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THE ANALYTICITY OF q-CONCAVE SETS OF LOCALLY FINITE HAUSDORFF *(2n-2q)*-MEASURE

by Viorel VÂJÂITU

1. Introduction.

Let A be a closed subset of a complex space X. The question of finding reasonable assumptions on A which guarantee its analyticity has been studied over the years by various authors.

Hartogs [14] considered a continuous function $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}^n$ is open, and showed that the graph G_f of f in $D \times \mathbb{C}$ is pseudoconcave (*i.e.*, the complement of G_f in $D \times \mathbb{C}$ is locally Stein) if and only if f is holomorphic, that is G_f is analytic.

Grauert revealed in his thesis [13] a new interesting aspect of the above question bringing into play thin complements of complete Kähler domains. This topic was afterwards thoroughly studied by Diederich and Fornæss ([6], [7]) and Ohsawa [19].

On the other hand, Hirschowitz [15] settled the case when X is nonsingular and A is pseudoconcave of locally finite Hausdorff $(2n-2)$-measure, where n is the complex dimension of X.

In this article, using q-convexity with corners we introduce the notion of *q-concavity.* (See §2 for definition. Note that for $q = 1$ we recover the usual pseudoconcavity as used in [15] and [18].) For instance, if X is a complex manifold of pure dimension n and $A \subset X$ is an analytic subset

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such that every irreducible of it has dimension $\geqslant n-q$, then A is q-concave [20]. Two more examples are given at the end of Section 2.

Our main result in this note, which establishes a converse of the above result due to M. Peternell and generalizes Hirschowitz's theorem already quotes above, is the following:

THEOREM 1. — *Let X be a complex space of pure dimension n and q a positive integer less than n. If $A \subset X$ is a q-*concave *subset such that its Hausdorff $(2n-2q)$-measure is locally finite, then A is analytic of pure dimension $n-q$.*

As an application (see also Example 2 in Section 2) we have:

COROLLARY 1. — *Let T be a closed positive current of bidimension (q,q) on a complex manifold M. If the Hausdorff $2q$-measure of* $\mathrm{Supp}(T)$ *is locally finite, then* $\mathrm{Supp}(T)$ *is an analytic subset of M of pure dimension q.*

On the other hand, using [16], Theorem 1 yields the following removability theorem. (For $q=1$ we recover the main result in [2].)

THEOREM 2. — *Let M be a complex manifold of pure dimension n, q a positive integer less than n, $E \subset M$ a closed subset of locally finite Hausdorff $(2n-2q)$-measure, and f a meromorphic mapping from $M \setminus E$ into a complex space Y. If E does not contain any $(n-q)$-dimensional analytic subset of M and Y* possesses the meromorphic extension property in bidimension $(q,n-q)$ *(e.g., if Y is q-complete), then f is continued to a meromorphic mapping from M into Y.*

The organization of this paper is as follows. After a preliminary section, we give in §3 the proofs of Theorems 1 and 2. The last section, §4, establishes connections with the q-pseudoconcavity notion introduced by M. Peternell [20].

2. Preliminaries.

(•) Let T be a metric space and S a subset of T. For $p>0$ and $\varepsilon>0$ let $h^p_\varepsilon(S)$ denote the infimum of all (infinite) sums of the form $\sum \delta(S_n)^p$ where $S=\cup S_n$ is an arbitrary decomposition of S with $\delta(S_n)<\varepsilon$ for all n ($\delta=$ diameter). For $p>0$ the Hausdorff p-measure h^p is defined by

$h^p(S) = \sup_{\varepsilon>0} h^p_\varepsilon(S) \leqslant +\infty$. We define $h^0(S)$ to be equal to the cardinality of S. The usual notion of k-dimensional volume in a Riemannian manifold agrees with h^k up to a constant factor depending only on n (for positive integers k). Thus, if A is a pure k-dimensional analytic set in a domain in $\mathbb{C}^n$, then $h^{2k}(A)$ is equal to a universal constant (depending on k) times the Riemannian volume of the set of regular points of A. For a detailed discussion on Hausdorff measure, see [11].

