delta \mapsto \zeta_p \otimes \left[\frac{x, \delta}{E}\right],\]

where \(\zeta_p\) is a primitive \(p\)th root of unity and \([\left(\frac{x, \delta}{E}\right)]\) denotes the isomorphism class of the cyclic algebra \(\left(\frac{\mu_p}{E}\right)\), with generators \(u, v\) and relations:

\[u^p = x, v^p = \delta, uv = \zeta_p u v.\]

In general, not much is known about the higher “Tate-kernels” \(D_E^{(i)}\) defined by \(K^{\text{ét}}_{2i-1}(E)/p \cong (D_E^{(i)}/E^{*p})\). However, for \(n\) large, the groups \(D_{E_n^{(i)}}/E_n^{*p}\) can be characterized in terms of local conditions, if we assume the Gross conjecture, which we describe next: Let \(F\) be any number field. For a finite prime \(v\) in \(F\) let \(\hat{F}_v = \lim_{\longleftarrow} F^\nu/F^\nu_{p^k}\), let \(\hat{U}_v = \mathbb{Z}_p \otimes U_v\) and let \(N_v \subset \hat{F}_v\) denote the group of norms from the cyclotomic \(\mathbb{Z}_p\)-extension of \(F_v\). Thus \(N_v = \hat{U}_v\) if \(v \not\in S_p\), and for \(v \in S_p\) we have the following characterization:

\[a \in N_v \iff \log_p(N_{F_v}/\mathbb{Q}_p)(a) = 0,\]

where \(\log_p\) denotes the \(p\)-adic logarithm normalized by \(\log_p(p) = 0\) (cp. [9], [19]). There is a natural homomorphism

\[g_F : \mathbb{Z}_p \otimes \hat{U}_F' \rightarrow \bigoplus_{v | p} \hat{F}_v^*/N_v\]

and the Gross kernel \(GK(F) := \ker g_F\) has \(\mathbb{Z}_p\)-rank \(r_1(F) + r_2(F) + \delta_F\), where \(\delta_F > 0\) is the Gross defect. \(GK(F)\) is therefore characterized by the
following local conditions:

\[ \epsilon \in GK(F) \Leftrightarrow \epsilon \in \mathbb{Z}_p \otimes U_F' \quad \text{and} \quad \log_p(N_{F_w/F}(\epsilon)) = 0 \quad \forall v \in S_p. \]

The Gross Conjecture postulates that \( \delta_F = 0 \), which is true for instance for abelian fields \( F \). Let - as before - \( E = F(\mu_p) \) and \( \Delta = \text{Gal}(E/F) \). The following result was proved for \( i = 2 \) in [19, Theorem 2.5]. The method was extended to higher étale \( K \)-theory in [5].

**Theorem 2.15.** — For \( n \) large there is an exact sequence

\[ 0 \rightarrow K_{2i-1}(E_n)/p \rightarrow (\ker g_{E_n}/p)(i - 1) \rightarrow (\mathbb{Z}/p\mathbb{Z})^{\delta_n} \rightarrow 0, \]

where \( \delta_n \) denotes the Gross defect for the field \( E_n \).

**Corollary 2.16.** — Assume that the Gross Conjecture holds for \( E_n \), \( n \) large. Then

\[ D^{(i)}_{E_n}/E_n^{*p} \cong GK(E_n)/p \quad \text{for} \quad n \text{ large}. \]

In particular, for \( n \) large, the groups \( D^{(i)}_{E_n}/E_n^{*p} \) are independent of \( i \).

So far in this section we have ignored the prime 2. Let us briefly discuss the case \( p = 2 \) in the classical situation \( i = 2 \), where special attention has to be paid to real infinite primes in \( F \). Let \( L = F(\sqrt{\delta}) \) be a quadratic extension of number fields with Galois group \( G \). Denote by \( T_{L/F} \) the set of finite primes in \( F \) which consists of all ramified non-dyadic primes and of all undecomposed dyadic primes \( v \) of \( F \), for which either \( \mu(L_w)\{2\} = \mu(F_v)\{2\} \) or \( L_w \) is not contained in the cyclotomic \( \mathbb{Z}_2 \)-extension of \( F_v \), where \( w \) is the prime above \( v \) in \( L \). Also, denote by \( D_F \) the subgroup of \( F^* \) of all elements \( x \), such that \( \{-1, x\} = 1 \) in \( K^{*2}(F) \). This is the classical Tate-kernel. Then the following results can be proved along the same lines as for odd \( p \):

**Proposition 2.17.** — The canonical map \( WK_2(L)\{2\} \rightarrow WK_2(F)\{2\} \) is surjective precisely in the following situations, and has cokernel of order 2 otherwise:

i) \( |\mu(L)\{2\}| > |\mu(F)\{2\}| \) and \( L \subset F_\infty \).

ii) \( |\mu(L)\{2\}| > |\mu(F)\{2\}| \), \( L \not\subset F_\infty \) and \( \mu(L_w)\{2\} = \mu(L)\{2\} \) for some \( w \mid v \), \( v \in T_{L/F} \).

iii) \( \mu(L)\{2\} = \mu(F)\{2\} \) and \( \mu(L_w)\{2\} = \mu(F_v)\{2\} \) for some \( v \in T_{L/F} \).
We note in particular that the map $WK_2(L)\{2\}_G \to WK_2(F)\{2\}$ is always surjective if a non-dyadic prime of $F$ is ramified in $L$.

**Theorem 2.18.** — Let $L/F$ be a relative quadratic extension with Galois group $G$.

a) If $|\mu(L)\{2\}| > |\mu(F)\{2\}|$ and $L \subset F_\infty$, then $WK_2(L)\{2\}_G \cong WK_2(F)\{2\}$.

b) If either $|\mu(L)\{2\}| = |\mu(F)\{2\}|$ or $L \not\subset F_\infty$, and if either a real infinite prime of $F$ ramifies in $L$ or if $|\mu(F_v)\{2\}| = |\mu(F)\{2\}|$ for some prime $v \in T_{L/F}$, then

$$\frac{|WK_2(L)\{2\}_G|}{|WK_2(F)\{2\}|} = \frac{2^{|T_{L/F}| - 1}}{|D_F : D_{F_n} \cap N_{L/F}(L^*)|}.$$

In [6], Browkin and Schinzel computed the 2-rank of the wild kernel of a quadratic number field and obtained a complete list of quadratic number fields with trivial 2-primary wild kernels. A combination of their results with the genus formula in Theorem 2.18 and methods of [12] yield a complete list of bi-quadratic fields with trivial 2-primary wild kernels. Details will appear elsewhere.

### 3. Capitulation kernels.

Let $p$ be an odd prime and let $F_\infty/F$ be an arbitrary $\mathbb{Z}_p$-extension of $F$ with finite layers $F_n$. Let $A'_n = A'(F_n)$ denote the $p$-part of the $p$-class group of $F_n$ and $A'_\infty = \varprojlim A'_n$. We define the capitulation kernel $\text{Cap}_0(F_\infty/F_n) = \ker(A'_n \to A'_\infty)$. As is well-known (cp. [13]) these kernels stabilize, more precisely, the norm $N_{F_m/F_n} : \text{Cap}_0(F_\infty/F_m) \to \text{Cap}_0(F_\infty/F_n)$ is an isomorphism for $n$ large and $m \geq n$ and we set $\text{Cap}_0(F_\infty) = \varprojlim \text{Cap}_0(F_\infty/F_n)$.

**Remark 3.1.** — Let $A_n$ denote the $p$-part of the (usual) class group of $F_n$ and let $A_\infty = \varprojlim A_n$. Once again, the capitulation kernels $\ker(A_n \to A_\infty)$ stabilize, and we can consider $\tilde{\text{Cap}}(F_\infty) = \varprojlim \ker(A_n \to A_\infty)$. We note that in general $\tilde{\text{Cap}}(F_\infty) \neq \text{Cap}_0(F_\infty)$. Indeed, from the explicit examples elaborated by Greenberg in [11, section 8], it is not hard to see
that if we take $F = \mathbb{Q}(\sqrt{142})$, $p = 3$, and let $F_\infty$ be the cyclotomic $\mathbb{Z}_3$-extension of $F$, then $\text{Cap}(F_\infty) \cong \mathbb{Z}/3\mathbb{Z}$, whereas $\text{Cap}_0(F_\infty)$ is trivial. From a $K$-theoretic point of view, $\text{Cap}_0(F_\infty)$ is the appropriate object to study.