(•) The definition of q-convexity is the same as in [1], namely; a function $\varphi \in C^\infty(D, \mathbb{R})$, where $D \subset \mathbb{C}^n$ is an open subset, said to be *q-convex* if its *Levi form*

$$\mathcal{L}_\varphi(z)(\xi) := \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i\bar{\xi}_j,\ \xi \in \mathbb{C}^n,$$

has at least $n-q+1$ positive (> 0) eigenvalues for every $z \in D$. This definition can be carried over to complex spaces by local restriction.

Let X be a complex space. X is said to be *q-complete* if there exists a q-convex function $\varphi \in C^\infty(X, \mathbb{R})$ which is *exhaustive*, *i.e.*, the sublevel sets $\{x \in X\,;\, \varphi(x) < c\}, c \in \mathbb{R}$, are relatively compact in X. We choose the normalization such that 1-complete spaces correspond to Stein spaces.

Following [8] and [20] a function $\varphi \in C^0(X, \mathbb{R})$ is said to be *q-convex with corners* on X if every point of X admits an open neighborhod U on which there are finitely many q-convex functions $f_1, \dots, f_k$ such that $\varphi|_U = \max(f_1, \dots, f_k)$. Denote by $F_q(X)$ the set of all functions q-convex with corners on X.

We say that X is *q-complete with corners* if there exists an exhaustion function $\varphi \in F_q(X)$.

DEFINITION 1. — *Let X be a complex space. A subset A of X is said to be* q-concave *(in X) if A is closed and every point of A has an open neighborhood Ω such that $\Omega \setminus A$ is q-complete with corners.*

From [24] (see also [25]) we deduce immediately:

COROLLARY 2. — *Let $\pi : X \to Y$ be a finite surjective holomorphic map of complex spaces and $A \subset Y$ a closed subset. Then A is q-concave in Y if and only if $\pi^{-1}(A)$ is q-concave in X.*

Subsequently we give some facts on q-completeness with corners which allow us to reduce the proof of Theorem 1 to the case when X is a domain in $\mathbb{C}^n$.

Proposition 1. — *Let Y be an analytic set in a complex space X. If Y is q-complete with corners, then Y has a neighborhood system of open sets which are q-complete with corners.*

Proof. — By ([3], Lemma 3) if $\varphi \in F_q(Y)$ and $\eta \in C^0(Y, \mathbb{R})$, $\eta > 0$, then there exists an open neighborhood V of Y in X and $\psi \in F_q(V)$ such that $|\psi - \varphi| < \eta$ on Y. The method of Colţoiu ([4], Theorem 2) or Demailly ([5], the proof of Theorem 1, p. 287) can easily be adapted to our case. □

Proposition 2. — *Let X be a complex space and φ, ψ be continuous exhaustion functions on X such that there is an open neighborhood Ω of the set $\{\varphi = \psi\}$ in X with $\varphi \in F_p(\Omega \cup \{\varphi < \psi\})$ and $\psi \in F_q(\Omega \cup \{\psi < \varphi\})$. Then X is $(p+q)$-complete with corners.*

Proof. — Let $\Lambda := \{\lambda \in C^\infty(\mathbb{R}, \mathbb{R}); \lambda' > 0, \lambda'' \geqslant 0\}$. For $\lambda \in \Lambda$ define $\Phi_\lambda : X \to \mathbb{R}$ by

$$\Phi_\lambda := 1/\big(\exp(-\lambda(\varphi)) + \exp(-\lambda(\psi))\big).$$

It is straightforward to see that Φ_λ is exhaustive for X and it is $(p+q)$-convex with corners on Ω. Now we let $\varepsilon > 0$ be continuous on X such that $\{|\varphi - \psi| \leqslant \varepsilon\} \subset \Omega$; define $W_- = \{\varphi - \psi \leqslant -\varepsilon\}$ and $W_+ = \{\varphi - \psi \geqslant \varepsilon\}$. Clearly W_-, W_+ are closed subsets of X and $W_- \cup W_+ \cup \Omega = X$. The proof is concluded if we show the next

Claim. — *There is $\lambda \in \Lambda$ such that Φ_λ is p-convex with corners on W_- and q-convex with corners on W_+.*

But this follows by adjusting the arguments in [22]. We omit the details. □

Proposition 3. — *Let U, V be open subsets of a complex space X such that U is p-complete with corners and V is q-complete with corners. Then $U \cup V$ is $(p+q)$-complete with corners.*

Proof. — Consider exhaustion functions $f \in F_q(U)$ and $g \in F_q(V)$ for U and V respectively. Let $a \in C^\infty(U, \mathbb{R})$ with $0 \leqslant a \leqslant 1$, $a(x) = 1$ if $x \in U \setminus V$ or $x \in U \cap V$ and $f(x) \leqslant g(x) + 1$; $a(x) = 0$ if $x \in U \cap V$ and $f(x) > g(x) + 2$. Set $D := U \cup V$. Define φ on D by setting

$$\varphi = \begin{cases} f & \text{on } U \setminus V, \\ af + (1-a)(1+g) & \text{on } U \cap V, \\ 1 + g & \text{on } V \setminus U. \end{cases}$$

Then φ is continuous and exhaustive for D.

Let $b \in C^\infty(V,\mathbb{R})$ with $0 \leqslant b \leqslant 1$, $b(x) = 1$ if $g(x) \leqslant \varphi(x) + 1$ and $b(x) = 0$ if $g(x) > \varphi(x) + 2$. Define ψ on D by setting

$$\psi = \begin{cases} bg + (1-b)(1+\varphi) & \text{on } V, \\ 1+\varphi & \text{on } U \setminus V. \end{cases}$$

Then ψ is continuous and exhaustive for D.

Finally, it easy to see that $S := \{\psi < 1 + \varphi\} \subset V$ and $\psi = g$ on S; hence $\psi \in F_q(S)$. Similarly, $T := \{\varphi < 1 + \psi\} \subset U$ and $\varphi = f$ on T; so $\varphi \in F_p(T)$. The conclusion then follows from Proposition 2. □

COROLLARY 3. — *Let A and B be p-concave and q-concave sets in the complex spaces X and Y respectively. Then $A \times B$ is $(p+q)$-concave in $X \times Y$.*

Proof. — Since the assertion is local, we may assume that X and Y are Stein spaces, $X \setminus A$ is p-complete with corners, and $Y \setminus B$ is q-complete with corners. Then $X \times Y \setminus A \times B = X \times (Y \setminus B) \cup (X \setminus A) \times Y$ is $(p+q)$-complete with corners by Proposition 3. □

For a complex space X we introduce [20] the set $G_q(X)$ as follows: For $x_o \in X$ let $G_q(x_o)$ be the set of all functions $g : X \to \mathbb{R}$ such that there are: an open neighborhood U of x_o (which may depend on g) and $f \in F_q(U)$ with $f(x_o) = g(x_o)$ and $f \leqslant g|_U$. Then put

$$G_q(X) := C^0(X,\mathbb{R}) \cap \bigcap_{x \in X} G_q(x).$$

Clearly $F_q(X) \subseteq G_q(X) \subset C^0(X,\mathbb{R})$.

Note that given an open set $D \Subset X$, an $\varepsilon > 0$, and a function $g \in G_q(X)$, there is a function $h \in F_q(D)$ such that $|h - g| < \varepsilon$ on D. See [20], Lemma 1. But we cannot use this fact and the classical perturbation procedure (see for instance [8]) to get a globally defined h since we do not know that given $v \in G_q(X)$ and $\theta \in C_o^\infty(X,\mathbb{R})$ there is $\varepsilon_o > 0$ such that $v + \lambda\theta \in G_q(X)$ for every $\lambda \in \mathbb{R}$, $|\lambda| < \varepsilon_o$. However we can avoid this difficulty since we show:

LEMMA 1. — *The set $F_q(X)$ is dense in $G_q(X)$ in the sense that given an arbitrary $g \in G_q(X)$ and $\eta \in C^0(X,\mathbb{R})$, $\eta > 0$, there is $f \in F_q(X)$ such that $|f - g| < \eta$.*

Proof. — We do this in three steps.

Step 1). Fix $x \in X$ and $\varepsilon > 0$. By definition there is an open neighborhood Ω of x and $\varphi \in F_q(\Omega)$ with $\varphi(x) = g(x)$ and $\varphi \leqslant g$ on Ω. Let W, U be open neighborhoods of x, $W \Subset U \Subset \Omega$, such that $\varphi \geqslant g - \varepsilon$ on U; then let $\theta \in C_o^\infty(U, \mathbb{R})$, $\theta = -1$ on ∂W and $\theta(x) = 1$. If $c > 0$ is small enough, then $\psi := \varphi + c\theta \in F_q(U)$, $\psi < g$ on ∂W, $\psi > g$ on a neighborhood V of x in W, and $|\psi - g| < 2\varepsilon$ on U.