We want to consider the analog of these kernels in higher étale $K$-theory.

Let again $S$ be a finite set of primes in $F$ containing $S_p$. To simplify notation, we put

$$\tilde{K}_{2i-1}^\text{ét}(F_\infty) = \lim_{\to} K_{2i-1}^\text{ét}(F_n)$$

and

$$\tilde{K}_{2i-2}^\text{ét}(o_\infty^S) = \lim_{\to} K_{2i-2}^\text{ét}(o_n^S),$$

where $o_n^S$ denotes the ring of $S$-integers in $F_n$, i.e. the integral closure of $o_F^S$ in $F_n$. We now define for $i > 2$:

$$\text{Cap}_{i-1}(F_\infty/F_n) = \ker(K_{2i-2}^\text{ét}(o_n^S) \to \tilde{K}_{2i-2}^\text{ét}(o_\infty^S)).$$

The following result implies in particular that the definition is independent of the choice of the finite set $S$ containing $S_p$. Let $\Gamma_n$ denote the Galois group of $F_\infty/F_n$ with the usual convention $\Gamma_0 = 1$.

**Proposition 3.2.** For $i > 2$ there is a short exact sequence

$$0 \to H^1(\Gamma_n, \tilde{K}_{2i-1}^\text{ét}(F_\infty)) \to K_{2i-2}^\text{ét}(o_n^S) \to \tilde{K}_{2i-2}^\text{ét}(o_\infty^S)^{\Gamma_n} \to 0.$$

**Proof.** For each $m \geq n$, Theorem 1.2 gives an exact sequence

$$0 \to H^1(\Gamma_n/\Gamma_m, K_{2i-1}^\text{ét}(F_m)) \to K_{2i-2}^\text{ét}(o_m^S) \to K_{2i-2}^\text{ét}(o_m^S)^{\Gamma_n/\Gamma_m} \to H^2(\Gamma_n/\Gamma_m, K_{2i-1}^\text{ét}(F_m)) \to 0.$$ 

From Corollary 1.4 we see that the orders of the groups $H^2(\Gamma_n/\Gamma_m, K_{2i-1}^\text{ét}(F_m))$ are bounded independently of $m$ by the order of $K_{2i-2}^\text{ét}(o_n^S)$, and therefore the limit

$$H^2(\Gamma_n, \tilde{K}_{2i-1}^\text{ét}(F_\infty)) = \lim_{\to} H^2(\Gamma_n/\Gamma_m, K_{2i-1}^\text{ét}(F_m))$$

is finite. On the other hand, this group is divisible, since $cd_p(\Gamma_n) = 1$, hence trivial.  

TOME 50 (2000), FASCICULE 1
In the classical case \( i = 1 \), it was shown by Iwasawa (cp. [13, Theorem 12]) that

\[
\text{Cap}_0(F_\infty/F_n) \cong H^1(\Gamma_n, U'_\infty),
\]

where \( U'_\infty = \lim U'_n \) and \( U'_n \) denotes the group of \( p \)-units of \( F_n \). Therefore Proposition 3.2 gives, in particular, the following higher-dimensional analog of this result:

**Corollary 3.3.** — For \( i \geq 2 \)

\[
\text{Cap}_{i-1}(F_\infty/F_n) \cong H^1(\Gamma_n, \tilde{K}_{2i-1}^{\text{ét}}(F_\infty)).
\]

To go further, we quote the following general result of Kahn (cp. [16, Proposition 6.2]), which he attributes to Nguyen Quang Do:

**Lemma 3.4.** — Let \( A \) be a discrete torsion free \( \Gamma \)-module. Assume that for all integers \( n \geq 0 \):

i) \( H^0(\Gamma_n, A) \) is finitely generated;

ii) \( H^1(\Gamma_n, A) \) is finite;

iii) \( H^2(\Gamma_n, A) = 0 \).

Then the groups \( H^1(\Gamma_n, A) \) stabilize, in particular \( \lim_{\to} H^1(\Gamma_n, A) \) is finite.

Let

\[
\tilde{K}_{2i-1}^{\text{ét}}(F_n) = K_{2i-1}^{\text{ét}}(F_n)/\text{torsion}
\]

and

\[
\tilde{K}_{2i-1}^{\text{ét}}(F_\infty) = \tilde{K}_{2i-1}^{\text{ét}}(F_\infty)/\text{torsion}.
\]

We want to apply the previous lemma with \( A = \tilde{K}_{2i-1}^{\text{ét}}(F_\infty) \). From the exact sequence

\[
0 \to H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)) \to \tilde{K}_{2i-1}^{\text{ét}}(F_\infty) \to \tilde{K}_{2i-1}^{\text{ét}}(F_\infty) \to 0,
\]

we deduce the exact sequence

\[
0 \to \tilde{K}_{2i-1}^{\text{ét}}(F_n) \to \tilde{K}_{2i-1}^{\text{ét}}(F_\infty) \to H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i))) \to H^1(\Gamma_n, \tilde{K}_{2i-1}^{\text{ét}}(F_\infty)) \to 0,
\]

as well as an isomorphism

\[
H^2(\Gamma_n, \tilde{K}_{2i-1}^{\text{ét}}(F_\infty)) \cong H^2(\Gamma_n, \tilde{K}_{2i-1}^{\text{ét}}(F_\infty)).
\]
The proof of Proposition 2.2 showed that $H^2(\Gamma_n, K_{2i-1}^{\text{et}}(F_{\infty})) = 0$, and hence we see that $K_{2i-1}^{\text{et}}(F_{\infty})$ satisfies the assumptions of the previous lemma. We obtain the fact that the groups $H^1(\Gamma_n, K_{2i-1}^{\text{et}}(F_{\infty}))$ stabilize and therefore that $\lim_{\rightarrow} H^1(\Gamma_n, K_{2i-1}^{\text{et}}(F_{\infty}))$ is finite. To obtain the same result for the groups $H^1(\Gamma_n, K_{2i-1}^{\text{et}}(F_{\infty}))$ and their limit, we look at the term $H^1(\Gamma_n, H^0(F_{\infty}, Q_p/Z_p(i)))$ in the above exact sequence: The group $H^0(F_{\infty}, Q_p/Z_p(i))$ is either $Q_p/Z_p(i)$ or finite. In the first case, Tate’s Lemma implies that $H^1(\Gamma_n, H^0(F_{\infty}, Q_p/Z_p(i))) = 0$, hence

$$H^1(\Gamma_n, K_{2i-1}^{\text{et}}(F_{\infty})) \cong H^1(\Gamma_n, K_{2i-1}^{\text{et}}(F_{\infty})).$$

In the second case, $H^1(\Gamma_n, H^0(F_{\infty}, Q_p/Z_p(i)))$ stabilizes for $n$ large, and hence in any case we obtain:

**Proposition 3.5.** — The groups $\text{Cap}_{i-1}(F_{\infty}/F_n)$ stabilize; more precisely, the corestriction maps

$$\text{Cap}_{i-1}(F_{\infty}/F_{n+1}) \to \text{Cap}_{i-1}(F_{\infty}/F_n)$$

are surjective for all $n$ and $\lim_{\rightarrow} \text{Cap}_{i-1}(F_{\infty}/F_n)$ is finite.

We now define

$$\text{Cap}_{i-1}(F_{\infty}) = \lim_{\rightarrow} \text{Cap}_{i-1}(F_{\infty}/F_n).$$

Now let us specialize and take $F_{\infty}/F$ to be the cyclotomic $Z_p$-extension. As in the case $i = 1$, the finite groups $\text{Cap}_{i-1}(F_{\infty})$ then have various characterizations in terms of Iwasawa-theory. Let $E = F(\mu_p)$, let $E_{\infty} = F(\mu_{p^\infty})$ be the cyclotomic $Z_p$-extension of $E$ and identify $\Gamma_n$ with the Galois group of $E_{\infty}/E_n$. We first describe $\text{Cap}_{i-1}(E_{\infty})$. Let $\mathcal{X}_{\infty}$ denote the standard Iwasawa-module for $E_{\infty}$, i.e. the Galois group over $E_{\infty}$ of the maximal abelian $p$-ramified pro-$p$-extension of $E_{\infty}$. Denote by $\text{tor}_{\Lambda} \mathcal{X}_{\infty}$ the torsion part of $\mathcal{X}_{\infty}$ as a module over $\Lambda = Z_p[[\Gamma]]$. As is well-known, there exists an injective homomorphism ([11, Theorem 3])

$$\mathcal{X}_{\infty}/\text{tor}_{\Lambda} \mathcal{X}_{\infty} \to \Lambda^{\gamma_2(E)}$$

with finite cokernel $H$. The following result is due to Iwasawa ([13]) for $i = 1$, to Coates ([7]) for $i = 2$ and to Nguyen Quang Do([27, section 4]) in general:

**Theorem 3.6.** — For all $i \geq 1$ and all $n \geq 0$, there are canonical isomorphisms

$$\text{Cap}_{i-1}(E_{\infty}/E_n) \cong H^*(i)_{\Gamma_n}.$$
Since $H$ is finite, the group $\Gamma_n$ acts trivially on $H^*(i)$ for all $i$ provided $n$ is large enough. Therefore, as abstract groups, all capitulation kernels $\text{Cap}_{i-1}(E_\infty)$ are isomorphic to $H$.

Let $\Delta = \text{Gal}(E/F)$ and let $d$ denote the order of $\Delta$. Now clearly

$$\text{Cap}_{i-1}(F_\infty) = \text{Cap}_{i-1}(E_\infty)^\Delta.$$  

Theorem 3.6 shows that $\text{Cap}_{i-1}(E_\infty)$ and $\text{Cap}_{j-1}(E_\infty)$ are isomorphic as $\Delta$-modules for $i \equiv j \mod d$. Therefore we obtain the following periodicity result:

**Corollary 3.7.** — Let $p$ be odd and let $F_\infty/F$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$. Then

$$\text{Cap}_{i-1}(F_\infty) \cong \text{Cap}_{j-1}(F_\infty)$$  

for all $i, j \geq 1$, $i \equiv j \mod d$.

Next we would like to discuss another well-known relation between capitulation kernels and Iwasawa-theory: We continue to assume that $E_\infty = F(\mu_p\infty)$ is the cyclotomic $\mathbb{Z}_p$-extension of $E = F(\mu_p)$, and that $p$ is odd. As usual, let $X'_\infty$ denote the Galois group over $E_\infty$ of the maximal abelian unramified pro-$p$-extension of $E_\infty$, in which all primes above $p$ are completely decomposed. Thus $X'_\infty \cong \lim_{\leftarrow} A'_n(E)$. The co-invariants $(X'_\infty)_r$ have been described by Jaulent as a group of logarithmic classes $cl(E)$ which can be interpreted as the class field theory analog of the wild kernels corresponding to the case $i = 1$. The Galois co-descent for these modules $cl(E)$ has been studied in [14]. Now, let $(X'_\infty)^0$ denote the maximal finite submodule of $X'_\infty$. It is well-known (cp. [21]) that

$$\text{Cap}_0(E_\infty) \cong (X'_\infty)^0.$$  

On the other hand, we have for all $n \geq 0$ and all $i \geq 2$, an isomorphism

$$(X'_\infty(i - 1))_{\Gamma_n} \cong WK^\text{ét}_{2i-2}(E_n)$$  

(cp. 33, section 6, Lemma 1), and therefore

$$\ker (WK^\text{ét}_{2i-2}(E_n) \to WK^\text{ét}_{2i-2}(E_m))$$  

$$\cong \ker ((X'_\infty(i - 1))_{\Gamma_n} \to (X'_\infty(i - 1))_{\Gamma_m})$$  

$$\cong (X'_\infty)^0(i - 1)$$
for \( n \) large and \( m \) sufficiently larger than \( n \). If we define
\[
\tilde{W}K^\text{ét}_{2i-2}(E_\infty) = \lim W\overline{K}^\text{ét}_{2i-2}(E_n),
\]
then we obtain

**Proposition 3.8.** — For \( i \geq 2 \) and \( n \) sufficiently large we have:

\[
\text{Cap}_{i-1}(E_\infty) \cong \ker (W\overline{K}^\text{ét}_{2i-2}(E_n) \to \tilde{W}K^\text{ét}_{2i-2}(E_\infty)) \cong (X'_\infty)^0(i - 1)
\]
as \( \Delta \)-modules.