Step 2). The above step shows that for all compact subsets K, L of X, L a neighborhood of K and $\varepsilon > 0$, there are: a finite set of indices I (which depends on K and L), open sets $V_i \Subset W_i \Subset U_i \subset L$ such that $\{V_i\}_{i\in I}$ cover K, functions $f_i \in F_q(U_i)$ with $|f_i - g| < 2\varepsilon$ on W_i, $f_i > g$ on V_i and $f_i < g$ on ∂W_i.

Step 3). Let $\{K_\nu\}_{\nu\in\mathbf{N}}$ be an exhaustion sequence for X by compact sets, $K_o = \emptyset$ (by convention set $K_{-1} = \emptyset$), and $K_\nu \subset \operatorname{int}(K_{\nu+1})$ for all ν. For each ν apply Step 2 to $K = K_\nu \setminus \operatorname{int}(K_{\nu-1})$, $L = K_{\nu+1} \setminus \operatorname{int}(K_{\nu-2})$, and $\varepsilon = (\min_L \eta)/2$. We therefore obtain open sets $V_{i\nu} \Subset W_{i\nu} \Subset U_{i\nu}$ such that the family $\{W_{i\nu}\}$ is locally finite, $\{V_{i\nu}\}_{i\nu}$ is a covering of X, and functions $f_{i\nu} \in F_q(U_{i\nu})$ as in Step 2 from above. Then define $f : X \to \mathbb{R}$ by $f(x) = \max\{f_{i\nu}(x)\,;\, x \in W_{i\nu}\}$, where the maximum is taken over all indices i, ν such that $W_{i\nu} \ni x$. It is straightforward to see that f is continuous, $f \in F_q(X)$, and $g < f < g + \eta$. □

Remark. — It can be shown that for $q > \dim(X)$ the set $F_q(X)$ is dense in the above sense even in $C^0(X, \mathbb{R})$.

From ([20], Lemma 4) we quote:

LEMMA 2. — *Let U be a complex space, V a complex manifold of pure dimension r, and $f \in F_{q+r}(U \times V)$ such that $\sup f < \infty$. Consider $g : U \to \mathbb{R}$ defined by*

$$g(x) = \sup\{f(x, y); y \in V\}, \quad x \in U.$$

Assume that for some $x_o \in U$ there is $y_o \in V$ with $g(x_o) = f(x_o, y_o)$. Then $g \in G_q(x_o)$.

The key proposition for the proof of Theorem 1 is:

PROPOSITION 4. — *Let X and Y be complex manifolds such that Y is of pure dimension r and p-complete with corners. Let A be a $(q + r)$-concave subset in $X \times Y$ such that the natural projection $\pi : A \to X$ is* *proper. Then $\pi(A)$ is $(q+p-1)$-concave in X. In particular, if Y is Stein (i.e. $p=1$), then $\pi(A)$ is q-concave.*

Proof. — Set $m := q+p-1$. We may assume without any loss in generality that X is Stein. The statement of the proposition follows from the next claim.

CLAIM. — *For every relatively compact Stein open subset U of X, the set $U \setminus \pi(A)$ is m-complete with corners.*

In order to show this, consider a relatively compact open subset V of Y which is p-complete with corners and such that $\pi^{-1}(\overline{U} \times \pi(A)) \subset \overline{U} \times V$. Then $K := \overline{U} \times \partial V$ is compact and disjoint from A. Now, since $U \times Y \setminus A$ is $(m+r)$-complete with corners by [20], there exists an exhaustion function $\psi \in F_{m+r}(U \times Y \setminus A)$.

Let $\lambda := \max_K \psi$ and define $\sigma : U \setminus \pi(A) \to \mathbb{R}$ by setting

$$\sigma(x) = \max\{\psi(x,y), y \in \overline{V}\}, x \in U \setminus \pi(A).$$

Clearly σ is continuous. Consider θ be a 1-convex exhaustion function on U and then define $\varphi : U \setminus \pi(A) \to \mathbb{R}$ by setting

$$\varphi = \theta + \max(\lambda, \sigma).$$

Then φ is continuous and exhaustive. To conclude the proof, in view of Lemma 1, it suffices to show that $\varphi \in G_m(x)$ for ever $x \in U \setminus \pi(A)$. Indeed, two cases may occur:

a) If $\sigma(x) > \lambda$, then $\sigma \in G_m(x)$ by Lemma 2. Since $\varphi = \sigma + \theta$ on a neighborhood of x, we get $\varphi \in G_m(x)$.

b) If $\sigma(x) \leqslant \lambda$, then $\theta(x) + \lambda = \varphi(x)$ and since $\lambda + \theta \leqslant \varphi$ on $U \setminus \pi(A)$, $\varphi \in G_1(x)$, *a fortiori*, $\varphi \in G_m(x)$.

The proof is complete. □

(•) Denotes by $\Delta^k(t)$ the open polydisc in $\mathbb{C}^k$ of polyradius $(t, \dots, t)$ centered at the origin. Let n and q be positive integers such that $q < n$. We define the $(q, n-q)$ *Hartogs figure* in $\mathbb{C}^n = \mathbb{C}^q \times \mathbb{C}^{n-q}$ to be the open set $H_q \subset \mathbb{C}^n$ given by

$$H_q := \Big((\Delta^q(1) \setminus \overline{\Delta^q(t)}) \times \Delta^{n-q}(1)\Big) \cup \Big(\Delta^q(1) \times \Delta^{n-q}(s)\Big)$$

where $0<t$, $s<1$. Put $\widehat{H}_q := \Delta^n(1)$, *i.e.* the envelope of holomorphy of H_q.

Following [16] we say that a complex space Y *possesses the meromorphic extension property (in bidimension* $(q, n-q)$*)* if every meromorphic map $f : H_q \to Y$ extends to a meromorphic map $\widehat{f} : \widehat{H}_q \to Y$.

By [16] every q-complete complex space possesses a meromorphic extension property in bidimension $(q, n-q)$ for every integer $n > q$.

DEFINITION 2. — *M be a complex manifold of pure dimension n. We say that a closed subset $A \subset M$ is* pseudoconcave of order q *if for every injective holomorphic map $f : \widehat{H}_q \to M$ such that $f(H_q) \cap A$ is empty, the set $f(\widehat{H}_q) \cap A$ is also empty.*

In this set-up, a variant of Proposition 4 for $Y = \mathbb{C}^r$ is straightforward. See ([10], Lemma 3.6).

Also by ([24], Corollary 5) one has: *A closed subset A of a pure dimensional complex manifold is* pseudoconcave of order q *if and only if A is* q-concave.

Pseudoconcavity of order q is easier to handle; though it does not suit to complex spaces. One has the next examples:

1) Let M be a Stein manifold of pure dimension n and $K \subset M$ a compact set. Then $\widehat{K} \setminus K$ is $(n-1)$-concave in $X \setminus K$. (See [23].)

2) The support of a closed positive current of bidegree (q, q) on a pure dimensional complex manifold is q-concave. (This follows by [12], Corollary 2.6 and the above remark.)

3. Proof of Theorems 1 and 2.

Proof of Theorem 1.

We remark that it suffices to show that A is analytic and for this we distinguish three steps.

Step 1). — Here we reduce the proof to the case when $X \subset \mathbb{C}^n$ is open. For this we need:

LEMMA 3. — *Let Z be a complex space, $X \subset Z$ an analytic subset, and $A \subset X$ a closed subset (not necessarily analytic). If A is q-concave in X and X is r-concave in Z, then A is $(q + r)$-concave in Z.*

Proof. — Let $x_o \in A$ and U be a Stein open neighborhood of x_o in Z such that $U \setminus X$ is r-complete with corners and $(U \setminus A) \cap X$ is q-complete with corners. Since $(U \setminus A) \cap X$ is analytic in $U \setminus A$, there is by Proposition 1 an open subset Ω of $U \setminus A$ which is q-complete with corners and contains $(U \setminus A) \cap X$. Therefore $U \setminus A = (U \setminus X) \cup \Omega$ is $(q+r)$-complete with corners by Proposition 3. □

To complete Step 1, we let $x \in A$, then take a coordinate patch $\iota : U \to D \subset \mathbb{C}^N$ around $x \in X$ with D Stein; hence U is isomorphic to the closed analytic subset $\iota(U)$ of D, hence $\iota(A \cap U)$ is q-concave in $\iota(U)$. Put $p := q+N-n$. Note that $N-p = n-q$. Therefore $\iota(A \cap U)$ is p-concave in D by Lemma 3 since $\iota(U)$ is $(N-n)$-concave in D. On the other hand, $\iota(A \cap U)$ as a closed subset of D has its Hausdorff $(2N-2p)$-measure locally finite.