For the original field \( F \) and the cyclotomic \( \mathbb{Z}_p \)-extension \( F_\infty/F \) this implies:

\[
\text{Cap}_{i-1}(F_\infty) = \ker (W\overline{K}^\text{ét}_{2i-2}(F_n) \to \tilde{W}K^\text{ét}_{2i-2}(F_\infty)) \cong ((X'_\infty)^0(i - 1))^\Delta.
\]

Again let \( \omega \) denote the Teichmüller character on \( \Delta \). We have

\[
((X'_\infty)^0(i - 1))^\Delta \cong ((X'_\infty)^0)^{[1-i]} \cong (X'_\infty)^{[1-i]}^0,
\]
and hence

\[
\text{Cap}_{i-1}(F_\infty) \cong (X'_\infty)^{[1-i]}^0 \cong \text{Cap}_0(E_\infty)^{[1-i]}
\]
for all \( i \geq 1 \). We therefore obtain a decomposition of \( \text{Cap}_0(E_\infty) \) into eigenspaces:

\[
\text{Cap}(E_\infty) \cong \bigoplus_{j=0}^{d-1} \text{Cap}(F_\infty)
\]
with \( \text{Cap}_j(F_\infty) \) being isomorphic to the \((d - j)\)-th eigenspace of \( \text{Cap}_0(E_\infty) \).

The following result gives the connection with Section 2:

**Proposition 3.9.** — For \( i \geq 2 \), the following statements are equivalent:

i) \( \text{Cap}_{i-1}(F_\infty) \cong W\overline{K}^\text{ét}_{2i-2}(F_n) \) for large \( n \).

ii) \( X'_\infty^{[1-i]} \) is finite.

**Proof.** — As already mentioned we have for \( i \geq 2 \):

\[
(X'_\infty(i - 1))_n \cong W\overline{K}^\text{ét}_{2i-2}(E_n),
\]
hence

\[
X'_\infty(i - 1) \cong \lim \ W\overline{K}^\text{ét}_{2i-2}(E_n),
\]

TOME 50 (2000), FASCICULE 1
and therefore
\[ X^{[1-i]}_{\infty} \cong \lim_{\to} WK^\text{et}_{2i-2}(F_n). \]

The equivalence of i) and ii) is now obvious. \(\square\)

Let us assume now that the base field \(F\) is totally real. Then \(E\) is a CM-field with maximal real subfield \(E^+\). Since obviously the plus-part of the group \(H\) is trivial in this situation, Theorem 3.6 implies that \(\text{Cap}_{i-1}(F_{\infty}) = 0\) for all even \(i \geq 2\), hence that the minus-part of \(\text{Cap}_0(E_{\infty})\) vanishes: \(\text{Cap}_0(E_{\infty})^- = 0\). Let \(X_{\infty}\) denote the Galois group of the maximal abelian unramified pro-\(p\)-extension of \(E_{\infty}\). Greenberg’s Conjecture (cp. [11]) for the cyclotomic \(\mathbb{Z}_p\)-extension \(F_{\infty}\) of the totally real field \(F\) is equivalent to the fact that \(X_{\infty}^{\Delta}\) is finite. Clearly this implies that \((X_{\infty}^\text{tr})^\Delta\) is also finite, and the converse implication is true if one assumes for example that Leopoldt’s Conjecture holds for the layers \(F_n\) of \(F_{\infty}/F\). We will refer to Greenberg’s Conjecture in the form: \((X_{\infty}^\text{tr})^\Delta\) is finite. In fact we will consider Greenberg’s Conjecture for the field \(E^+\). Using Proposition 3.9, we can summarize:

**Proposition 3.10.** — Let \(F\) be a totally real number field, \(p\) an odd prime, \(E = F(\mu_p)\) and \(E^+\) the maximal real subfield of \(E\). Furthermore, let \(F_{\infty}\) denote the cyclotomic \(\mathbb{Z}_p\)-extension of \(F\) and \(E_{\infty}\) the cyclotomic \(\mathbb{Z}_p\)-extension of \(E\). Then:

i) \(\text{Cap}_0(E_{\infty})^- = 0\), i.e. \(\text{Cap}_{i-1}(F_{\infty}) = 0\) for all even \(i \geq 2\).

ii) \(\text{Cap}_{i-1}(F_{\infty}) \cong WK^\text{et}_{2i-2}(F_n)\) for large \(n\) and all odd \(i \geq 3\), if and only if Greenberg’s Conjecture holds for \(E^+\).