Step 2). — We give here some general facts for further reduction of the proof of Theorem 1.

Let $E \subset \mathbb{C}^n$ be a locally closed set with $h^{2n-2q+1}(E) = 0$ and suppose $0 \in E$. Then there is a complex $(n-q)$-plane Γ through 0 such that $h^1(E \cap \Gamma) = 0$ ([21], Lemma 2). Hence for a suitable unitary transformation σ of $\mathbb{C}^n$ we have $h^1(\sigma(E) \cap (\mathbb{C}^{n-q} \times \{0\})) = 0$. By ([21], Corollary 2), $\sigma(E) \cap (\partial B(r) \times \{0\})$ is empty for (h^1)-almost all $r > 0$. (Here $B(r)$ denotes the open unit ball in $\mathbb{C}^{n-q}$ of radius r.) Since $\sigma(E)$ is also locally closed in $\mathbb{C}^n$ and $0 \in \sigma(E)$, there is $r > 0$ arbitrary small and a polydisc P in $\mathbb{C}^q$ centered at the origin such that $\sigma(E) \cap (\overline{B(r)} \times \overline{P})$ is closed in $\overline{B(r)} \times \overline{P}$ and $\sigma(E) \cap (\partial B(r) \times \overline{P})$ is empty. In particular, the canonically induced projection map π from $\sigma(E) \cap (B(r) \times P)$ into $B(r)$ is proper.

If furthermore $h^{2n-2q}(E) < \infty$, then $\pi^{-1}(z)$ is finite for (h^{2n-2q})-almost all $z \in B(r)$ ([21], Corollary 4).

Recall that a set $\Gamma \subset \mathbb{C}^n$ is said to be *locally pluripolar* if for every $a \in \Gamma$ there is a connected neighborhood $U \ni a$ and a plurisubharmonic function φ on U, $\varphi \not\equiv -\infty$, such that $\Gamma \cap U \subset \{\varphi = -\infty\}$. In fact, if Γ is locally pluripolar then by [17] one can take $U = \mathbb{C}^n$, so Γ is pluripolar. Note that for $n = 1$ pluripolarity of a set in $\mathbb{C}$ means that it is of *zero-capacity* as used in [18]. Also it is easy to check that for $U \subset \mathbb{C}^n$ open and $S \subset \mathbb{C}^n$ of zero Lebesgue measure, the set $U \setminus S$ is not pluripolar.

Step 3). — Here we conclude the proof.

By Steps 1, 2, and Proposition 4 it remains to show the next lemma.

LEMMA 4. — *Let $U \subset \mathbb{C}^{n-q}$ be an open set, Δ the open unit disc in $\mathbb{C}$, and $A \subset U \times \Delta^q$ a closed subset such that the canonical projection $\pi : A \to U$ is proper. If A is q-concave and $\pi^{-1}(z)$ is finite for z in a non pluripolar subset of U, then A is analytic of pure dimension $n-q$.*

Proof. — For $q = 1$ this is precisely the lemma due to Hartogs-Oka-Nishino [18]. For $q > 1$ we proceed as follows. Notice that it suffices to show the analyticity of A. In order to do this we let $p_j : \Delta^q \to \Delta$, $j = 1, \dots, q$, denote the projection onto the j^{th} component of Δ^q, then let $\sigma_j : A \to U \times \Delta$ naturally induced by p_j. Then σ_j is proper and Proposition 4 implies that $\sigma_j(A)$ is 1-concave in $U \times \Delta$ for all indices $j = 1, \dots, q$. Furthermore if we consider $\pi_j : \sigma_j(A) \to U$ canonically induced, we arrive at the case $q = 1$. So the sets $\sigma_j(A)$ are analytic for all j.

Now, if $\iota : U \times \Delta^q \to (U \times \Delta) \times \cdots \times (U \times \Delta)$ (the product is taken q-times) is given by $\iota(z, t_1, \dots, t_q) = ((z, t_1), \dots, (z, t_q))$, then $A = \iota^{-1}(\sigma_1(A) \times \cdots \sigma_q(A))$, whence the lemma. Thus the proof of Theorem 1. □

Proof of Theorem 2.