As an immediate consequence of part ii), we obtain that under Greenberg’s Conjecture the étale wild kernels \(WK^\text{et}_{2i-2}(F_n)\) show the same periodic behaviour as the capitulation kernels for \(n\) large and \(i \geq 3\) odd. On the other hand, under Greenberg’s Conjecture for \(E^+\), we also have \(\text{Cap}_0(E_{\infty}^+) = \text{Cap}_0(E_{\infty})^+ = A_n'(E)^+\) for \(n\) large; hence for all \(i \geq 3\) odd:

\[ \text{Cap}_{i-1}(F_{\infty}) \cong A_n'(E)^{[1-i]} \cong WK^\text{et}_{2i-2}(F_n) \]  

for \(n\) large.

Therefore, the Galois co-descent results of Section 2 also apply to both \(\text{Cap}_{i-1}(F_{\infty})\) and the eigenspaces \(A_n'(E)^{[1-i]}\) of \(A_n'(E^+)\) for \(n\) large. In particular:

**Theorem 3.11.** — Let \(L/F\) be a cyclic extension of totally real number fields of degree \(p\), \(p\) odd, with Galois group \(G\) and let \(E = F(\mu_p)\). Assume Greenberg’s conjecture holds for \(E^+\), \(LE^+\) and the Gross
conjecture holds for $E_n$, $n$ large. Then for $i \geq 3$ odd, $n$ large and $T_{L_n/F_n} \neq \emptyset$, Galois co-descent holds for $\text{Cap}_{i-1}(L_\infty)$ and $A'_{n}(LE)^{[1-i]}$ if and only if the cup-product
\[(GK(E_n/p)^{[1-i]} \otimes H^1(G, Z/pZ) \to p\text{Br}^T(E_n)^{[1-i]} \]
is surjective.

Remark 3.12.— If $i \equiv 1 \mod d$, then, under the assumptions of Theorem 3.11, we can compare the genus formulae for $WK^{\text{et}}_{2i-2}(L_n)$ and $A'_{n}(L)$ to obtain for large $n$:
\[\left[ U'_n : U'_n \cap N_{L_n/F_n}(L_n^*) \right] = \left[ GK(F_n) : GK(F_n) \cap N_{L_n/F_n}(L_n^*) \right], \]
a result which one can also prove directly.

4. Galois co-descent for the étale tame kernel.

In this final section we briefly discuss how the methods of Section 2 can be used to study the much easier problem of Galois co-descent for the étale tame kernels again for cyclic extensions $L/F$ of degree $p$, $p$ odd. Results for arbitrary finite Galois $p$-extensions have been obtained by Assim (cp. [1], [2]) in terms of primitive ramification, however under the assumption that Leopoldt’s Conjecture holds for the fields $L(p^n)$ for all $n$. Let $S$ be the finite set of primes of $F$, consisting of the set $S_p$ and the tamely ramified primes in $L/F$. We have the following exact sequence:
\[0 \to K^{\text{et}}_{2i-2}(o_F) \to K^{\text{et}}_{2i-2}(o_L) \to \bigoplus_{v \in S \setminus S_p} H^2(F_v, Z_p(i)) \to 0, \]
which, combined with Proposition 1.3, shows that the canonical map
\[K^{\text{et}}_{2i-2}(o_L)_G \to K^{\text{et}}_{2i-2}(o_F) \]
is always surjective and that the kernel of this map is isomorphic to the cokernel of the map
\[K^{\text{et}}_{2i-2}(o_L')^G \to \left( \bigoplus_{w \in S'_L} H^2(L_w, Z_p(i)) \right)^G, \]
where $S'_L$ consists of the primes in $L$ above $S \setminus S_p$. We recall that $S \setminus S_p$ is always contained in $T^{(i)}_{L/F}$. The following is now clear from the results in Section 2:
THEOREM 4.1.— The kernel of the surjective map $K^{\text{ét}}_{2i-2}(\mathcal{O}_L)_G \to K^{\text{ét}}_{2i-2}(\mathcal{O}_F)$ is isomorphic to the cokernel of the map

$$K^{\text{ét}}_{2i-1}(F)/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{v \in S \setminus S_F} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

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