Denote by $A^0 :=$ the set of points $x \in A$ such that f extends meromorphically onto a neighborhood of x. Then $A' := A \setminus A^0$ is closed and as the complement to A is locally connected in M these local meromorphic continuations of f in points of A^0 glue together to a unique meromorphic map from $M \setminus A'$ into Y.

Now, we assert that A' is pseudoconcave of order q. For this we let $\Phi : \widehat{H}_q \to M$ be an injective holomorphic map with $\Phi(H_q) \cap A' = \emptyset$. Then $f \circ \Phi$ is meromorphic from H_q into Y, hence it extends to $\widehat{H}_q$; therefore f extends over $\Phi(\widehat{H}_q)$, and by definition $\Phi(\widehat{H}_q) \subset A^0$; whence the desired assertion.

Finally, by Theorem 1, if A' is not the empty set, then A' is analytic of pure dimension $n-q$. But this contradicts the hypothesis, whence the proof. □

4. A final remark.

Motivated by M. Peternell's work ([20], §7) we give:

DEFINITION 3. — *Let X be a complex space of pure dimension n. A closed subset A of X is said to be* q-pseudoconcave *if there is an analytic subset $B \subset X$ such that*

1) $\overline{A \setminus B} = A$.

2) *For each point $x \in A \setminus B$ there is a locally closed analytic subset Y of X which passes through x, $Y \subset A$, and Y is a complex manifold of dimension $n-q$.*

As an example, if A is analytic and $\dim_x A \geqslant n-q$, $\forall\, x \in A$, then A is q-pseudoconcave.

Let now r be a non-negative integer and suppose X is purely dimensional. We say that X has property (E_r), if there is $\varphi \in F_{n+r}(X \times X \setminus \Delta_X)$, where Δ_X is the diagonal set of $X \times X$, such that $\varphi(x_\nu, x) \to +\infty$ if $x_\nu \to x, x_\nu \neq x$, $\forall\, x \in X$. Condition (E_r) holds locally on X if every point of X admits an open neighborhood U which satisfies (E_r).

The next proposition is an easy consequence of ([20], Lemma 9).

PROPOSITION 5. — *Let X be a pure dimensional complex space such that (E_r) holds locally. Then every q-*pseudoconcave *subset of X is* $(q+r)$-concave.

The importance of the condition (E_r) resides in the fact that, for example, if a Stein space X fulfils (E_0), then every locally Stein open subset of X is Stein. It is easy to check for a Stein manifold that (E_0) holds. However, this fails, in general, if we allow singularities. For example, we let X be the Segre cone in $\mathbb{C}^4$, $X = \{xy = zw\}$. Clearly the hypersurface $A = \{x = z = 0\}$ is 1-pseudoconcave. Now, if (E_0) would hold locally on X, then A will be 1-concave; and as X has isolated singularities $X \setminus A$ will be Stein. But this is absurd since $X \setminus A$ is biholomorphic to $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$.

COROLLARY 4. — *If X is a complex manifold, then every q-*pseudoconcave *subset of X is also q-*concave.

Example 3. — For every positive integer q there is an open subset X of $\mathbb{C}^{q+1}$ and a *q-concave* subset $A \subset X$ which is **not** *q-pseudoconcave.*

To do this we consider a compact subset K of $\mathbb{C}^2$ such that $\widehat{K} \setminus K$ contains no analytic disc. See [26] for the existence of K. Put $X := (\mathbb{C}^2 \setminus K) \times \mathbb{C}^{q-1}$ and $A := (\widehat{K} \setminus K) \times \{0\}$. Then A is **not** q-pseudoconcave in X; however, by Example 1 in §2 and Corollary 3 it is easily seen that $\widehat{K} \setminus K$ is q-concave in X. □

The corresponding version of Theorem 1 reads:

THEOREM 3. — *Let A be a closed subset of a pure n-dimensional complex space X such that A is* q-pseudoconcave *and its Hausdorff $(2n-2q)$-measure is locally finite. Then A is analytic of pure dimension $n-q$.*

Proof. — If $\iota : U \to D$ is a local path of X, where D is an open subset of $\mathbb{C}^N$, then $\iota(A \cap U)$ is $(N-n+q)$-pseudoconcave in D. Now we conclude by the above corollary and Theorem 1. □

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